Precalculus Lecture Notes

David M. McClendon

Department of Mathematics
Ferris State University

Spring 2024 edition

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Chapter 1

Preliminaries

1.1 Basic mathematics terminology

Expressions

> **Definition 1.1** *An* **expression** *(or* **mathematical expression***) is a combination of numbers, variables and/or symbols that represents a single object.*

EXAMPLES OF EXPRESSIONS

- 8
- t
- $4\sin\theta\cos 2\theta$
- $17.35 - 12.82 + 3.51$
- $\dfrac{(x+2)(y^2-3\cdot 2^y)}{xy^5-\sqrt{3x^4\cos 3x}} + \left(\dfrac{16+7tx}{3-\frac{1}{4-7y}}\right)^{-3/5}$
- $\Gamma(\mathfrak{G})/\widetilde{H}_2(\mathbb{R},\mathbb{Z})$

Terms

Definition 1.2 *The* **terms** *of an expression are the individual parts of that expression that are added (i.e. separated by $+$ (or $-$) signs). Reading from left to right, we call the terms the* **first term (of the expression)**, *the* **second term**, *etc.*

A **constant term** *is a term of an expression that has no variables in it.*

The **coefficient** *of a term is the constant being multiplied by variables in the term.*

Two terms are called **like (terms)** *if, other than their coefficients, they are the same. Otherwise the terms are called* **unlike (terms)**.

EXAMPLE 1

Consider this expression:

$$3x + 5 - 7x^2 + 9 - x$$

1. What is the first term of this expression?
2. What is the third term of this expression?
3. Identify the constant term(s) of this expression.
4. What is the coefficient on the first term?
5. What is the coefficient on the last term?
6. Are the second and last terms of this expression like?
7. Identify all the like terms (if any) in this expression.

EXAMPLE 2

Consider this expression:

$$4xy + 7x + 10xy - 2yx.$$

1. What is the coefficient on the third term?
2. Identify all the like terms (if any) in this expression.

Some general rules for legally manipulating terms and/or expressions

Seven things you can always legally do to any expression at any time

1. Write terms in a different order
2. Combine like terms
3. Replace equals with equals
4. Add 0 creatively
5. Multiply by 1 creatively
6. FOIL, distribute and/or factor
7. Apply other legal math rules

Let's go through these seven things, one-by-one:

1. **Write terms in a different order:** you can write the terms of an expression in any order you want. For instance,

   $$7x - 5 + 2x^2 \text{ is the same thing as } 2x^2 + 7x - 5,$$

   which we write as

   $$7x - 5 + 2x^2 = 2x^2 + 7x - 5.$$

2. **Combine like terms** using addition and subtraction:

   $$1 + 2x + 3x^2 + 7x - 5 + 2x^2 = 5x^2 + 9x - 4.$$

   WARNING: Be careful not to combine unlike terms:

   $$5 + 8\cos x \neq 13\cos x$$

3. **Replace equals with equals:** this means that you can substitute/replace any term or expression with an equal quantity. Here are some examples:

You can replace this...	...with this:
$2^2x - 3t$	$4x - 3t$
$7x - 3t$	$4x + 3x - 3t$

   Another example: if you know $s = 8$, then you can replace $s^2 - 4$ with $8^2 - 4$.

We often write these replacements using $=$ signs. For example, the first line of the chart above would usually be written

$$2^2x - 3t = 4x - 3t.$$

Often, we chain together several steps where, in each step, we replace equals with equals:

$$\begin{aligned} 4(2 \cdot 3^2 - 5) + 7 &= 4(2 \cdot 9 - 5) + 7 \\ &= 4(18 - 5) + 7 \\ &= 4(13) + 7 \\ &= 52 + 7 \\ &= 59. \end{aligned}$$

Make sure when doing this that you follow universally agreed upon rules for order of operations (see Section 1.3).

4. You can **creatively add** 0 to a term or expression at any time.

   An uncreative addition of 0: $\quad 7x = 7x + 0$

   More creative: $\quad x^2 + 8x + 7 = x^2 + 8x + 7 + 16 - 16$

   Why might you do this?

5. You can **creatively multiply by** 1 to a term or expression at any time.

$$\frac{3x+2}{\sqrt{t-4}} = \frac{3x+2}{\sqrt{t-4}} \cdot 1$$

$$\frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\tan\theta} \cdot \frac{\cos\theta}{\cos\theta}$$

$$x^2 + 5 = (x^2 + 5)\left(3^2 - 8\right)$$

$$\frac{3x+2}{\sqrt{t-4}} + 7x^2 = \frac{3x+2}{\sqrt{t-4}} \cdot \frac{\sqrt{t-4}}{\sqrt{t-4}} + 7x^2$$

Why might you do this? One reason is to add/subtract fractions:

$$\frac{2}{3} + \frac{3}{5} =$$

6. You can **FOIL, distribute and/or factor** terms or expressions (more on this in Section 1.4).

7. You can **apply other legal rules** of arithmetic and algebra to a term or expression (more on these later, at various points in the course).

