Spring 2017
Math 122 Exams

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Contents

0.1 General comments on these exams

These are the exams I gave in a Math 122 (business analysis) course at Ferris State University in Spring 2017. Each exam is followed by solutions (there may be minor errors in the solutions).

These exams correspond to the material in the 11^{th} edition of the Tan textbook as follows:

Exam 1: sections 1.1-1.5, 2.1-2.7

Exam 2: sections 3.1-3.3, 4.1-4.3

Exam 3: sections 5.1-5.4

Final Exam: chapters 1-5 except section 3.4

0.2 Spring 2017 Exam 1

1. Write the equation of the line passing through the points $(-5, 12)$ and $(7, 4)$. Write your answer in $y = mx + b$ form.

2. Sketch the graph of each line:

   a) $x = 2$

   b) $y = 4x + 7$

   c) $-3x + 12y = 24$

3. The value of a stock x days after purchase is given in the following table:

days after purchase	value
0	12
2	15
5	20
7	21
12	26
15	25

   a) Find the equation of the least-squares line fitting these points.

   b) Use the equation you wrote in part (a) to estimate the value of the stock 24 days after purchase.

4. A company produces widgets. Each widget costs the company $\$3.50$ to make, but the widgets can be sold for $\$5.00$ each. If the fixed costs of the company are $\$6250$, how many widgets need to be sold for the company to have a profit of $\$3500$?

5. A company has two interacting sectors called Sector X and Sector Y. For Sector X to produce 1 unit of output, it must consume $.05$ units of its own output and $.02$ units of output of Sector Y. For Sector Y to produce 1 unit of output, it must consume $.1$ units of output of Sector X and $.05$ units of Sector Y output. How much output must each sector produce to satisfy a monthly external demand of 500 units of Sector X output and 400 units of Sector Y output?

6. Solve the following system:

$$\begin{cases} 2x - 7y &= 4 \\ 3x - 11y &= 2 \end{cases}$$

7. Solve the following system:

$$\begin{cases} x + y + z &= 9 \\ 2x + 3y + z &= 20 \\ x - y - z &= -3 \end{cases}$$

8. Suppose $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$.

   a) Compute $5B$.
   b) Compute B^{-1}.
   c) Compute AB.
   d) Compute BA.

9. Write a system of equations which models the following story problem (you do not have to solve the system of equations). Be sure to clearly define your variables.

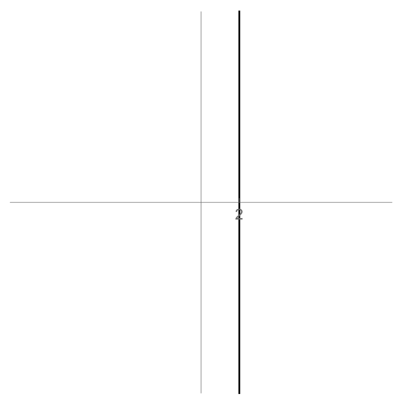
   Three investors, Al, Beth and Carl, each own some stock in three companies: Google, Facebook and Microsoft. Al owns 1000 shares of Google, 2500 shares of Facebook and 800 shares of Microsoft. Beth owns 2000 shares of Google and 1500 shares of Facebook, but no shares of Microsoft. Carl owns 1500 shares of each of the three stocks. If Al's stock is worth a total of $\$2,500,000$, Beth's stock is worth $\$1,800,000$ and Carl's stock is worth $\$2,250,000$, what is each stock's share price?

Solutions

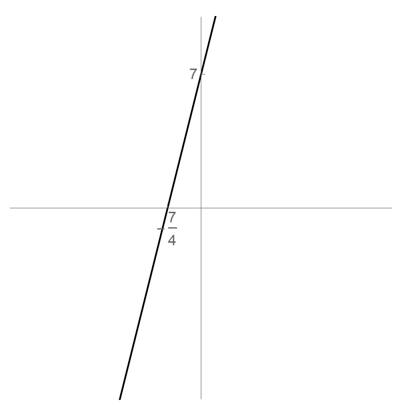
1. The slope is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12-4}{-5-7} = \frac{8}{-12} = \frac{-2}{3}$. Therefore, by writing the point-slope formula and then solving for y by simplifying the right-hand side, the line has equation

$$y = 4 - \frac{2}{3}(x-7)$$
$$y = 4 - \frac{2}{3}x + \frac{14}{3}$$
$$y = -\frac{2}{3}x + \frac{26}{3}.$$

2. a) $x = 2$ is a vertical line at x-coordinate 2:



   b) This is a line with y-intercept $(0, 7)$ and slope 4 (so you go over 1 and up 4):

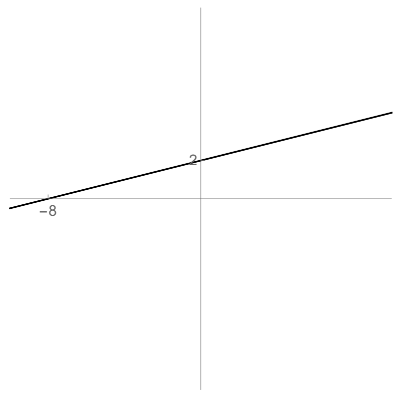


   c) *Method 1:* find the intercepts:
   - for the x-intercept, set $y = 0$ and solve for x: $-3x + 12(0) = 24$ gives $-3x = 24$ which gives $x = -8$, so the x-intercept is $(-8, 0)$.
   - for the y-intercept, set $x = 0$ and solve for y: $-3(0) + 12y = 24$ gives $12y = 24$ which gives $y = 2$, so the y-intercept is $(0, 2)$.

Method 2: solve for y:

$$\begin{aligned} -3x + 12y &= 24 \\ 12y &= 3x + 24 \\ y &= \frac{1}{4}x + 2 \end{aligned}$$

Therefore the line has slope $\frac{1}{4}$ (i.e. over 4 and up 1) and y-intercept $(0, 2)$. Whether you use Method 1 or Method 2, the graph ends up looking like this:



3. a) First, complete the chart as follows:

x	y	xy	x^2
0	12	0	0
2	15	30	4
5	20	100	25
7	21	147	49
12	26	312	144
15	25	375	225
$\sum x = 41$	$\sum y = 119$	$\sum xy = 964$	$\sum x^2 = 447$

Using the formulas given in the problem, we get

$$\begin{aligned} m &= \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \\ &= \frac{6\,(964) - (41)\,(119)}{6\,(447) - (41)^2} \\ &= \frac{905}{1001} \approx .904096 \end{aligned}$$

and

$$\begin{aligned} b &= \frac{\left(\sum y\right)\left(\sum x^2\right) - \left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \\ &= \frac{119\,(447) - (41)\,(964)}{6\,(447) - (41)^2} \\ &= \frac{13669}{1001} \approx 13.6553 \end{aligned}$$

Therefore the least-squares line is $y = mx + b$, i.e. $y = .904096x + 13.6553$.

b) Plug in $x = 24$ to the solution to part (a) to get $y = .904096(24) + 13.6553 = 35.356$.

