1. Consider a two-player strategic game with the following payoff matrix:

$$\begin{array}{c} \\ R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} (4,2) & (1,4) & (0,1) \\ (1,5) & (0,3) & (2,0) \\ (3,4) & (7,3) & (4,1) \end{pmatrix} \end{array}$$

   (a) Show that this game has no pure strategy Nash equilibria.

   (b) Find the mixed strategy Nash equilibrium of this game.

2. Suppose that in Cournot's duopoly game, the two firms have cost functions

$$C_1(q_1) = \frac{1}{6}q_1^2 \quad \text{and} \quad C_2(q_2) = q_2^2$$

and that the demand function D is given by

$$D(q_1, q_2) = \begin{cases} 18 - q_1 - 2q_2 & \text{if } q_1 + 2q_2 \leq 18 \\ 0 & \text{if } q_1 + 2q_2 > 18 \end{cases}$$

Find the Nash equilibrium of this game (remember that in Cournot's game, the utility functions π_1 and π_2 for firms 1 and 2 respectively are

$$\pi_1(q_1, q_2) = q_1\, D(q_1, q_2) - C_1(q_1) \quad \text{and} \quad \pi_2(q_1, q_2) = q_2\, D(q_1, q_2) - C_2(q_2)$$

where q_1 and q_2 are the number of units produced by firms 1 and 2, respectively).

3. Find all subgame perfect equilibria of the extensive game with perfect information whose game tree is given in Figure 1. In this game, player 1 chooses an edge at the root R and at the vertex C, and player 2 chooses an edge at the vertices A and B.

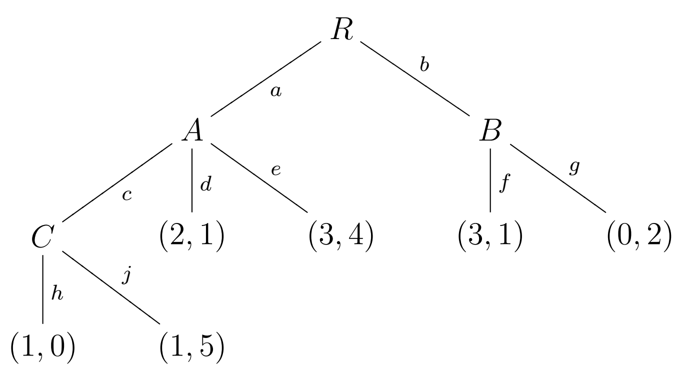

Figure 1: Diagram of the extensive game with perfect information of Problem 3.

4. Consider the three-player extensive game whose game tree is given in Figure 2. In this game, player 1 chooses an edge at the root R and on the information set $\{X, Y\}$. Player 2 chooses only at the vertex A; player 3 chooses on the information set $\{B, C, D\}$.

   (a) Suppose player 1 chooses from the edges (a, c, d) according to the probability distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and player 2 chooses edge b with probability $1/4$ and chooses edge x with probability $3/4$. What belief system of player 3 is consistent with this behavior of player 1 and player 2?

   (b) What beliefs of player 1 on the information set $\{X, Y\}$ justify player 1 always selecting edge p (as opposed to edge q)?

   (c) Find a weak sequential equilibrium, including the behavioral profile and belief system, where each player uses a pure strategy (there is more than one correct answer). Please clearly identify your final answer.