Equations

Definition 1.3 *An* **equation** *is a statement that asserts that two expressions are equal. In this situation, we put an = between the two expressions.*

In an equation, the expression before the = is called the **left-hand side (LHS)** *and the expression after the = is called the* **right-hand side (RHS)**.

EXAMPLES OF EQUATIONS

- $3 + 5 = 8$
- $8x + 3xy - 4y^2 = 17$
- $3 \sin 2\theta + 4 \cos^2 \theta = 5 \tan 3\theta - 1$
- $\dfrac{(x+2)(y^2 - 3 \cdot 2^y)}{xy^5 - \sqrt{3x^4 \cos 3x}} + \left(\dfrac{16 + 7tx}{3 - \frac{1}{4-7y}} \right)^{-3/5} = 3x^6 + y^3 \tan(x - 3y^2) - \dfrac{4\sqrt[3]{y+5}}{7t^3 - 2t^2 + 5e^x}$

Equals signs and arrows

An = in math has a very specific meaning. It means that what precedes the = is an expression (i.e. a quantity), and that what follows the = is also an expression, and that those expressions are the same (i.e. equal).

EXAMPLES OF HORRIBLE MISUSE OF EQUALS SIGNS

$$x + 8 = 15 = \boxed{7} \quad \text{or} \quad x + 8 = 15 = x = \boxed{7}.$$

WHAT THAT PERSON ACTUALLY MEANT

$$\begin{aligned} x + 8 &= 15 \\ x &= \boxed{7}. \end{aligned}$$

Arrows in mathematical sentences also have specific meaning. A double arrow $\Rightarrow$ is short for the word "therefore". The arrow $\Rightarrow$ means that whatever comes after the $\Rightarrow$ follows as a logical consequence of what comes before the $\Rightarrow$.

A single arrow $\rightarrow$ is used for functions (more on this later) or limits (more in calculus), not logical implication or to indicate steps in a problem.

Examples of horrible misuse of arrows

$$5(x+4) \Rightarrow 5x+20 \qquad 5(x+4) \rightarrow 5x+20$$

Examples of correct usage of $\Rightarrow$

$$x = 7 \;\Rightarrow\; x^2 = 49.$$

$$x + 8 = 15 \;\Rightarrow\; x = 7.$$

WARNING: I penalize the misuse of arrows and equals signs. This is a pet peeve of mine.

Example 3

Consider this equation:

$$5x^2 + 3x - 7 = 4x^2 - 2x^2 + 9$$

1. What is the second term on the left-hand side?

   Answer: $3x$

2. What is the constant term on the right-hand side?

   Answer: 9

3. Which side of the equation contains like terms?

   Answer: the RHS ($4x^2$ and $-2x^2$ are like)

4. What is the x term on the left-hand side?

   Answer: $3x$

5. What is the coefficient on the first term of the left-hand side?

   Answer: 5

General rules for manipulating equations

Equations are like balanced scales. Whenever you change the value of one side of an equation, you must change the value of the other side in the same way (to keep the scale balanced).

EXAMPLE 4

Consider this equation:

$$3x^2 + 4x - 2 = 10x^2 - 9x + 12$$

1. Add $3x$ to both sides of this equation.

2. If this equation is manipulated so that the left-hand side becomes $4x-2$, what must the right-hand side be?

3. If this equation is manipulated so that the right-hand side becomes $10x^2-5x$, what must the left-hand side be?

4. Add or subtract something from both sides of this equation so that the right-hand side becomes 12.

Note: Combining like terms, creatively adding 0 and creatively multiplying by 1 do not change the value of an expression.

So you can do these types of things on one side of an equation without doing them on the other side.

Here is an example of valid math:

$$3\sin 2\theta + 4\cos^2\theta = 5\tan 3\theta - 1$$
$$3\sin 2\theta + 4\cos^2\theta \cdot \frac{\tan^2\theta}{\tan^2\theta} = 5\tan 3\theta - 1 + \cos\theta - \cos\theta$$

But this next "algebra" isn't valid, because the LHS changed in a way the RHS didn't:

$$3\sin 2\theta + 4\cos^2\theta = 5\tan 3\theta - 1$$
$$3\sin 2\theta + 4\cos^2\theta \cdot \tan^2\theta = 5\tan 3\theta - 1$$

Inequalities

Definition 1.4 *An* **inequality** *is a statement that asserts one expression is more than (and/or equal to) another. In this situation, we put $<$, $\leq$, $>$ or $\geq$ between the expressions.*

An inequality is called **strong** *or* **strict** *if it has $<$ or $>$, and is called* **weak** *if it has $\leq$ or $\geq$.*

EXAMPLES OF INEQUALITIES

a) $3 < 5 + 6$ is a strong inequality.

b) $5x + 7 \geq 3 - 4y$ is a weak inequality.