4. Suppose the company produces (and sells) x widgets. The costs of the company are $C(x) = 6250 + 3.50x$ and the revenues are $R(x) = 5x$, so the profit of the company is $P(x) = R(x) - C(x) = 5x - (6250 + 3.5x) = 1.5x - 6250$. Set the profit equal to 3500 and solve for x:

$$\begin{aligned} 3500 &= 1.5x - 6250 \\ 9750 &= 1.5x \\ \frac{9750}{1.5} &= x \\ 6500 &= x \end{aligned}$$

5. This is a Leontief input-output problem; the input-output matrix is

$$A = \begin{pmatrix} .05 & .1 \\ .02 & .05 \end{pmatrix}$$

and the external demand is

$$D = \begin{pmatrix} 500 \\ 400 \end{pmatrix}.$$

The solution is therefore

$$\begin{aligned} X &= (I-A)^{-1}D \\ &= \begin{pmatrix} 1-.05 & 0-.1 \\ 0-.02 & 1-.05 \end{pmatrix}^{-1} \begin{pmatrix} 500 \\ 400 \end{pmatrix} \\ &= \begin{pmatrix} .95 & -.1 \\ -.02 & .95 \end{pmatrix}^{-1} \begin{pmatrix} 500 \\ 400 \end{pmatrix} \\ &= \frac{1}{(.95)^2-(-.1)(-.02)} \begin{pmatrix} .95 & .1 \\ .02 & .95 \end{pmatrix} \begin{pmatrix} 500 \\ 400 \end{pmatrix} \\ &= \frac{1}{.9005} \begin{pmatrix} .95 & .1 \\ .02 & .95 \end{pmatrix} \begin{pmatrix} 500 \\ 400 \end{pmatrix} \\ &= 1.11049 \begin{pmatrix} 515 \\ 390 \end{pmatrix} \\ &= \begin{pmatrix} 571.902 \\ 433.091 \end{pmatrix} \end{aligned}$$

Therefore the company needs Sector X to produce 571.902 units of output and Sector Y to produce 433.091 units of output.

6. I'll use the elimination method on this (there are other valid methods of solution):

$$\begin{cases} 2x-7y &= 4 \\ 3x-11y &= 2 \end{cases} \begin{matrix} \xrightarrow{\times -3} \\ \xrightarrow{\times 2} \end{matrix} \begin{cases} -6x+21y &= -12 \\ 6x-22y &= 4 \end{cases}$$

Add the two equations on the right to get $-y=-8$, i.e. $y=8$. Then plug back into the first equation and solve for x:

$$\begin{aligned} 2x-7(8) &= 4 \\ 2x-56 &= 4 \\ 2x &= 60 \\ x &= 30 \end{aligned}$$

So the solution is $(30,8)$.

7. Analyze this system by writing the matrix and doing row operations (you could have used the elimination method instead):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 3 & 1 & 20 \\ 1 & -1 & -1 & -3 \end{array}\right) \xrightarrow[-1\cdot R_1+R_3]{-2\cdot R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & -2 & -12 \end{array}\right) \xrightarrow{2\cdot R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -4 & -8 \end{array}\right)$$

This produces the equations

$$\begin{cases} x+y+z &= 9 \\ y-z &= 2 \\ -4z &= -8 \end{cases}$$

From the third equation, $z = 2$. Plugging this into the second equation and solving for y, we get $y = 4$. Plugging y and z into the first equation and solving for x, we get $x = 3$. So the solution is $(3, 4, 2)$.

8. a) Multiply each entry by 5 to get $5B = \begin{pmatrix} 15 & 20 \\ 10 & 25 \end{pmatrix}$.

   b) From the formula for the inverse of a 2×2 matrix,

$$B^{-1} = \frac{1}{3(5) - 4(2)} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 5 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{-4}{7} \\ \frac{-2}{7} & \frac{3}{7} \end{pmatrix}.$$

   c) Compute AB by the usual method of matrix multiplication:

$$\begin{array}{cc} & \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \\ \begin{pmatrix} 2 & 1 \\ 5 & 3 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 8 & 13 \\ 21 & 35 \\ 3 & 4 \end{pmatrix} \end{array}$$

   d) $BA = B_{2\times 2}A_{3\times 2}$ does not exist because the number of columns of B does not equal the number of rows of A.

9. Let x be the price of a share of Google; y be the price of a share of Facebook and z be the price of a share of Microsoft. From the given information, we have the system

$$\begin{cases} 1000x + 2500y + 800z & = 2500000 \\ 2000x + 1500y & = 1800000 \\ 1500x + 1500y + 1500z & = 2250000 \end{cases}$$

0.3 Spring 2017 Exam 2

1. Graph the following inequalities on the same xy-plane, shading the false side of each inequality:

$$\begin{cases} 4x+y & \leq 12 \\ 2x+3y & \leq 18 \\ x & \geq 0 \\ y & \geq 0 \end{cases}$$

2. Find the corner points of the region described by the following inequalities:

$$\begin{cases} 12x+5y & \leq 60 \\ -2x+y & \leq 8 \\ x & \leq 4 \\ x & \geq 0 \\ y & \geq 0 \end{cases}.$$

3. Find the maximum and minimum of $U = 5x + 2y$ subject to the constraints

$$\begin{cases} x+y & \geq 8 \\ 4x+y & \geq 14 \\ x & \geq 0 \\ y & \geq 2 \end{cases}.$$

   If the maximum and/or minimum do not exist, say so.

4. Given the minimization problem described in the next paragraph, construct the dual problem (that is, give a formula for the utility, whether you want to maximize or minimize the utility, and a list of constraints the utility must satisfy).