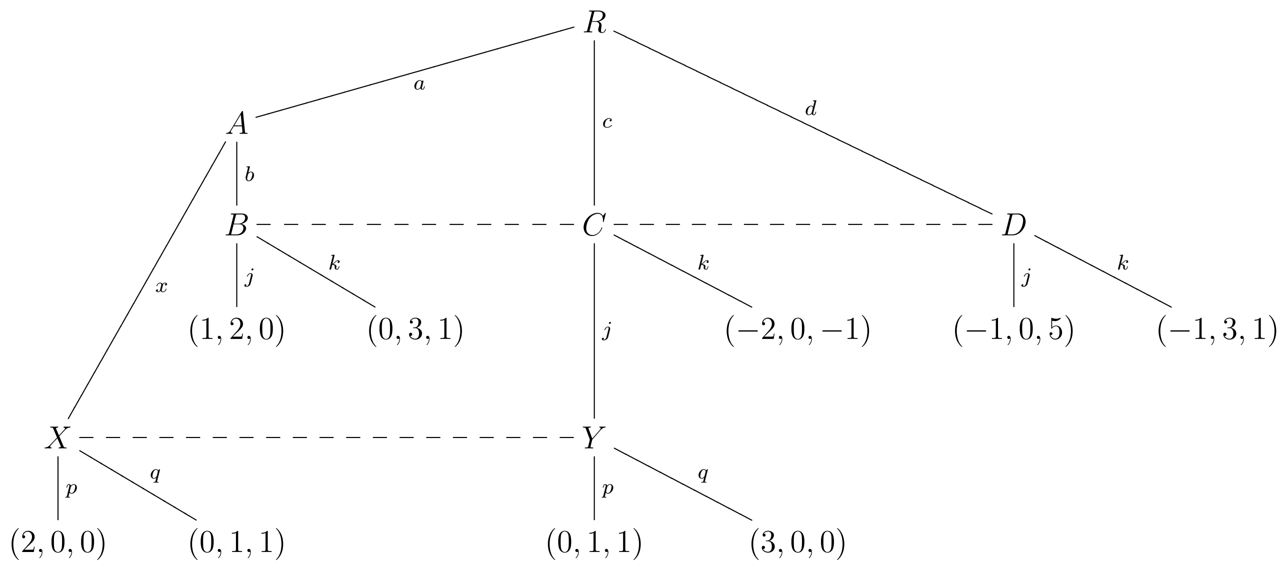

Figure 2: Diagram of the extensive game with imperfect information of Problem 4.

5. Consider the infinitely repeated game with discount parameter $\delta \in (0, 1)$ of Prisoner's Dilemma where each playing of the game has payoffs given by

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (5,5) & (0,6) \\ (6,0) & (3,3) \end{pmatrix} \end{array}.$$

Consider the strategy $\mathcal{S}$ where action C is chosen in the first stage and action C is chosen if the outcome of the previous stage was (C, C) or (D, D) (action D is chosen if the outcome of the previous stage was (D, C) or (C, D)).

   (a) Suppose both players use strategy $\mathcal{S}$, and suppose that $\delta = \frac{1}{2}$. Find the discounted payoff for player 1.

(b) (The assumptions of part (a) do not apply to this question.) Suppose player 2 uses strategy $\mathcal{S}$. For what values of δ (if any) does player 1 prefer using strategy $\mathcal{S}$ to using the strategy where he (player 1) chooses D at every stage?

6. Consider a $n-$player (assume $n \geq 2$) strategic game which works as follows: first, each of the n players antes \$1 into a pot. Then each of the players select a whole number greater than or equal to 0, but less than or equal to 160. The player whose number is closest to, but without going over, three-fourths of the average of the numbers selected by the players wins the pot (in case of a tie, all tied players split the pot evenly).

   Note: When the exam was given, I forgot the phrase "without going over". This changed the problem slightly; the way I worded it there were actually two Nash equilibria.

   (a) Find a Nash equilibrium (in pure strategies) of this $n-$player game. Explain why your answer is in fact a Nash equilibrium.

   (b) Show that the Nash equilibrium you found in part (a) is the only pure strategy Nash equilibrium of the game.

1. (a) If (R_j, C_k) is a pure strategy Nash equilibrium, then $R_j = B_1(C_k)$ and $C_k = B_2(R_j)$ where B_1 and B_2 are the best response functions. Now calculate B_1 for each row and column:

$$B_1(C_1) = R_1; \quad B_1(C_2) = R_3; \quad B_1(C_3) = R_3$$

$$B_2(R_1) = C_2; \quad B_2(R_2) = C_1; \quad B_2(R_3) = C_1$$

We see that there is no row and column for which they are best responses to one another. So there is no pure strategy Nash equilibrium.