General rules for manipulating inequalities

You manipulate inequalities the same as you would equations, with one exception: *when you multiply or divide by a negative number, the direction of the inequality changes*.

$$5x < 25 \qquad\qquad -\frac{x}{3} \geq -4$$

1.2 Quick review of arithmetic

Multiplication of fractions

To multiply fractions, multiply both the numerators and denominators:

§1.2 EXAMPLE 1

a) $\frac{3}{8} \cdot \frac{5}{7}$

b) $\frac{-8}{9} \cdot \frac{3}{20}$

Negative signs in fractions

A negative sign in a fraction can be in the top, bottom, or out in front; it doesn't change the fraction:

$$-1 \cdot \frac{a}{b} = -\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

Of course, negatives in both the numerator and denominator cancel:

$$\frac{-a}{-b} = \frac{a}{b}$$

Division of fractions

The **reciprocal** of fraction $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}} = \frac{b}{a}$ (i.e. flip the fraction over). To divide one fraction by another, multiply the first by the reciprocal of the second:

§1.2 EXAMPLE 2

a) $\frac{3}{8} \div \frac{5}{7}$

b) $\frac{\frac{7}{6}}{\frac{-5}{3}}$

Solution: $\frac{\frac{7}{6}}{\frac{-5}{3}} = \frac{7}{6} \div \frac{-5}{3} = \frac{7}{6} \cdot \frac{3}{-5} = \frac{21}{-30} = \boxed{-\frac{7}{10}}$.

c) $4 \div \left(\frac{3}{8}\right)$

Solution: $4 \div \left(\frac{3}{8}\right) = 4 \cdot \frac{8}{3} = \frac{4}{1} \cdot \frac{8}{3} = \boxed{\frac{32}{3}}$.

d) $\dfrac{\frac{3}{8}}{4}$

e) $\dfrac{3}{\frac{8}{4}}$

WARNING: The answers to (d) and (e) above are different. In general,

$$\frac{\frac{a}{b}}{c} \neq \frac{a}{\frac{b}{c}},$$

so you should avoid writing this:

Addition and subtraction of fractions

Adding/subtracting fractions is harder than multiplying/dividing them, because to add fractions you need to find a ____________________________ .

§1.2 EXAMPLE 3

a) $\frac{2}{7} + \frac{3}{4}$

b) $\frac{3}{5} - \frac{1}{2} + \frac{7}{8}$

Solution:

$$\frac{3}{5} - \frac{1}{2} + \frac{7}{8} = \frac{3(8)}{5(8)} - \frac{1(20)}{2(20)} + \frac{7(5)}{8(5)} = \frac{24}{40} - \frac{20}{40} + \frac{35}{40} = \frac{24-20+35}{40} = \boxed{\frac{39}{40}}.$$

c) $\frac{5}{12} + \frac{1}{8}$

Solution: $\frac{5}{12} + \frac{1}{8} = \frac{5(2)}{12(2)} + \frac{1(3)}{8(3)} = \frac{10}{24} + \frac{3}{24} = \boxed{\frac{13}{24}}.$

Whole number exponents

First, a **whole number** is a number like $1, 2, 3, 4$.....

> **Definition 1.5** *If n is a whole number, then the expression x^n is short for the product of n copies of x:*
> $$x^n = x \cdot x \cdot x \cdots x$$
> *In this context, x^n is called the n^{th}* **power** *of x; x is called the* **base**; *and n is called the* **exponent**.

QUICK EXAMPLES

a) $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = \boxed{32}$

We pronounce this "2 to the 5^{th} power is 32".

In this expression, 2 is the base and 5 is the exponent.

b) $\left(\frac{5}{3}\right)^3 = \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{5}{3} = \boxed{\frac{125}{9}}$

c) $-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3) = \boxed{-81}$

d) $(-3)^4 = (-3)(-3)(-3)(-3) = \boxed{81}$.

