Quiz 1: Basic Probability

1. Ten Scrabble tiles are placed in a bag. Four of the tiles have the letter A printed on them, and there are two tiles each with the letters B, C and D on them.
   (a) Suppose one tile is drawn from the bag at random. What is the probability that a letter other than C is drawn?
   (b) Suppose two tiles are drawn from the bag without replacement. What is the probability that the drawn tiles have the same letter on them (simplify your answer)?
2. Suppose E and F are independent events with $P(E) = \frac{1}{2}$ and $P(F) = \frac{2}{3}$. Find $P(E \cup F)$ (simplify your answer).
3. Suppose a factory has two machines (call them A and B) which produce a certain part. Machine A produces 60% of the parts; machine B produces 40% of the parts. Unfortunately, 3% of the parts produced by machine A are defective and 5% of the parts produced by machine B are defective. What is the probability that a part was produced by machine B, given that it is defective? (This answer does not need to be simplified.)

Quiz 2: Permutations and Combinations

1. You have seven different paintings and choose four of them to hang on your wall, from left to right. In how many different ways can you hang the four paintings? Simplify your answer.
2. Suppose you have six paintings: three identical copies of a work by Renoir, and three identical copies of a work by Monet. In how many distinguishable ways can you hang these six paintings on your wall, from left to right? Simplify your answer.
3. Suppose six playing cards are chosen randomly from a standard deck of 52 cards.
   (a) How many different six card hands are possible?
   (b) What is the probability that two of the six chosen cards are clubs?
   (c) What is the probability that at least three of the six chosen cards are aces?

Quiz 3: Binomial, Multinomial and Continuous Distributions

1. A cube has three of its faces painted red, two painted white, and one painted black. The cube is rolled 15 times and on each roll, the color of the face which faces upward is recorded. (Assume all faces are equally likely to land upward on any one roll.)
   (a) What is the probability that among the 15 rolls, the black face lands up twice? Leave your answer in terms of combinatorics notation.
   (b) What is the probability that among the 15 rolls, a red face lands up 10 times and a white face lands up 3 times? Leave your answer in terms of combinatorics notation.
2. Suppose a number X is chosen from the interval $[10, 20]$ uniformly.
   (a) What is the probability that $X = 12$?

(b) What is the probability that $X \geq 17$?

Quiz 4: Normal Distributions

1. The Math SAT scores are scaled so that the mean score is 500 and the standard deviation of the scores is 100; assume that these scores are normally distributed. Johnny Johnson scored 640 on his Math SAT.

   (a) Calculate the $z-$score of Johnny's result.

   (b) Estimate (to four decimal places) the probability that another student scored worse than Johnny on their Math SAT.

2. There are ten beads in a bag. Three of them are red. Suppose you draw 150 beads from the bag, replacing each bead before you draw the next one. Use the normal approximation to the binomial distribution to estimate (to four decimal places) the probability that of the 150 beads drawn, at least 45 but at most 60 are red.

Quiz 5: Expected Value and Linear Equations

1. Jane estimates that the chances of her house being flooded in the coming year are 2%. If she buys flood insurance for her house, the insurance company will pay her $\$500,000$ if her house floods and will pay her nothing if her house does not flood.

   (a) Calculate the expected value of the amount the insurance company will pay to Jane.

   (b) If the flood insurance costs Jane \$7500, is it worth the cost (ignoring issues unrelated to the mathematics)? Why or why not?

2. Find the slope of each of the following lines:

   (a) The line passing through $(0, 4)$ and $(-2, 8)$.

   (b) The line $5x + 2y = 7$.

   (c) A line perpendicular to $y = -3x + 2$.

Quiz 6: Systems of equations in two variables and augmented matrices

1. Solve the system of equations

$$\begin{cases} 4x - 3y = 1 \\ -5x + 2y = 4 \end{cases}$$

2. Consider the system of equations

$$\begin{cases} v = 2x + z \\ w - x + 3y = 7 \\ x + 5z = 4 \\ z = 2 \end{cases}$$

   Write the augmented matrix for this system of equations. Is the matrix in row-echelon form? Why or why not?

Quiz 7: Solving systems of equations in row-echelon form

For each matrix, state the number of solutions the corresponding system of equation and solve the system.

