Name:

Directions: This exam has six questions, numbered 1 to 6, spread across six pages. Show all work and clearly mark your final answers. All final answers should be exact, written either as whole numbers, exact decimals, or fractions in lowest terms. Calculators may be used (indicate where you used a calculator to perform complicated calculations), but other study aids are not permitted.

1. (10 pts) Find the equation of the line passing through the points $(-2, 3)$ and $(4, -9)$. Write the equation in slope-intercept form.

2. (12 pts) Suppose $A = \begin{pmatrix} -1 & 2 & 2 \\ 4 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$.

   (a) (6 pts) Find $3A - B$ (if this expression is undefined, say so).

   (b) (6 pts) Find BA (if this expression is undefined, say so).

3. (20 pts) Find all solutions to the following system of linear equations (if there is no solution, say so):

   (a) (10 pts)

$$\begin{cases} 2x - 5y = 13 \\ 3x + 3y = 9 \end{cases}.$$

   (b) (5 pts) *Note:* The first two equations of this system are the same equations as those in part (a).

$$\begin{cases} 2x - 5y = 13 \\ 3x + 3y = 9 \\ 4x - 2y = 18 \end{cases}.$$

(c) (5 pts) *Note:* The first three equations of this system are the same equations as those in part (b).

$$\left\{\begin{array}{rcrcr} 2x & - & 5y & = & 13 \\ 3x & + & 3y & = & 9 \\ 4x & - & 2y & = & 18 \\ -2x & + & y & = & -5 \end{array}\right. .$$

4. (15 pts) Given each of the following matrices, state the next row operation that should be performed on the matrix if one was trying to transform the matrix into row-echelon form (you only need to give one answer, even though there is more than one reasonable choice of operation in some cases):

(a) (5 pts) $\begin{pmatrix} 1 & 2 & 0 & -2 & 1 & 2 \\ 0 & 4 & 2 & 1 & 4 & -1 \\ 0 & 1 & 3 & -2 & 0 & 2 \\ 0 & -2 & 4 & 1 & 3 & -7 \end{pmatrix}$

(b) (5 pts) $\begin{pmatrix} 1 & 1 & 5 & -2 & 1 & 2 & 4 \\ 0 & 1 & 2 & -3 & 4 & -1 & -3 \\ 0 & 3 & 1 & 1 & 0 & 2 & 15 \\ 0 & -2 & 6 & -2 & 3 & -4 & -7 \end{pmatrix}$

(c) (5 pts) $\begin{pmatrix} 1 & -2 & 4 & -7 & 1 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}$

5. (36 pts) Consider a Markov chain with the following transition matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

(a) (7 pts) Show this Markov chain is regular.

(b) (7 pts) Find the probability that you are in state 3 after two steps, given that you start in state 1.

(c) (10 pts) Recall from the previous page that $P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix}$. Suppose your initial distribution is given by $X_0 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \end{pmatrix}$. Find the probability you are in state 3 after two steps.

(d) (12 pts) What is the probability that you are in state 1 at some arbitrary time in the distant future?

6. (7 pts) Find the expected value of a random variable with the following distribution:

X	2	4	6	8
$P(X)$	.5	.2	.1	.2

1. First find the slope: $m = \frac{y_2-y_1}{x_2-x_1} = \frac{-9-3}{4-(-2)} = \frac{-12}{6} = -2$. Then by the point-slope formula $y-y_1 = m(x-x_1)$, we have $y-(-9) = -2(x-4)$. Distribute the -2 on the left-hand side of this expression and subtract 9 from both sides to get $y = -2x-1$.

2. (a) This is undefined because you cannot add or subtract matrices of different sizes.