(b) First, observe that Row 3 strictly dominates Row 2 and Column 2 strictly dominates Column 3. So if $(\overrightarrow{\alpha}, \overrightarrow{\beta})$ is a mixed strategy Nash equilibrium, we must have $\overrightarrow{\alpha} = (p, 0, 1-p)$ and $\overrightarrow{\beta} = (q, 1-q, 0)$ and we can therefore associate $\overrightarrow{\alpha} \leftrightarrow p$ and $\overrightarrow{\beta} \leftrightarrow q$. Now calculate best response functions, starting with expected value calculations:

$$E_1(R_1, q) = 4q + 1(1-q) = 3q + 1; \quad E_1(R_3, q) = 3q + 7(1-q) = 7 - 4q.$$

Setting these equal and solving for q, we obtain $q = \frac{6}{7}$. So the best response function for player 1 is

$$B_1(q) = \begin{cases} 1 & \text{if } q < \frac{6}{7} \\ [0,1] & \text{if } q = \frac{6}{7} \\ 0 & \text{if } q > \frac{6}{7} \end{cases}.$$

Repeating this for player 2, we have:

$$E_2(C_1, p) = 2p + 4(1-p) = 4 - 2p; \quad E_2(C_2, p) = 4p + 3(1-p) = 3 + p.$$

Setting these equal and solving for p, we obtain $p = \frac{1}{3}$. So the best response function for player 2 is

$$B_2(p) = \begin{cases} 0 & \text{if } p < \frac{1}{3} \\ [0,1] & \text{if } p = \frac{1}{3} \\ 1 & \text{if } p > \frac{1}{3} \end{cases}.$$

Graphing these best response functions on the same axes (or by inspection), the only intersection of these functions is at $p = \frac{1}{3}, q = \frac{6}{7}$. Hence the mixed strategy Nash equilibrium is at this point, i.e. is

$$(\overrightarrow{\alpha}, \overrightarrow{\beta}) = \left(\left(\frac{1}{3}, 0, \frac{2}{3}\right), \left(\frac{6}{7}, \frac{1}{7}, 0\right)\right).$$

2. First, explicitly write out the utility functions for each player using the provided information:

$$\pi_1(q_1,q_2) = \begin{cases} q_1(18-q_1-2q_2)-\frac{1}{6}q_1^2 & \text{if } q_1+2q_2 \le 18 \\ -\frac{1}{6}q_1^2 & \text{if } q_1+2q_2 > 18 \end{cases}$$

$$\pi_2(q_1,q_2) = \begin{cases} q_2(18-q_1-2q_2)-q_2^2 & \text{if } q_1+2q_2 \le 18 \\ -q_2^2 & \text{if } q_1+2q_2 > 18 \end{cases}$$

To find the Nash equilibrium, we use best response functions. First, $B_1(q_2)$ is the choice of q_1 maximizing $\pi_1(q_1,q_2)$; to find this we use partial derivatives:

$$\frac{\partial \pi_1}{\partial q_1} = \begin{cases} 18-2q_1-2q_2-\frac{1}{3}q_1 & \text{if } q_1+2q_2 \le 18 \\ -\frac{1}{3}q_1 & \text{if } q_1+2q_2 > 18 \end{cases}$$

Setting this derivative equal to zero and solving for q_1 in terms of q_2, we obtain

$$q_1 = \begin{cases} \frac{3}{7}(18-2q_2) & \text{if } q_1+2q_2 \le 18 \\ 0 & \text{if } q_1+2q_2 > 18 \end{cases}.$$

It is clear that these choices of q_1 yield maxima, since the utility function is quadratic with negative leading coefficient. Therefore these values of q_1 are best responses. Next, use similar calculations to find $B_2(q_1)$:

$$\frac{\partial \pi_2}{\partial q_2} = \begin{cases} 18-q_1-4q_2-2q_2 & \text{if } q_1+2q_2 \le 18 \\ -2q_2 & \text{if } q_1+2q_2 > 18 \end{cases}$$

Set this equal to zero and solve for q_2 to obtain

$$q_2 = \begin{cases} \frac{1}{6}(18-q_1) & \text{if } q_1+2q_2 \le 18 \\ 0 & \text{if } q_1+2q_2 > 18 \end{cases}.$$

Again, these choices of q_2 yield maxima, and are therefore best responses. Finally, to find the Nash equilibrium we solve the system of equations relating q_1 and q_2 obtained above:

$$\begin{cases} B_1(q_2) = q_1 = \frac{3}{7}(18-q_2) \\ B_2(q_1) = q_2 = \frac{1}{6}(18-q_1) \end{cases}$$

This yields $q_1 = 6$, $q_2 = 2$ so the Nash equilibrium is $(6,2)$. This is a valid answer since $q_1 + 2q_2 = 6 + 2 \cdot 2 = 10 \le 18$.