1. $\begin{pmatrix} 1 & 4 \end{pmatrix}$ 2. $\begin{pmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & -1 & 3 & 11 \\ 0 & 1 & -3 & 8 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Quiz 8: Markov chains

A soccer team's performance in a game depends only on its performance in its previous game. In particular:

- If the soccer team wins a game, then it is 60% likely to win its next game, 10% likely to tie its next game, and 30% likely to lose its next game.
- If the soccer team ties a game, then it is 60% likely to win its next game and 40% likely to lose its next game.
- If the soccer team loses a game, then it is 50% likely to win its next game, 20% likely to tie its next game, and 30% likely to lose its next game.

We model the soccer team's performance by a Markov chain where state 1 corresponds to the team winning, state 2 corresponds to the team tieing, and state 3 corresponds to the team losing.

1. Write the transition matrix P for this Markov chain.
2. Find the time 2 transition matrix P^2.
3. Find the probability that the team wins the second game it plays from now, given that it ties the game it is playing now.
4. Suppose the team will win the game it is playing now with probability 80% and tie the game it is playing now with probability 10%. What is the probability the team will win its next game?

Quiz 1 Solutions:

1. (a) $P(C') = 1 - P(C) = 1 - \frac{2}{10} = \frac{8}{10}$.

   (b)

$$\begin{aligned} P(\text{same letter}) &= P(AA) + P(BB) + P(CC) + P(DD) \\ &= \frac{4}{10}\left(\frac{3}{9}\right) + \frac{2}{10}\left(\frac{1}{9}\right) + \frac{2}{10}\left(\frac{1}{9}\right) + \frac{2}{10}\left(\frac{1}{9}\right) \\ &= \frac{12}{90} + 3\left(\frac{2}{90}\right) \\ &= \frac{18}{90} = \frac{1}{5}. \end{aligned}$$

2. Since $E \perp F$, $P(E \cap F) = P(E)P(F) = \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}$. Now by Inclusion-Exclusion,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{5}{6}.$$

3. Let A and B represent the event that the part was produced by machine A or B, respectively. Let D represent the event that the part is defective. By Bayes' Law,

$$\begin{aligned} P(B|D) &= \frac{P(D|B)P(B)}{P(D|B)P(B) + P(D|A)P(A)} \\ &= \frac{(.05)(.40)}{(.05)(.40) + (.03)(.60)} \\ &= \frac{.02}{.02 + .018} \\ &= \frac{.02}{.038} = \frac{20}{38} = \frac{10}{19}. \end{aligned}$$

Quiz 2 Solutions:

1. $P(7,4) = \frac{7!}{(7-4)!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$

2. $\frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{6} = 20.$

3. (a) $\binom{52}{6}$.

   (b) $\dfrac{\binom{13}{2}\binom{39}{4}}{\binom{52}{6}}$.

   (c) $\dfrac{\binom{4}{3}\binom{48}{3}}{\binom{52}{6}} + \dfrac{\binom{4}{4}\binom{48}{2}}{\binom{52}{6}} = \dfrac{4\binom{48}{3}}{\binom{52}{6}} + \dfrac{\binom{48}{2}}{\binom{52}{6}}.$

Quiz 3 Solutions:

1. (a) This is binomial with $n = 15$, $k = 2$, and $p = 1/6$ so

$$P = \binom{15}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{13}.$$

   (b) This is multinomial with $n = 15$. Let the outcomes O_1, O_2 and O_3 of each trial be rolling a red face, a white face, and a black face, respectively. Then $p_1 = 3/6 = 1/2$, $p_2 = 2/6 = 1/3$, $p_3 = 1/6$ and $k_1 = 10$, $k_2 = 3$ and $k_3 = 15 - 10 - 3 = 2$. So

$$P = \frac{15!}{10!3!2!}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{3}\right)^3\left(\frac{1}{6}\right)^2.$$

2. (a) 0 (the probability of choosing any single number in any continuous probability distribution is zero).

(b) The density function here is constant on the interval $[10, 20]$ because the number is chosen uniformly; it must have height $1/10$ since the total area under the density function must be 1. The probability that $X \geq 17$ is the area under this density function between $x = 17$ and $x = 20$; this is the area of a rectangle with width 3 and height $1/10$ which is $3/10$.

Quiz 4 Solutions:

1. (a) $z = \frac{x-\mu}{\sigma} = \frac{640-500}{100} = 1.40$.