   Find the minimum of $C = 40x + 35y$ subject to the constraints

$$\begin{cases} 40x+18y & \geq 375 \\ 33x+30y & \geq 425 \\ 25x+32y & \geq 500 \\ x & \geq 0 \\ y & \geq 0 \end{cases}.$$

5. Given the linear programming problem described in the paragraph below:

   a) write the matrix you would start with when solving the problem by the simplex method;

   b) draw a box around the element in your matrix that would serve as the first pivot for your matrix operations;

c) describe in words the first matrix operation you would do (i.e. "add 3 times Row 2 to Row 1" or "multiply Row 1 by 6", etc.).

Find the maximum of $P = 3x + y + 4z$ subject to the constraints

$$\begin{cases} x + 5y + 8z & \leq 38 \\ 3x + 4y + 6z & \leq 42 \\ 7z + 2y + z & \leq 27 \\ x & \geq 0 \\ y & \geq 0 \\ z & \geq 0 \end{cases}.$$

6. Find the maximum value of $P = 2x - y + z$ subject to the constraints

$$\begin{cases} 2x + y & \leq 10 \\ x + 2y - 2z & \leq 20 \\ y + 2z & \leq 5 \\ x & \geq 0 \\ y & \geq 0 \\ z & \geq 0 \end{cases}.$$

7. Given the story problem in the next paragraph, formulate the story problem as a linear programming problem.

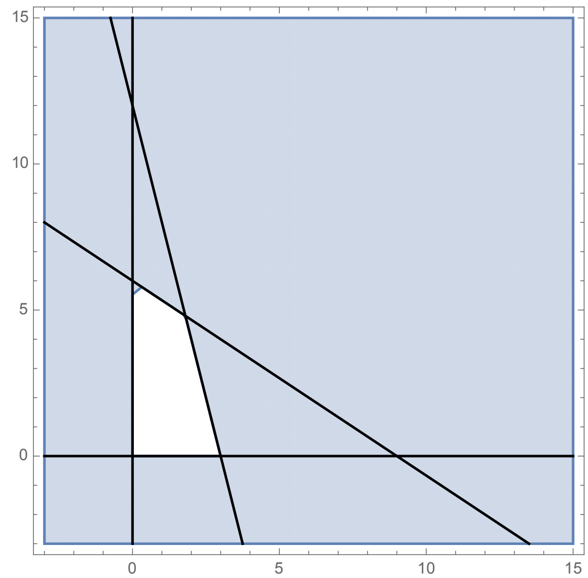
   You need to buy some filing cabinets for your business. A cabinet manufactured by Miller costs $\$100$ per unit, requires six square feet of floor space, and holds eight cubic feet of files. A cabinet manufactured by Jones costs $\$200$ per unit, requires eight square feet of floor space, but holds twelve cubic feet of files. If you have $\$1400$ to spend on file cabinets and only have 72 square feet of floor space, how many of each cabinet should you buy, to maximize storage volume?

8. Given the story problem in the next paragraph, formulate the story problem as a linear programming problem.

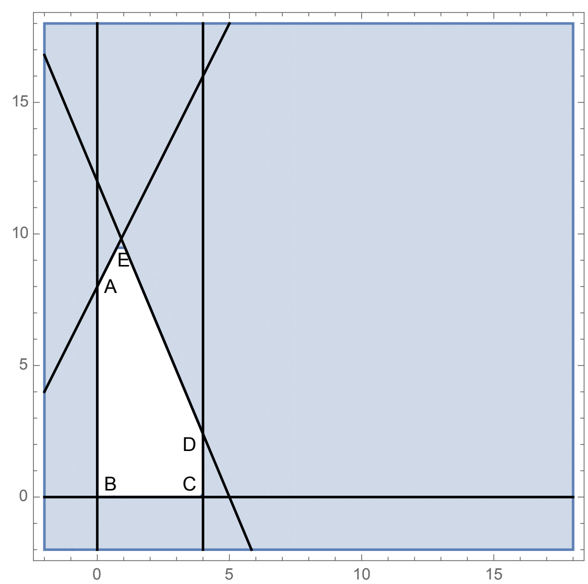
   A manufacturer produces four types of plastic fixtures. Each Type A fixture requires 60 minutes for molding, 40 minutes for trimming, and 30 minutes for packaging. Each Type B fixture requires 100 minutes for molding, 60 minutes for trimming and 15 minutes for packaging. Each Type C fixture requires 80 minutes for molding, 75 minutes for trimming and 25 minutes for packaging. Each Type D fixture requires 140 minutes for molding, 100 minutes for trimming and 45 minutes for packaging. The company makes a profit of $\$13$ on each Type A fixture, $\$16$ on each Type B fixture, $\$19$ on each Type C fixture and $\$22$ on each Type D fixture. If they only have 12000 minutes of molding, 4600 minutes of trimming and 2800 hours of packaging available, how many of each type of fixture should be produced to maximize the company's profit?

Solutions

1. The line $4x + y = 12$ has intercepts $(3, 0)$ and $(0, 12)$. The line $2x + 3y = 18$ has intercepts $(9, 0)$ and $(0, 6)$. This gives you a picture like this:



2. The line $12x + 5y = 60$ has intercepts $(5, 0)$ and $(0, 12)$. The line $-2x + y = 8$ has slope 2 and $y-$intercept 8. The line $x = 4$ is a vertical line through $(4, 0)$. These lines, when graphed, give the following picture:

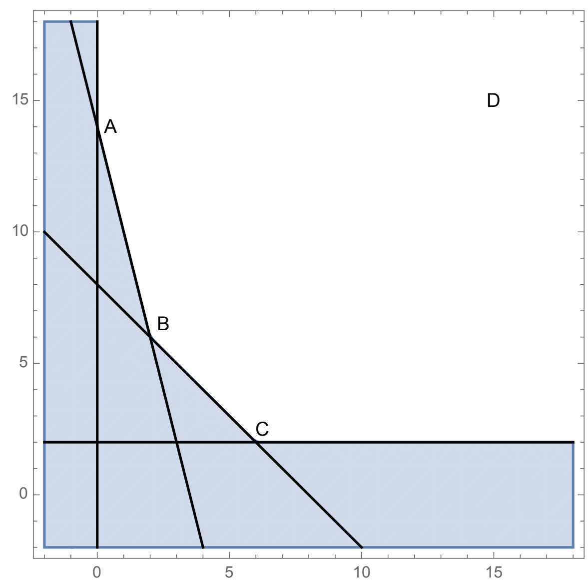


There are five corner points:

- A, which is the y-intercept of $-2x + y = 8$, i.e. $(0, 8)$;

- B, which is clearly $(0,0)$;
- C, which is the $x-$intercept of $x=4$, i.e. $(4,0)$;
- D, which is the intersection of $x=4$ and $12x+5y=60$, i.e. $(4,2.4)$;
- E, which is the intersection of $12x+5y=60$ and $-2x+y=8$, i.e. $x=\frac{10}{11}, y=\frac{108}{11}$, i.e. $\left(\frac{10}{11}, \frac{108}{11}\right)$.