Exponent rules

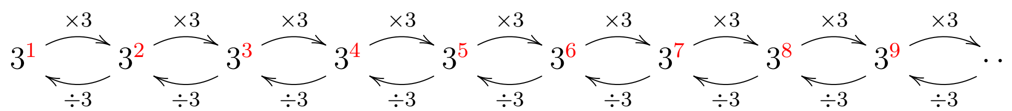
In calculus, it is vital to be able to manipulate expressions involving exponents. This manipulation involves appliction of what are called *rules (or laws) of exponents*. To understand exponent rules, let's start by considering a picture which lists the powers of 3:

$$3 \qquad 9 \qquad 27 \qquad 81 \qquad 243 \qquad 729 \qquad \cdots$$

Writing these numbers as exponents, the same picture is

$$3^1 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^2 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^3 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^4 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^5 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} 3^6 \underset{\div 3}{\overset{\times 3}{\rightleftarrows}} \cdots$$

The exponents (shown in red) make a number line (of sorts), where multiplying by 3 moves you one unit to the right and dividing by 3 moves you one unit to the left.



Observation # 1

Multiplying by 3^b corresponds to moving b places to the right on this diagram. So if you start at 3^a and multiply by 3^b, you go from position a to position ________ . Writing this idea as an exponent rule, this is

Observation # 2

Suppose you multiply by 3^a b times. Each time you multiply by 3^a, you move to the right a units, so if you do this b times, you have moved to the right a total of $\square$ units, which corresponds to multiplying by $\square$. Writing this as an exponent rule, we have

Observation # 3

If I wanted to know what 3^0 is, for the diagram to make sense I would have to get to 3^0 by starting at _____ and ________________ by 3. This makes

$$3^0 = 3^1 \div 3 = \square .$$

Observation # 4

Dividing by 3^b is equivalent to moving b units to the left. So if you start at 3^a and divide by 3^b, you go from position a to position ________ . Writing this idea as an exponent rule, we have

There's nothing special about the 3 in these observations. They hold for any base x, meaning that we've derived these rules:

Theorem 1.6 (Laws of exponents)

Anything[a] to zero power is 1: $x^0 = 1$

Anything to first power is itself: $x^1 = x$

To multiply powers, add the exponents: $x^a x^b = x^{a+b}$.

Iterated exponents multiply: $(x^a)^b = x^{ab}$

To divide powers, subtract the exponents: $\dfrac{x^a}{x^b} = x^{a-b}$

[a] One caveat: if $x = 0$, then $x^0 = 0^0$ which is tricky... you need calculus to evaluate 0^0.

Theorem 1.7 (Exponents respect multiplication and division)

$$(xy)^a = x^a y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

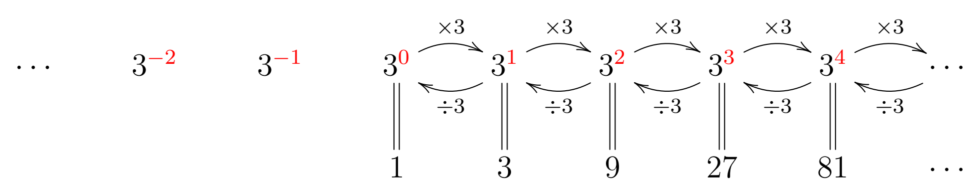
WARNING: Exponents do not respect addition and subtraction: in general,

$$(x+y)^a \neq x^a + y^a \qquad (x-y)^a \neq x^a - y^a$$

Integer exponents

First, an **integer** is a positive or negative whole number, or zero. Integers are numbers like ..., $-3, -2, -1, 0, 1, 2, 3$, ...

We know what happens when an exponent is either a positive whole number or zero. What if the exponent is negative?



$$\begin{array}{ccccccccccccc} \cdots & 3^{-a} & \cdots & 3^{-3} & 3^{-2} & 3^{-1} & 3^{0} & 3^{1} & 3^{2} & 3^{3} & \cdots & 3^{a} & \cdots \\ & \| & & \| & \| & \| & \| & \| & \| & \| & & \| & \\ \cdots & \frac{1}{3^a} & \cdots & \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & 1 & 3 & 9 & 27 & \cdots & 3^a & \cdots \end{array}$$

Again, there's nothing special about 3 (we could use any base). This leads to the following rule:

Theorem 1.8 (Negative exponents)

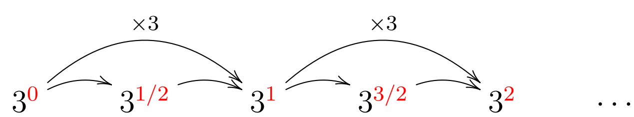
$$x^{-a} = \frac{1}{x^a} = \left(\frac{1}{x}\right)^a.$$

Rational exponents, square roots and other radicals

First, a **rational number** is any number that can be written as the quotient of two integers. These include