   (b) Since the number of columns of B equals the number of columns of A, the product BA is defined. It is equal to

$$\begin{pmatrix} -2 & 4 & 4 \\ 6 & -4 & -3 \end{pmatrix}.$$

3. (a) Multiply the first equation by 3 and the second equation by 5 to get

$$\left\{ \begin{array}{rcrcr} 2x & - & 5y & = & 13 \\ 3x & + & 3y & = & 9 \end{array} \right. \begin{array}{c} \rightarrow \\ \rightarrow \end{array} \left\{ \begin{array}{rcrcr} 6x & - & 15y & = & 39 \\ 15x & + & 15y & = & 45 \end{array} \right. .$$

   Then add the equations on the right to get $21x = 84$; therefore $x = 4$. Substitute this into any equation which contains x and y and solve for y; in particular $2(4) - 5y = 13$ so $-5y = 5$ and $y = -1$. There is one solution: $(4, -1)$.

   (b) From part (a), the first two equations of this system have one solution: $(4, -1)$. Check this solution in the third equation:

$$4(4) - 2(-1) = 16 + 2 = 18.$$

   We see that $(4, -1)$ also satisfies the third equation, so this system has one solution: $(4, -1)$.

   (c) From part (b), the first three equations of this system have one common solution: $(4, -1)$. Check this solution in the fourth equation:

$$-2(4) + (-1) = -9 \neq -5.$$

   We see that $(4, -1)$ is not a solution of the fourth equation, so this system has no solution.

4. (a) We need a row operation which produces a 1 in the entry in row 2 and column 2. Furthermore, we cannot use a row operation involving row 1 since column 1 is already "finished". So some possible answers are switching Row 2 and Row 3, or adding (-3) times Row 3 to Row 2, or multiplying Row 2 by $1/4$.

   (b) Here we need to produce a zero underneath the two ones in column 2. To do this, we can add (-3) times Row 2 to Row 3 (producing a zero in row 3) or add 2 times Row 2 to Row 4 (producing a zero in row 4).

(c) Here, since the first three columns are "finished", we can only do a row operation on the fourth row; we want to turn the 2 into a 1 so we multiply Row 4 by (1/2).

5. (a) First, calculate P^2 by direct matrix multiplication:

$$P^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 3/16 & 3/4 & 1/16 \\ 3/8 & 7/16 & 3/16 \end{pmatrix}.$$

Since all entries of P^2 are positive, P is regular.

(b) This is the entry of P^2 (since two steps have occurred) in row 1 (your initial state) and column 3 (your state in the future); from part (a) this is 1/4.

(c) First, calculate the distribution after two steps by matrix multiplication:

$$X_0 P^2 = \begin{pmatrix} 1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 3/16 & 3/4 & 1/16 \\ 3/8 & 7/16 & 3/16 \end{pmatrix} = \begin{pmatrix} 25/64 & 27/64 & 3/16 \end{pmatrix}$$

The probability you are in state 3 after two steps is the third entry of this distribution, which is 3/16.

(d) First, since the Markov chain is regular, there is an equilibrium vector V which we can find by setting $V = \begin{pmatrix} a & b & c \end{pmatrix}$ and solving $VP = V$:

$$VP = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} b/2 + c/4 & a + b/4 + 3c/4 & b/4 \end{pmatrix}.$$

Setting this equal to $\begin{pmatrix} a & b & c \end{pmatrix}$, and remembering that $a+b+c = 1$, we obtain the system of equations

$$\begin{cases} \phantom{a + {}} b/2 + c/4 = a \\ a + b/4 + 3c/4 = b \\ \phantom{a + {}} b/4 = c \\ a + b + c = 1 \end{cases}$$

The third equation yields $b = 4c$; substituting this into the first equation we get $4c/2 + c/4 = a$ which means $a = 9c/4$. Finally, substituting into the last equation we get $9c/4 + 4c + c = 1$; solve for c to obtain $c = 4/29$. Then

$$V = \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} 9c/4 & 4c & c \end{pmatrix} = \begin{pmatrix} 9/29 & 16/29 & 4/29 \end{pmatrix}.$$

The probability you are in state 1 at some time in the distant future is the first entry of this vector, namely 9/29.

6. $EX = \sum XP(X) = 2(.5) + 4(.2) + 6(.1) + 8(.2) = 4.$