3. First, graph the inequalities using the methods of the previous problems. This gives the following picture:



There are three corner points, and since the region is unbounded, you need a fourth "dummy" point out in the unbounded region:

- A, which is $(0,14)$;
- B, which is $(2,6)$;
- C, which is $(6,2)$;
- D, which is $(20,20)$ (or any two other large numbers).

Now test each of these points in the utility:

- $A=(0,14) \Rightarrow U=5(0)+2(14)=28$
- $B=(2,6) \Rightarrow U=5(2)+2(6)=22$
- $C=(6,2) \Rightarrow U=5(6)+2(2)=34$
- $D=(20,20) \Rightarrow U=5(20)+2(20)=140$

Thus there is no maximum value of U (since the largest value came from the dummy point), and the minimum value of U is 22, occurring at $(2,6)$.

4. Write the matrix associated to the minimization problem and take its trans-

pose:

$$\begin{pmatrix} 40 & 18 & 375 \\ 33 & 30 & 425 \\ 25 & 32 & 500 \\ 40 & 35 & C \end{pmatrix} \longrightarrow \begin{pmatrix} 40 & 33 & 25 & 40 \\ 18 & 30 & 32 & 35 \\ 375 & 425 & 500 & C \end{pmatrix}$$

Then interpret this is a maximization problem. The dual problem is therefore to maximize $375u+425v+500w$ subject to the constraints $40u+33v+25w \leq 40$, $18u + 30v + 32w \leq 35$, $u \geq 0$, $v \geq 0$, $w \geq 0$.

5. a) First, rewrite the nontrivial constraints and the utility as equations:

$$\begin{cases} x + 5y + 8z + u = 38 \\ 3x + 4y + 6z + v = 42 \\ 7x + 2y + z + w = 27 \\ -3x - y - 4z + P = 0 \end{cases}$$

Then the matrix is

$$\left(\begin{array}{ccc|ccc|c|c} 1 & 5 & 8 & 1 & 0 & 0 & 0 & 38 \\ 3 & 4 & 6 & 0 & 1 & 0 & 0 & 42 \\ 7 & 2 & 1 & 0 & 0 & 1 & 0 & 27 \\ -3 & -1 & -4 & 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

b) The most negative entry in the bottom row is -4, in the third column. Then the smallest positive ratio of the last-column entry to third-column entry is $\frac{38}{8} = 4.75$, so the box should go around the 8 in the third column of the first row.

c) The first row operation would be **multiplying Row 1 by** $\frac{1}{8}$.

6. First, set the matrix up using the ideas from Problem 5 (a). The first pivot is the 2 in the upper left-hand corner:

$$\left(\begin{array}{ccc|ccc|c|c} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 10 \\ 1 & 2 & -2 & 0 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 5 \\ -2 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \times \frac{1}{2}} \left(\begin{array}{ccc|ccc|c|c} 1 & .5 & 0 & .5 & 0 & 0 & 0 & 5 \\ 1 & 2 & -2 & 0 & 1 & 0 & 0 & 20 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 5 \\ -2 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{-R_1+R_2, 2R_1+R_4} \left(\begin{array}{ccc|ccc|c|c} 1 & .5 & 0 & .5 & 0 & 0 & 0 & 5 \\ 0 & 1.5 & -2 & -.5 & 1 & 0 & 0 & 15 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

Now, the next pivot is the 2 in the third column of the third row:

$$\left(\begin{array}{ccc|ccc|c|c} 1 & .5 & 0 & .5 & 0 & 0 & 0 & 5 \\ 0 & 1.5 & -2 & -.5 & 1 & 0 & 0 & 15 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 5 \\ 0 & 2 & -1 & 1 & 0 & 0 & 1 & 10 \end{array}\right) \xrightarrow{R_3\times\frac{1}{2}} \left(\begin{array}{ccc|ccc|c|c} 1 & .5 & 0 & .5 & 0 & 0 & 0 & 5 \\ 0 & 1.5 & -2 & -.5 & 1 & 0 & 0 & 15 \\ 0 & .5 & 1 & 0 & 0 & .5 & 0 & 2.5 \\ 0 & 2 & -1 & 1 & 0 & 0 & 1 & 10 \end{array}\right)$$

$$\xrightarrow{2R_3+R_2, R_3+R_4} \left(\begin{array}{ccc|ccc|c|c} 1 & .5 & 0 & .5 & 0 & 0 & 0 & 5 \\ 0 & 2.5 & 0 & -.5 & 1 & 1 & 0 & 20 \\ 0 & .5 & 1 & 0 & 0 & .5 & 0 & 2.5 \\ 0 & 2.5 & 0 & 1 & 0 & .5 & 1 & 12.5 \end{array}\right)$$

Since the bottom row has only positive numbers, we are done. Set $u = v = w = 0$ and set $y = 0$ (since the bottom-row entry of the y column is greater than zero); this gives $P = 12.5$ when $x = 5$ and $z = 2.5$. Therefore the maximum is $P = 12.5$, occurring at $(5, 0, 2.5)$.

7. The goal is to maximize the storage volume $V = 8x + 12y$ where x is the number of Miller cabinets and y is the number of Jones cabinets. The constraints come from the cost and floor space: $100x + 200y \leq 1400$ and $6x + 8y \leq 72$ (together with $x \geq 0$ and $y \geq 0$).

8. The goal is to maximize the profit $P = 13a + 16b + 19c + 22d$ where a, b, c and d are the numbers of Type A, B, C and D fixtures produced, respectively. The constraints come from the minutes for each process: $60a + 100b + 80c + 140d \leq 12000$; $40a + 60b + 75c + 100d \leq 4600$; $30a + 15b + 25c + 45d \leq 2800$ (and $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$).