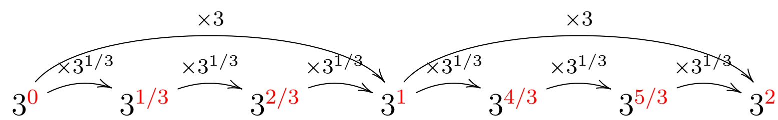
$$\frac{5}{3}; \quad \frac{-1985445}{979438}; \quad 19 \qquad\qquad 8.27$$

Continuing with our diagram with powers of 3, let's try to think about what 3^a would be if a was a rational number that wasn't an integer. Let's start with $a = \frac{1}{2}$:



So $3^{1/2}$ must be a number which, when multiplied by itself, gives 3. This is what we call the **square root** of 3 and denote by $\sqrt{3}$. We conclude:

Now, what about $3^{1/3}$? Here, we think of the diagram



We can see from this picture that $3^{1/3} = \boxed{\quad}$ and that $3^{4/3} = \boxed{\qquad\qquad}$.

Similarly, $3^{1/n} = \boxed{\quad}$ and that $3^{m/n} = \boxed{\qquad\qquad}$.

More generally, there is nothing special about using 3 as a base. We have these rules for rational (fractional) exponents:

Theorem 1.9 (Rational exponents)

$$\sqrt{x} = x^{1/2} \qquad \sqrt[n]{x} = x^{1/n} \qquad x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Expressions involving a $\sqrt{\ }$, $\sqrt[3]{\ }$ or $\sqrt[n]{\ }$ are called **radicals** (these signs are called **radical signs**). Since radicals are effectively rewritten exponents, they respect multiplication and division but not addition and subtraction:

Theorem 1.10 (Radicals respect multiplication and division)

$$\sqrt{xy} = \sqrt{x}\,\sqrt{y} \qquad \sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

WARNING: Radicals do not respect addition and subtraction: in general,

$$\sqrt{x \pm y} \neq \sqrt{x} \pm \sqrt{y} \qquad \sqrt[n]{x \pm y} \neq \sqrt[n]{x} \pm \sqrt[n]{y}$$

One more thing about square roots:. There is a difference between the statements

$$x^2 = c \qquad \text{and} \qquad x = \sqrt{c}.$$

Square roots (and other even roots) are never allowed to be negative. So

$$\sqrt{25} = 5 \text{ (not } \pm 5\text{).}$$

However, if we have the equation $x^2 = 25$, then x might be negative, so $x^2 = 25$ leads to $x = \pm\sqrt{25} = \pm 5$ (not just 5).

Examples

§1.2 EXAMPLE 1

Evaluate each expression:

1. 3^0
2. 0^3
3. 0^{-3}
4. $\sqrt{24}\sqrt{6}$
5. $\frac{4^8}{4^5}$
6. $32^{2/5}$
7. $27^{-2/3}$
8. $7^{4/5}\sqrt[5]{7}$
9. $\sqrt{(-8)^3}$

§1.2 EXAMPLE 1

Simplify each expression (by combining the exponents as much as possible):

1. $\sqrt[4]{b^3}\sqrt{b}\sqrt[4]{b^7}$

2. $(3x^2y^3)y^2$

   Solution: $(3x^2y^3)y^2 = 3x^2y^{3+2} = \boxed{3x^2y^5}$.

3. $\left(\frac{2x^2}{3\sqrt{y}}\right)^4 \left(\frac{4y^2}{12x}\right)$

§1.2 EXAMPLE 3

Rewrite each expression using radical signs, so that it contains no fractional or negative exponents:

1. $x^{1/3}$

   Solution:. $x^{1/3} = \boxed{\sqrt[3]{x}}$.

2. $x^{-2/3}$

   Solution:. $x^{-2/3} = \frac{1}{x^{2/3}} = \boxed{\frac{1}{\sqrt[3]{x^2}}}$. ($\boxed{\frac{1}{(\sqrt[3]{x})^2}}$ is also correct.)

3. $19^{1/2}$

   Solution:. $19^{1/2} = \boxed{\sqrt{19}}$.

4. $17^{2/3}50^{3/4}x^{1/8}$

   Solution:. $17^{2/3}50^{3/4}x^{1/8} = \boxed{\sqrt[3]{17^2}\sqrt[4]{50^3}\sqrt[8]{x}}$.