3. Use backward induction. First observe that player 1 can choose either h or j at vertex C.

   If player 1 chooses h at C, then player 2 will choose e at A and g at B. Finally, player 1 will then choose a at R so we obtain the subgame perfect equilibrium (a,e,g,h).

   If player 1 chooses j at C, then player 2 will choose c at A and g at B. Finally, player 1 will then choose a at R so we obtain the subgame perfect equilibrium (a,c,g,j).

4. (a) We are given elements of the behavioral profile β; in particular $\beta(a) = \beta(b) = \beta(c) = \frac{1}{3}$ and $\beta(b) = \frac{1}{4}$ and $\beta(x) = \frac{3}{4}$. Now we can find the probabilities $\beta(B), \beta(C), \beta(D)$ that the vertices B, C, D are reached given these behaviors. In particular:

$$\beta(B) = \beta(a)\beta(b) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}; \quad \beta(C) = \beta(c) = \frac{1}{3}; \quad \beta(D) = \beta(d) = \frac{1}{3}.$$

Now let $\mu = (\mu(B), \mu(C), \mu(D))$ represent the belief of player 3 on the information set $\{B, C, D\}$. For this to be consistent, we must have

$$\begin{aligned}
\mu(B) &= \frac{\beta(B)}{\beta(B)+\beta(C)+\beta(D)} = \frac{1/12}{1/12+1/3+1/3} = \frac{1/12}{3/4} = \frac{1}{9} \\
\mu(C) &= \frac{\beta(C)}{\beta(B)+\beta(C)+\beta(D)} = \frac{1/3}{3/4} = \frac{4}{9} \\
\mu(D) &= \frac{\beta(D)}{\beta(B)+\beta(C)+\beta(D)} = \frac{1/3}{3/4} = \frac{4}{9}
\end{aligned}$$

so $\mu = (\frac{1}{9}, \frac{4}{9}, \frac{4}{9})$.

(b) Let $\mu = (\mu(X), \mu(Y)) = (x, 1-x)$ be the belief of player 1 on the information set $\{X, Y\}$. If edge p is preferred to edge q, then the expected payoff for player 1 should be greater for choosing p than it is for choosing q (or the payoffs should be equal). With respect to the belief μ, these expected payoffs are

$$\begin{aligned}
E_1(p) &= 2x + 0(1-x) = 2x \\
E_1(q) &= 0x + 3(1-x) = 3 - 3x
\end{aligned}$$

The first expression is greater than or equal to the second expression when $x \geq \frac{3}{5}$, so player 1 should have believe that the probability they choose at vertex X (as opposed to Y) is at least $\frac{3}{5}$.

(c) To establish notation, we write the system of beliefs as

$$\mu = ((\mu(B), \mu(C), \mu(D)), (\mu(X), \mu(Y))).$$

First, two simplifying observations:

- Player 2 always picks b at vertex A because his worst eventual payoff he gets by choosing b is greater than the best eventual payoff he can get by choosing x.
- Player 1 never chooses d from the root because the best payoff he eventually gets by choosing d is less than the worst payoff he can get by choosing a.

Suppose player 1 chooses c. Then, μ must be of the form $((0,1,0),(x,1-x))$ (if it is to be consistent with the behaviors) and in particular, player 3 must choose believing that they are choosing from vertex C and not from vertices B or D. If player 3 knows they are at vertex C, he will choose j since the worst payoff he will get by choosing j is greater than the payoff he gets by choosing k. Now, we know that by consistency, $\mu = ((0,1,0),(0,1))$ since player 1's second action will be at vertex Y. Here, player 1 chooses q to maximize his payoff. To summarize, we have the weak sequential equilibrium