0.4 Spring 2017 Exam 3

1. Find the present value of an annuity consisting of daily payments of $\$10$ for 30 years at 2% annual, compounded daily.

2. Suppose you want to have $\$1000000$ (one million dollars) in an annuity when you turn 65 (assume that you are currently 22). If the annuity earns annual interest rate 4.5%, compounded monthly, how much do you have to put into this annuity each month to reach your goal?

3. You have $\$5000$ to invest.

   a) If you invest the money at 7.25% annual simple interest, how much will you have in eight years?
   b) If you invest the money at 5.8% annual, compounded quarterly, how much will you have in eight years?
   c) If you invest the money at 6% annual, compounded continuously, how much will you have in eight years?
   d) What is the effective rate of interest earned in the situation of part (b) of this problem? Express the answer as a percentage.

4. An accountant takes a job at a firm, where, as part of her salary, the firm deposits $\$420$ per month into a 401(k) retirement account. Supposing that the 401(k) earns interest at an annual rate of 3.75% compounded monthly, how much money will the accountant have in her 401(k) after working for 16 years?

5. To purchase a car, you take out a loan of $\$20,000$. The interest rate of the loan is 2.5% annual, compounded monthly, and the loan is structured so that it is amortized after six years. What is the size of the monthly payment you have to make on the car?

6. A worker starts working at a company with a salary of $\$42,000$. If the worker receives a 3% raise every year, how much will the worker earn in total across his first 12 years of employment?

7. An arithmetic progression begins with 7 and has as its sixth term 92.

   a) Find the thirty-seventh term.
   b) Find the sum of the first eighteen terms.

8. You decide to buy a house valued at $\$158,000$. You have saved enough to put 20% down, but finance the rest with a 30-year loan with annual interest rate 5%, compounded monthly. After making payments for 20 years, what is your equity in the home?

Solutions

1. We are given $R = 10$, $n = 365(30) = 10950$ and $i = .02/365 = .0000547945$. Using the present value formula, we obtain

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right] = 10\left[\frac{1-(1.0000547945)^{-10950}}{.0000547945}\right] = \$82340.20.$$

2. Use either the sinking fund formula or the annuity formula and solve for R. We are given $S = 1000000$, $i = .045/12 = .00375$ and $n = 43(12) = 516$. Therefore

$$R = \frac{iS}{(1+i)^n - 1} = \frac{.00375(1000000)}{1.00375^{516} - 1} = \$635.70.$$

3. a) By the simple interest formula with $P = 5000$, $r = .0725$ and $t = 8$, we have

$$A = P(1 + rt) = 5000(1 + .0725(8)) = \$7900.$$

   b) By the compound interest formula, using $r = .058$, $m = 4$, $P = 5000$ and $n = 8(4) = 32$, we have

$$A = P\left(1 + \frac{r}{m}\right)^n = 5000\left(1 + \frac{.058}{4}\right)^{32} = \$7925.66.$$

   c) By the continuously compounded interest formula, with $r = .06$ and $t = 8$, we have

$$A = Pe^{rt} = 5000e^{.06(8)} = \$8080.37.$$

   d) Using the same variables as in part (b), by the effective rate formula we have

$$r_{eff} = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{.058}{4}\right)^4 - 1 = .05927 = 5.927\%.$$

4. Use the annuity formula with $R = 420$, $i = .0375/12 = .003125$, and $n = 16(12) = 192$. Then

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 420\left[\frac{1.003125^{192} - 1}{.003125}\right] = \$110264.$$

5. Use the amortization formula with $P = 20000$, $i = .025/12 = .0020833$, and $n = 12(6) = 72$. Then

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{20000(.0020833)}{1 - 1.0020833^{-72}} = \$299.42.$$

6. Use the sum formula for geometric progressions: we have $a = 42000$, $r = 1.03$ and $n = 12$ so

$$a_1 + ... + a_{12} = a\left(\frac{1-r^n}{1-r}\right) = 42000\left(\frac{1-1.03^{12}}{1-1.03}\right) = \$596095.$$

7. a) We are given $a = a_1 = 7$ and $a_6 = 92$. Since $a_6 = a_1 + (6-1)d$, we have $92 = 7 + 5d$ so $5d = 85$, i.e. $d = 17$. That means $a_{37} = a + (37-1)d = 7 + 36(17) = 619$.

   b) By the sum formula for arithmetic progressions with $a = 7$, $d = 17$ and $n = 18$,

$$a_1 + ... + a_n = \frac{n}{2}\left[2a + (n-1)d\right] = 9\left[14 + 17(17)\right] = 2727.$$

8. First, your down payment is $.20(158000) = 31600$ so the amount of your loan is $P = 158000 - 31600 = 126400$. Therefore, using the amortization formula with $i = .05/12 = .0041666$, $n = 30(12) = 360$ and $P = 126400$, we can find the monthly payment:

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{126400(.00416666)}{1-1.00416666^{-360}} = \$678.54.$$

Now, after making payments for 20 years, the remaining principal is the present value of an annuity with regular payment $R = 678.54$ for $30-20 = 10$ years (i.e. $n = 10(12) = 120$ and $i = .05/12 = .0041666$). This remaining principal is

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right] = 678.54\left[\frac{1-1.0041666^{-120}}{.0041666}\right] = \$63973.70.$$

Thus your equity is your down payment plus the difference between what you borrowed and what you still owe:

$$31600 + (126400 - 63973.70) = \$94026.30.$$

0.5 Spring 2017 Final Exam

1. a) Write equation of the line passing through the point $(5, 14)$ with slope $\frac{-2}{5}$.
   b) Graph the line described in part (a).
   c) Find the x-intercept of the line described in part (a).

2. The value of a rare Hummel statue by year is as follows:

Year	2000	2003	2009	2014
Value	$\$120$	$\$136$	$\$141$	$\$160$

   Use the least-squares line for this data to project the value of the Hummel statue in the year 2020.

3. A company manufactures widgets. Assume that if the widgets are priced at x per widget, the company is able to manufacture $.4x - 70$ widgets. Assume also that if the widgets are priced at x per widget, then customers are willing to purchase $410 - .2x$ widgets. What is the equilibrium price of a widget, and how many widgets will be produced at the equilibrium price?