§1.2 EXAMPLE 4 (VERY IMPORTANT IN CALCULUS)

Rewrite each given expression in the form $\square x^{\square}$, where the boxes are numbers:

1. $\dfrac{8x^7}{4x^2}$
2. $2x^2(3x^3)^2$
3. $\dfrac{1}{5\sqrt{x}}$
4. $\dfrac{3}{x}$
5. $\sqrt[5]{x}$
6. $\dfrac{4}{x^5}$
7. $\dfrac{3}{12x^{-2}}$
8. $\dfrac{1}{\sqrt{x^5}}$
9. $\sqrt{2x^5}$
10. $\left(\dfrac{3x}{5}\right)^{-1}$
11. $x^3\sqrt{x}$
12. $\sqrt[3]{64x^3}$
13. $\dfrac{3x^{-3}}{5x^{-2}}$
14. 4

1.3 Order of operations

MOTIVATION

Consider these two expressions:

$$4 + 3 \cdot 5 \qquad\qquad 18 \div 3 \times 2$$

A reasonable person might interpret either of these expressions in two ways:

	$4 + 3 \cdot 5$	$18 \div 3 \times 2$
Reasonable person # 1:		
Reasonable person # 2:		

If we don't decree one of these people to be wrong, this is a big problem!

We need universally agreed-upon procedures for evaluating complicated expressions like these.

This means we have to know what to do first, and in what order to do things.

In other words, we have to have a standard **order of operations** that everyone agrees (has agreed) to follow.

Problems with PEMDAS

You may be aware of the expression "PEMDAS" which is frequently used to teach order of operations. This is not a bad thing to know, but it is dangerous.

- There is some batshit crazy wrong stuff related to PEMDAS / order of operations floating around on places like YouTube. Be very careful looking at this stuff.
- PEMDAS doesn't mention functions like $\sin$, $\log$, $\cos$, etc. How do functions fit into order of operations? PEMDAS doesn't say (we'll talk about this in Chapter 2).
- PEMDAS makes it seem like M comes before D. But in reality, M and D are "tied", in that you work out all multiplication and division from left to right (you don't do all the multiplication before all the division). A and S are similar.

In other words, PEMDAS should really be thought of as

$$\mathrm{P} - \mathrm{E} - \begin{pmatrix} \mathrm{M} \\ \mathrm{D} \end{pmatrix} - \begin{pmatrix} \mathrm{A} \\ \mathrm{S} \end{pmatrix}.$$

Put another way, here is our order of operations (for now):

1. **P**arentheses (and other grouping symbols like brackets) supersede everything else.

   Be careful! There are often invisible parentheses that are present even if they aren't written. Here is an example:

   $$8 + \frac{3+7}{9-4} =$$

   When writing a horizontal fraction bar, there are always invisible parentheses around the top and bottom of that fraction.

2. **E**xponents, from left-to-right
3. **M**ultiplication and **D**ivision, from left-to-right
4. **A**ddition and **S**ubtraction, from left-to-right

Quick Examples

a) $8 - 12 + 2 = -10 + 2 = \boxed{-8}.$

b) $8 - (12 + 2) = 8 - 10 = \boxed{-2}.$

c) $24 \div 4 \times 2 = 6 \times 2 = \boxed{12}.$

d) $3 + \dfrac{7-3}{2} = 3 + \dfrac{4}{2} = 3 + 2 = \boxed{5}.$

e) $5 \cdot 2^3 - (3^2 + 2) = 5 \cdot 8 - (9 + 2) = 40 - 11 = \boxed{29}.$

f) $(5 - 1)^2 - 4 \cdot 2 = 4^2 - 8 = 16 - 8 = \boxed{8}.$

1.4 Distributing, FOILing and factoring

Distributing

By **distributing**, we mean applying what is called the **distributive law**, which says that for numbers a, x and y,

$$a(x+y) = ax + ay.$$

This law is based on an area calculation, where we figure the area of a rectangle two different ways:

§1.4 EXAMPLE 1

Distribute, and then simplify by combining like terms:

a) $5(x+3)$

Solution: $5(x+3) = \boxed{5x+15}$.

b) $2x^{3/2}(5x^2 + 8\sqrt{x})$

c) $5(x+3) + x(x-2)$

Solution: $5(x+3) + x(x-2) = 5x + 15 + x^2 - 2x = \boxed{x^2 + 3x + 15}$.

d) $3(x^2+4) - 5(x+1) + 4(2x^2 + x)$

WARNING:. Distributing is special to multiplication. Operations other than multiplication **do not** distribute!

$$
\begin{aligned}
(x+y)^2 &\neq x^2+y^2 \\
\sin(x+y) &\neq \sin x + \sin y \\
\cos(x+y) &\neq \cos x + \cos y \\
\sqrt{x+y} &\neq \sqrt{x} + \sqrt{y} \\
\frac{1}{x+y} &\neq \frac{1}{x} + \frac{1}{y} \\
|x+y| &\neq |x| + |b| \\
\log(x+y) &\neq \log x + \log y
\end{aligned}
$$

The **only** thing that distributes is multiplication:

$$5(x+y) = 5x + 5y \qquad\qquad a(x+y) = ax + ay$$

In general, if you see $+$ or $-$ inside parentheses or otherwise grouped, you should think "Aw, crap - I probably cannot do anything simple with this". The presence of addition and/or subtraction makes algebraic manipulation harder, generally speaking.