$$\beta = (c, b, j, q),\ \mu = ((0,1,0),(0,1)).$$

Suppose player 1 chooses a. Then, again by consistency, μ must be of the form $((1,0,0),(x,1-x))$ and in particular, player 3 must choose believing they are at vertex B. They will subsequently choose k to maximize their payoff. What is left to determine is the action of player 1 on $\{X, Y\}$. Any belief they have on these two vertices is weakly consistent with the behaviors of the players since these vertices will never be reached. Calculate the expected payoff player 1 gets for choosing p and q, respectively, based on their belief $(x, 1-x)$:

$$E_1(p) = 2x + 0(1-x) = 2x; \quad E_1(q) = 0x + 3(1-x) = 3 - 3x.$$

The first payoff is maximal when $x \geq \frac{3}{5}$; the second payoff is maximal when $x \leq \frac{3}{5}$. So we obtain two classes of weak sequential equilibria

$$\beta = (a, b, k, p),\ \mu = ((1,0,0),(x,1-x)) \text{ where } x \geq \frac{3}{5}$$

$$\beta = (a, b, k, q),\ \mu = ((1,0,0),(x,1-x)) \text{ where } x \leq \frac{3}{5}.$$

5. (a) If both players use strategy $\mathcal{S}$, then the action profile at every stage will be (C, C) and thus player 1 receives payoff 5 at every stage. So the discounted payoff for player 1 is

$$U_1 = \sum_{j=1}^{\infty} \delta^{j-1} u_1(C, C) = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{j-1} 5 = \cdot 5 \cdot \frac{1}{1 - 1/2} = 10.$$

(b) Suppose player 1 uses strategy $\mathcal{S}$. Then the outcome of every stage is (C, C) and player 1 receives payoff 4 at every stage. So the discounted payoff for player 1 is

$$\sum_{j=1}^{\infty} \delta^{j-1}(5) = \frac{5}{1-\delta}.$$

Suppose player 1 uses the strategy of choosing D at every stage. Then player 2's actions will alternate between C and D (starting with C in the first stage) so player 1's payoffs will alternate between 6 and 3 (starting with 6 in the first stage). So the discounted payoff for player 1 is

$$\begin{aligned} U_1 &= 6 + 3\delta + 6\delta^2 + 3\delta^3 + \ldots \\ &= 6 \sum_{j=0, j \text{ even}}^{\infty} \delta^j + 3 \sum_{j=0, j \text{ odd}}^{\infty} \delta^j \\ &= 6 \sum_{k=0}^{\infty} \delta^{2k} + 3\delta \sum_{k=0}^{\infty} \delta^{2k} \\ &= 6 \frac{1}{1-\delta^2} + 3\delta \frac{1}{1-\delta^2} \\ &= \frac{6+3\delta}{1-\delta^2}. \end{aligned}$$

Now, strategy $\mathcal{S}$ is preferred to the strategy of choosing all Ds if and only if

$$\frac{5}{1-\delta} \geq \frac{6+3\delta}{1-\delta^2}.$$

Multiply through both sides by $(1-\delta)$ (which is positive since $\delta \in (0,1)$ to get

$$5 \geq \frac{6+3\delta}{1+\delta}.$$

Now solve the inequality for δ to obtain $\delta \geq \frac{1}{2}$.

6. (a) The strategy profile where all players choose 0 is a Nash equilibrium because if any one player changes their number from 0 to $x > 0$, then three-fourths of the average becomes $\frac{3}{4} \cdot \frac{x}{n}$ which is closer to 0 than x. So that player goes from receiving $1/n$ of the pot (i.e. getting their stake back) to receiving 0 (i.e. losing their stake). So no player has any incentive to change their strategy alone when all players pick zero; this is precisely what it means for a strategy profile to be a Nash equilibrium.

   Note: As worded on the original exam (but not on this revised version), the situation where all players choose 1 is a Nash equilibrium.

   (b) Since the players can choose no numbers greater than 160, three-fourths of the average can be no more than three-fourths of 160, which is 120. So if any player chooses a number greater than 120, they can do better by changing their number to 120 (because inevitably they will get closer to three-fourths of the average).

   But if all players play this way (and they will if they are rational), then three-fourths of the average can then be no more than three-fourths of 120 which is 90. So players can improve by changing numbers greater than 90 to 90.

   Keep going with this reasoning, players can improve by choosing smaller and smaller numbers until all the players pick zero. So the Nash equilibrium from part (a) is the only Nash equilibrium of this game.