4. An island's economy consists of two sectors: agriculture and construction. To produce each unit of agriculture output, $.15$ units of agriculture and $.05$ units of construction are required. To produce each unit of construction output, $.05$ units of agriculture and $.2$ units of construction are required. How much total output from each sector is required to meet an external demand of 417 units of agriculture and 645 units of construction?

5. Write a system of equations which models the story problem in the next paragraph. Be sure to clearly define your variables. **You do not need to actually solve the problem.**

   A computer company produces three types of laptops: standard, family and professional. They sell these laptops for $\$600$, $\$725$ and $\$1050$, respectively. They sell 5000 laptops to a retailer at a total cost of $\$3,516,050$. If they sell twice as many standard laptops to the retailer as family laptops, how many of each type of laptop did they sell to the retailer?

6. a) Solve the following system:

$$\begin{cases} 15x + 17y &= 179 \\ 10x + 11y &= 112 \end{cases}$$

b) Solve the following system:

$$\begin{cases} 14x - 12y &= 20 \\ -21x + 18y &= -30 \end{cases}$$

7. Solve the following system:

$$\begin{cases} x - y + 3z &= 4 \\ 2x - 3y - z &= -11 \\ -2x + y + 4z &= 7 \end{cases}$$

8. a) Let $M = \begin{pmatrix} 3 & -2 \\ 1 & 4 \\ -2 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 4 \\ -3 & -2 \\ 0 & 1 \end{pmatrix}$. Find $3M - P$.

   b) Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$. Find BA.

9. Graph the system of linear inequalities, shading the false side of each inequality:

$$\begin{cases} 7x + 11y &\leq 77 \\ 10x + 7y &\leq 70 \\ x &\geq 0 \\ y &\geq 0 \end{cases}$$

10. Find the maximum value of $U = 7x + y$, subject to the constraints $2x + 5y \leq 150$, $6x + y \leq 72$, $x \geq 0$ and $y \geq 0$.

11. Given the story problem in the next paragraph, formulate the story problem as a linear programming problem (i.e. give a formula for the utility, describe whether you want to maximize or minimize the utility, explicitly describe all variables used, and give a list of constraints). **You do not need to actually solve the problem.**

    A company makes two types of valves: small and large. Small valves sell for $\$6.50$ each, and large valves sell for $\$11$ each. Each small valve requires $.3$ units of metal, $.2$ units of hard plastic and $.15$ units of rubber, and each large valve requires $.8$ units of metal, $.35$ units of hard plastic and $.3$ units of rubber. If the company has 4000 units of metal, 3250 units of hard plastic and 3100 units of rubber available, how many of each valve should they plan to produce to maximize their profit?

12. Find the maximum value of $U = 4x + 5y + 3z$ subject to the following constraints:

$$\begin{cases} x + y + z & \leq 8 \\ 2x + 4y + 3z & \leq 24 \\ x & \geq 0 \\ y & \geq 0 \\ z & \geq 0 \end{cases}$$

13. Calculate the accumulated amount in each case if $\$2500$ is invested (once) at the indicated rates:

    a) 7% annual, simple interest, for six years

    b) 6% annual, compounded weekly, for nine years

    c) 8% annual, compounded continuously, for 3.5 years

14. The parts of this problem are unrelated to one another.

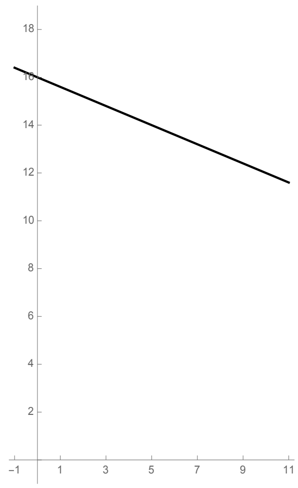
    a) An employer contributes $\$175$ per month into an employee's 401(k) retirement fund. If the fund earns an annual interest rate of 4.75%, compounded monthly, how much will the employee have in their 401(k) after 25 years of employment?

    b) A company retains an attorney who charges them $\$5000$ during the first year of service. If the attorney's fees increase by 8% each year, how much will the company pay the attorney in total over the first ten years?

15. You purchase a house worth $\$155,000$. You finance the entire purchase by taking out a 30-year fixed rate mortgage at annual rate 5.75%, compounded monthly. After making payments on the mortgage for 8 years, you come into a sum of money and decide to refinance your mortgage. To do so, you make a $\$25,000$ payment on the remaining principal and refinance the rest at 4.5% annual, compounded monthly, for 20 years. What is your monthly payment after refinancing?

Solutions

1. a) From the point-slope formula, this is $y = 14 - \frac{2}{5}(x-5)$. (If you wrote this in $y = mx + b$ form, the equation is $y = \frac{-2}{5}x + 16$.)

   b) 

   c) Set $y = 0$ and solve for x:

$$\begin{aligned} 0 &= 14 - \frac{2}{5}(x-5) \\ 0 &= 14 - \frac{2}{5}x + 2 \\ 0 &= 16 - \frac{2}{5}x \\ \frac{2}{5}x &= 16 \\ x &= 16\left(\frac{5}{2}\right) = 40. \end{aligned}$$

   Therefore the x-intercept is $(40, 0)$.

2. First, complete the chart as follows (let x be the number of years after 2000):

x	y	xy	x^2
0	120	0	0
3	136	408	9
9	141	1269	81
14	160	2240	196
$\sum x = 26$	$\sum y = 557$	$\sum xy = 3917$	$\sum x^2 = 286$

Using the formulas given in the problem, we get

$$\begin{aligned} m &= \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \\ &= \frac{4\,(3917) - (26)\,(557)}{4\,(286) - (26)^2} \\ &= \frac{593}{234} \approx 2.53419 \end{aligned}$$

and

$$\begin{aligned} b &= \frac{\left(\sum y\right)\left(\sum x^2\right) - \left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \\ &= \frac{557\,(286) - (38)\,(3917)}{4\,(286) - (26)^2} \\ &= \frac{1105}{9} \approx 122.778. \end{aligned}$$

Therefore the least-squares line is $y = mx + b$, i.e. $y = 2.53419x + 122.778$.

Last, to find the value of the Hummel in 2020, plug in $x = 20$ to the solution to part (a) to get $y = 2.53419(20) + 122.778 = \$173.46$.

3. The equilibrium price x is when the supply and demand are equal:

$$\begin{aligned} .4x - 70 &= 410 - .2x \\ .6x &= 480 \\ 6x &= 4800 \\ x &= 800. \end{aligned}$$

Therefore the equilibrium price is $\$800$; at this price, the supply and demand are both equal to $.4(800) - 70 = 320 - 70 = 250$ widgets.