   (b) Let X represent the score from some other randomly chosen student. We have

$$\begin{aligned} P(X < 640) &= \text{area under } n(500, 640) \text{ to left of } x = 640 \\ &= \text{area under } n(0,1) \text{ to left of } z = 1.40 \\ &\approx .9192. \end{aligned}$$

2. Let X be the number of red beads drawn; this is binomial with $n = 150$ and $p = 3/10 = .3$. We assume X is normally distributed with mean $\mu = np = 150(.3) = 45$ and standard deviation $\sigma = \sqrt{np(1-p)} = \sqrt{150(.3)(.7)} = \sqrt{31.5} \approx 5.567$. Now,

$$\begin{aligned} P(45 \leq X \leq 60) &= \text{area under } n(45, 5.567) \text{ between } x_1 = 44.5 \text{ and } x_2 = 60.5 \\ &= \text{area under } n(0,1) \text{ between } z_1 = -.09 \text{ and } z_2 = 2.78 \\ &= \text{area under } n(0,1) \text{ to left of } z_2 = 2.78 - \\ &\qquad \text{area under } n(0,1) \text{ to left of } z_1 = -.09 \\ &\approx .9973 - .4641 \\ &= .5332. \end{aligned}$$

Quiz 5 Solutions:

1. (a) Let X represent the amount the insurance company will pay Jane. X takes the value $500,000$ with probability .02 and takes the value 0 with probability .98. So $EX = (500000)(.02) + (0)(.98) = 10000$.

   (b) Yes, because the expected value she will be repaid is greater than the cost.

2. (a) $y = \frac{y_2-y_1}{x_2-x_1} = \frac{8-4}{-2-0} = \frac{4}{-2} = -2$.

   (b) Solve for y to get $y = (-5/2)x + 7/2$; the slope is the coefficient on the x term which is $-5/2$.

   (c) The given line has slope -3, so a line perpendicular to this line has slope $1/3$, the negative reciprocal of -3.

Quiz 6 Solutions

1. Multiplying the top equation by 5 and the bottom equation by 4, we have

$$\begin{cases} 4x - 3y = 1 & \rightarrow \quad 20x - 15y = 5 \\ -5x + 2y = 4 & \rightarrow \quad -20x + 8y = 16 \end{cases}$$

   Add the two equations on the right to produce $-7y = 21$, so $y = -3$. Then substitute this into the first equation to obtain $4x - 3(-3) = 1$, i.e. $4x = -8$ so $x = -2$. The solution is $(-2, -3)$.

2. First, rewrite the first equation in standard form as $v - 2x - z = 0$. Then, if we let the columns represent the corresponding variables in alphabetical order, we have

$$\begin{pmatrix} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

This is the augmented matrix for this system, and it is in row-echelon form because the diagonal entries are ones followed by zeros and all entries below the diagonal are zero.

Quiz 7 Solutions:

1. One solution: $x = 4$ (could also be written (4)).
2. Infinitely many solutions; solve by back substitution to get $y = 2z + 1$ and $x = 2y + 3z + 2 = 2(2z + 1) + 3z + 2 = 7z + 4$. So all solutions are of the form $(7z + 4, 2z + 1, z)$.
3. One solution: $z = -2$; substituting this into the second equation we obtain $y - 3(-2) = 8$ so $y = 2$; substituting this into the first equation we obtain $x - 2 + 3(-2) = 11$ so $x = 19$. The solution is $(19, 2, -2)$.

Quiz 8 Solutions:

1. There are three states, so the matrix is 3×3:

$$P = \begin{pmatrix} .6 & .1 & .3 \\ .6 & 0 & .4 \\ .5 & .2 & .3 \end{pmatrix}.$$

2. By multiplying P by itself, we get

$$P^2 = \begin{pmatrix} .57 & .12 & .31 \\ .56 & .14 & .3 \\ .57 & .11 & .32 \end{pmatrix}.$$

3. This is the entry in the first column and second row of the matrix P^2, which is .56.
4. We are given an initial probability vector $X_0 = (.8\ .1\ b)$ where b is not explicitly given. Of course, since the sum of the entries of a probability vector add to 1, it must be the case that $b = 1 - .8 - .1 = .1$. So $X_0 = (.8\ .1\ .1)$. Now the time 1 distribution is

$$X_0 P = \begin{pmatrix} .8 & .1 & .1 \end{pmatrix} \begin{pmatrix} .6 & .1 & .3 \\ .6 & 0 & .4 \\ .5 & .2 & .3 \end{pmatrix} = \begin{pmatrix} .59 & .1 & .31 \end{pmatrix}.$$

Since winning corresponds to state 1, the probability the team wins its next game is the first entry of this row vector, which is .59.