On the other hand, $\times$ or $\div$ inside parentheses is often not so bad:

$$
\begin{aligned}
(xy)^2 &= x^2y^2 & \left(\frac{x}{y}\right)^a &= \frac{x^a}{y^a} \\
|xy| &= |x|\,|y| & \frac{1}{xy} &= \frac{1}{x}\cdot\frac{1}{y} \\
\sqrt{xy} &= \sqrt{x}\,\sqrt{y} & \sqrt{\frac{x}{y}} &= \frac{\sqrt{x}}{\sqrt{y}} \\
\log(xy) &= \log x + \log y & \log\left(\frac{x}{y}\right) &= \log x - \log y
\end{aligned}
$$

etc.

FOILing

By **FOILing**, we mean applying this law:

$$(a+b)(x+y) = ax + ay + bx + by.$$

We call this **FOILing** because four terms you get come from the **F**irst, **O**utside, **I**nside and **L**ast terms of those being rewritten.

Like distributing, FOILING is also justified by an area calculation:

FOILing a sum and difference

Something special happens when you FOIL the sum and difference of the same terms:

$$(a+b)(a-b) =$$

It is useful to remember this fact:

> **Theorem 1.11 (FOILing a sum and difference)** *For any terms a and b,*
>
> $$(a+b)(a-b) = a^2 - b^2.$$

§1.4 EXAMPLE 2

FOIL (and/or distribute), and then simplify by combining like terms:

a) $(2x+1)(x-3)$

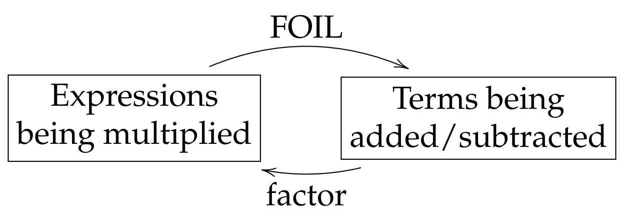
b) $(2x+5)(2x-5)$

c) $(x+7)(3x-1)+2x(x-5)$

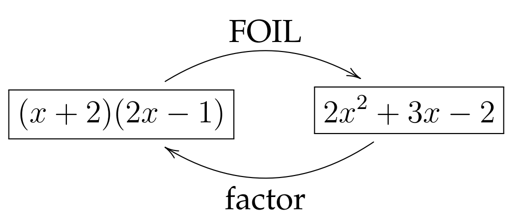
d) $(\sqrt{x}+2)\left(3x^{3/2}-5x^{-1/2}\right)$

Factoring

Factoring the "opposite" or "inverse" procedure of FOILING/distributing. Here's the idea:



Here's a specific example of the concept:



To factor an expression, pull out common factors, look for a difference of squares, and then try to factor what's left.

Pulling out common factors

§1.4 EXAMPLE 3

Completely factor each expression:

a) $3x - 12$

b) $5x^3 + 10x^2$

c) $x^2 + 5x$

Factoring a difference of squares

If we see any expression of the form $a^2 - b^2$, we can undo the FOILING rule we saw earlier to write

$$a^2 - b^2 = (a + b)(a - b).$$

This is called **factoring a difference of squares**.

§1.4 EXAMPLE 4

Completely factor each expression:

a) $x^2 - 9$

b) $x^2 - 14$

c) $25t^2 - 81$

d) $x^4 - x^2$

Factoring $x^2 \pm bx \pm c$

Expressions like this should be factored as $(x \pm \square)(x \pm \triangle)$. I tend to factor these by trial and error, with the object of finding two numbers $\square$ and $\triangle$ that multiply to the constant term c and add or subtract to give the coefficient b on the x term.

§1.4 EXAMPLE 5

Completely factor each expression:

a) $x^2 - 7x + 12$

b) $x^2 + 8x - 33$

c) $x^5 - 12x^4 + 36x^3$

d) $3x^2 + 9x - 30$

e) $x^2 + x + \frac{1}{4}$

Factoring $ax^2 \pm bx \pm c$

These tend to factor into something like $(\diamondsuit x \pm \Box)(\heartsuit x \pm \triangle)$, where the shapes are numbers. If the a isn't a common factor that can be pulled out, then you try to factor these by trial and error, or use the "box method".