4. This is a Leontief input-output problem; the input-output matrix is

$$A = \begin{pmatrix} .15 & .05 \\ .05 & .2 \end{pmatrix}$$

and the external demand is

$$D = \begin{pmatrix} 417 \\ 645 \end{pmatrix}.$$

The solution is therefore

$$\begin{aligned}
X &= (I-A)^{-1}D \\
&= \begin{pmatrix} 1-.15 & 0-.05 \\ 0-.05 & 1-.2 \end{pmatrix}^{-1} \begin{pmatrix} 417 \\ 645 \end{pmatrix} \\
&= \begin{pmatrix} .85 & -.05 \\ -.05 & .8 \end{pmatrix}^{-1} \begin{pmatrix} 417 \\ 645 \end{pmatrix} \\
&= \frac{1}{(.85)(.8)-(-05)(-.05)} \begin{pmatrix} .8 & .05 \\ .05 & .85 \end{pmatrix} \begin{pmatrix} 417 \\ 645 \end{pmatrix} \\
&= \frac{1}{.6775} \begin{pmatrix} .8 & .05 \\ .05 & .85 \end{pmatrix} \begin{pmatrix} 417 \\ 645 \end{pmatrix} \\
&= 1.47601 \begin{pmatrix} 365.85 \\ 569.1 \end{pmatrix} \\
&= \begin{pmatrix} 540 \\ 840 \end{pmatrix}
\end{aligned}$$

Therefore the island needs to produce 540 units of agriculture and 840 units of construction to meed the external demand.

5. Let x be the number of standard laptops sold. Let y be the number of family laptops sold, and let z be the number of professional laptops sold. Then the system of equations is

$$\begin{cases} x+y+z &= 5000 \\ 600x+725y+1050z &= 3516050 \\ x &= 2y \end{cases}$$

(The last equation could have been written $x-2y=0$.)

6. a) Multiply the top equation by -2 and the bottom equation by 3 to obtain

$$\begin{cases} -30x-34y &= -358 \\ 30x+33y &= 336 \end{cases}$$

Then add these two equations to get $-y=-22$, i.e. $y=22$. Then, solve for x by substituting into an earlier equation (I'll use the second one): $10x+11(22)=112$ so $10x+242=112$ so $10x=-130$ so $x=-13$. Thus the solution is $(-13,22)$.

b) Multiply the top equation by 3 and the bottom equation by 2 to get

$$\begin{cases} 42x-36y &= 60 \\ -42x+36y &= -60 \end{cases}$$

Then add the two equations to get $0 = 0$. This tells you that there are infinitely many solutions; to describe them, solve for one variable in terms of the other (I'll solve for x in the first equation): $14x - 12y = 20$ so $14x = 12y + 20$ so $x = \frac{12}{14}y + \frac{20}{14}$, i.e. $x = \frac{6}{7}y + \frac{10}{7}$. Thus the solutions are $\left(\frac{6}{7}y + \frac{10}{7}, y\right)$.

7. You can solve this with matrix methods, but I'll use addition/elimination:

$$\begin{array}{rl} -3x+3y-9z & = -12 \\ 2x-3y-z & = -11 \end{array} \quad \begin{array}{c} \xleftarrow{\times -3} \\ \xleftarrow{\times 1} \\ \end{array} \left\{ \begin{array}{rl} x-y+3z & = 4 \\ 2x-3y-z & = -11 \\ -2x+y+4z & = 7 \end{array} \right. \quad \begin{array}{c} \xrightarrow{\times 1} \\ \\ \xrightarrow{\times 1} \end{array} \quad \begin{array}{rl} x-y+3z & = 4 \\ & \\ -2x+y+4z & = 7 \end{array}$$

Adding the equations on the left produces $-x - 10z = -23$; adding the equations on the right produces $-x + 7z = 11$. So now we solve the system

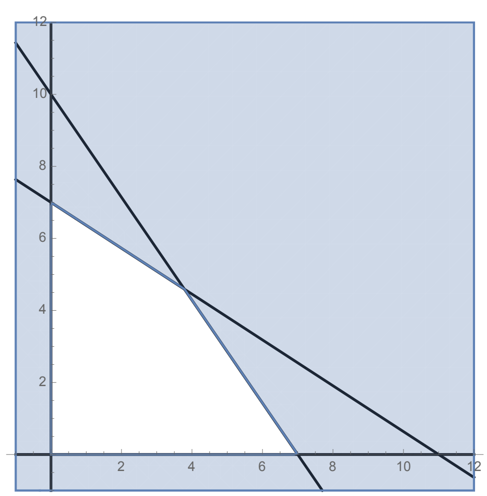
$$\left\{ \begin{array}{rl} -x-10z & = -23 \\ -x+7z & = 11 \end{array} \right.$$

by methods similar to Problem 6 to obtain $x = 3$, $z = 2$. Last, back-substitute into an original equation to solve for y. If you use the third equation, you get $-2(3) + y + 4(2) = 7$, i.e. $y = 5$. So the solution is $(3, 5, 2)$.

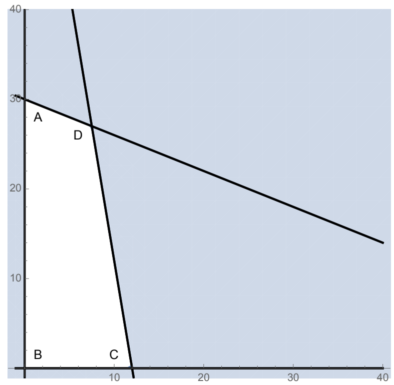
8. a) $3M - P = \begin{pmatrix} 3(3)-1 & 3(-2)-4 \\ 3(1)-(-3) & 3(4)-(-2) \\ 3(-2)-0 & 3(5)-1 \end{pmatrix} = \begin{pmatrix} 8 & -10 \\ 6 & 14 \\ -6 & 14 \end{pmatrix}$.

b) $BA = \begin{pmatrix} 5(1)+2(3) & 5(2)+2(0) & 5(4)+2(2) \\ 1(1)+3(3) & 1(2)+3(0) & 1(4)+3(2) \end{pmatrix} = \begin{pmatrix} 11 & 10 & 24 \\ 10 & 2 & 10 \end{pmatrix}$.

9. The first line has x-intercept $(11, 0)$ and y-intercept $(0, 7)$; the second line has x-intercept $(7, 0)$ and y-intercept $(0, 10)$. Putting these on the same x, y-axes with $x \geq 0$ and $y \geq 0$, we get this picture:



10. You can solve this with the simplex method, but I'll do it graphically: the first constraint has x-intercept $(75, 0)$ and y-intercept $(0, 30)$; the second constraint has x-intercept $(12, 0)$ and y-intercept $(0, 72)$. This produces a picture of the feasible region like this:



Next, find the corner point D by solving the system

$$\begin{cases} 2x + 5y &= 150 \\ 6x + y &= 72 \end{cases}$$

by methods similar to Problem 6, obtaining $x = \frac{15}{2}$ and $y = 27$. Now, test the corner points in the utility $U = 7x + y$:

$$\begin{array}{rl} A : (0, 30) & U = 7(0) + 30 = 30 \\ B : (0, 0) & U = 0 \\ C : (12, 0) & U = 7(12) + 0 = 84 \\ D : (7.5, 27) & U = 7(7.5) + 27 = 79.5 \end{array}$$

Therefore the maximum value of U is 84, occurring at $(12, 0)$.

11. The goal is to maximize $P = 6.5x+11y$, where x is the number of small valves produced and y is the number of large valves produced. The constraints are

$$\begin{cases} .3x + .8y \leq 4000 \\ .2x + .35y \leq 3250 \\ .15x + .3y \leq 3100 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

12. Since this problem has three variables, it needs to be solved with the simplex method. Rewrite the nontrivial constraints and the utility as equations:

$$\begin{cases} x + y + z + u &= 8 \\ 2x + 4y + 3z + v &= 24 \\ -4x - 5y - 3z + P &= 0 \end{cases}$$

Then write the corresponding matrix and pivot (the first pivot is the 4 in the second column of the second row

$$\left(\begin{array}{ccc|ccc|c|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ 2 & 4 & 3 & 0 & 1 & 0 & 0 & 24 \\ -4 & -5 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \overset{R_2\times\frac{1}{4}}{\longrightarrow} \left(\begin{array}{ccc|ccc|c|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 8 \\ .5 & 1 & .75 & 0 & .25 & 0 & 0 & 6 \\ -4 & -5 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\overset{-R_2+R_1,5R_1+R_3}{\longrightarrow} \left(\begin{array}{ccc|ccc|c|c} .5 & 0 & .25 & 1 & -.25 & 0 & 0 & 2 \\ .5 & 1 & .75 & 0 & .25 & 0 & 0 & 6 \\ -1.5 & 0 & .75 & 0 & 1.25 & 0 & 1 & 30 \end{array}\right)$$

The next pivot is the $.5$ in the upper-left corner:

$$\left(\begin{array}{ccc|ccc|c|c} .5 & 0 & .25 & 1 & -.25 & 0 & 0 & 2 \\ .5 & 1 & .75 & 0 & .25 & 0 & 0 & 6 \\ -1.5 & 0 & .75 & 0 & 1.25 & 0 & 1 & 30 \end{array}\right) \overset{R_1\times 2}{\longrightarrow} \left(\begin{array}{ccc|ccc|c|c} 1 & 0 & .5 & 2 & -.5 & 0 & 0 & 4 \\ .5 & 1 & .75 & 0 & .25 & 0 & 0 & 6 \\ -1.5 & 0 & .75 & 0 & 1.25 & 0 & 1 & 30 \end{array}\right)$$

$$\overset{-.5R_1+R_2,1.5R_1+R_3}{\longrightarrow} \left(\begin{array}{ccc|ccc|c|c} 1 & 0 & .5 & 2 & -.5 & 0 & 0 & 4 \\ 0 & 1 & .5 & -1 & .5 & 0 & 0 & 4 \\ 0 & 0 & 1.5 & 3 & .75 & 0 & 1 & 36 \end{array}\right)$$

Since the bottom row has only nonnegative numbers, we are done. Set $u = v = 0$ and set $z = 0$ (since the bottom-row entry of the z column is greater than zero); this gives $P = 36$ when $x = 4$ and $y = 4$. Therefore the maximum is $P = 36$, occurring at $(4, 4, 0)$.

13. a) $A = P(1 + rt) = 2500(1 + .07(6)) = \$3550$.

    b) $A = P\left(1 + \frac{r}{m}\right)^n = 2500\left(1 + \frac{.06}{52}\right)^{52\cdot 9}$ $\$4288.68$.

    c) $A = Pe^{rt} = 2500e^{.08(3.5)} = \$3307.82$.

14. a) We have $i = .0475/12 = .00395833$. By the annuity formula,

$$S = R\left[\frac{(1+i)^n - 1}{i}\right] = 175\left[\frac{1.00395833^{25\cdot 12} - 1}{.00395833}\right] = \$100411.$$

    b) Since the increase is a percentage each year, the amount charged is a geometric progression with $a = 5000$ and $r = 1.08$. The sum over the first ten terms is

$$a_1 + ... + a_{10} = a\left(\frac{1 - r^{10}}{1 - r}\right) = 5000\left(\frac{1 - 1.08^{10}}{1 - 1.08}\right) = \$72432.80.$$

15. This problem has several steps.

    - First, you need to determine the monthly payment R on the original loan. This is done as follows:

$$R = \frac{Pi}{1 - (1+i)^{-n}} = \frac{155000(\frac{.0575}{12})}{1 - \left(1 + \frac{.0575}{12}\right)^{-12(30)}} = \$904.54.$$

- Second, you need to determine how much you still owe after 8 years. This is done with the present value formula with n equal to the number of remaining payments (which is $22 \cdot 12 = 264$):

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right] = 904.54\left[\frac{1-\left(1+\frac{.0575}{12}\right)^{-264}}{\frac{.0575}{12}}\right] = \$135333.$$

- Third, you make the $\$25,000$ payment. This leaves you owing $\$135333 - \$25000 = \$110333$, which is the amount you finance with the new loan.
- Last, you need to figure the monthly payment on the new loan. This is done like the first part of the problem:

$$R = \frac{Pi}{1-(1+i)^{-n}} = \frac{110333(\frac{.045}{12})}{1-\left(1+\frac{.045}{12}\right)^{-12(20)}} = \$698.02.$$