§1.4 EXAMPLE 6

Completely factor each expression:

a) $2x^2 + 3x - 20$

b) $18x^2 + 21x - 4$

$18x^2$	
	-4

b) $12x^3 + 4x^2 - 16x$

1.5 The coordinate plane

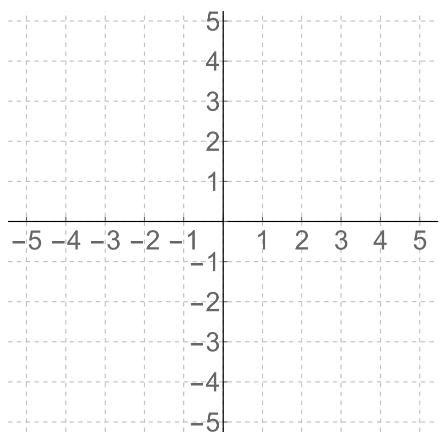
Ordered pairs

Just as real numbers can be thought of as points on a number line, **ordered pairs** (x, y) can be thought of as points in a plane. The first number in an ordered pair is called the **x-coordinate** and measures the horizontal distance the point (x, y) is from the origin; the second number in the pair is called the **y-coordinate** and measures the vertical distance that (x, y) is from the origin.

§1.5 EXAMPLE 1

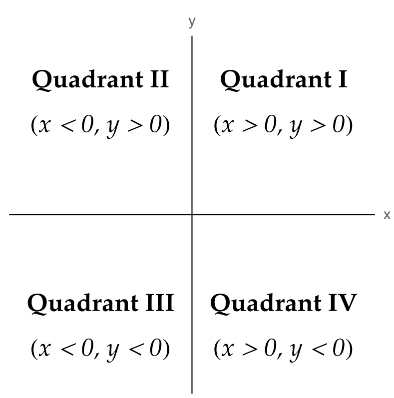
Graph the following points on the provided axes:

$$(0, -3) \qquad (-2, 0) \qquad (5, -1) \qquad (-3, 4)$$



Quadrants

The x- and y-axes divide the coordinate plane into four **quadrants**, numbered by Roman numerals:

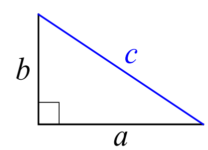


Distance formula & Pythagorean theorem

Recall the Pythagorean Theorem, which relates the lengths of the three sides of a right triangle:

Theorem 1.12 (Pythagorean Theorem) *If $\triangle ABC$ is a right triangle with legs a and b and hypotenuse c, then*

$$a^2 + b^2 = c^2.$$

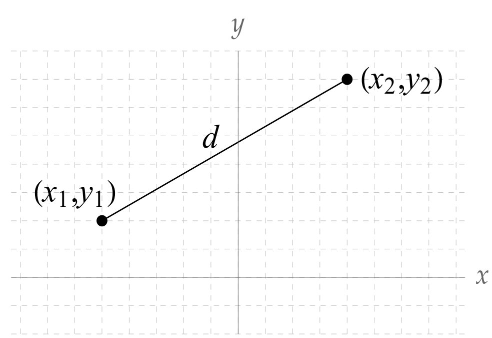


The Pythagorean Theorem can be used to find a formula for the distance between two points in the xy-plane:

Theorem 1.13 (Distance formula) *The distance between points (x_1, y_1) and (x_2, y_2) is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

WHY THIS FORMULA IS TRUE



§1.5 EXAMPLE 2

Compute the distance between each given pair of points:

a) $(3, 7)$ and $(-5, 10)$

b) $(2, -3)$ and $(9, 1)$

Solution: by the distance formula, this is

$$\begin{aligned} d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{(1 - (-3))^2 + (9 - 2)^2} \\ &= \sqrt{4^2 + 7^2} \\ &= \sqrt{16 + 49} = \boxed{\sqrt{65}}. \end{aligned}$$

c) $(4, 0)$ and (x, x^2)

Note: horizontal and vertical distances do not require the distance formula:

§1.5 EXAMPLE 3

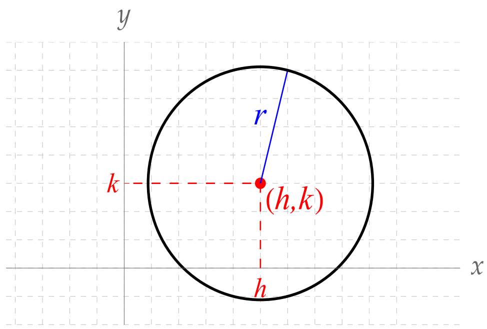
Compute the distance between each given pair of points:

a) $(5, 8)$ and $(5, -3)$

b) $(-2, -6)$ and $(-2, -1)$

Equations of circles

The distance formula can be used to describe the set of points in the xy-plane that lie on any circle. Suppose we have a circle centered at (h, k) with radius r:

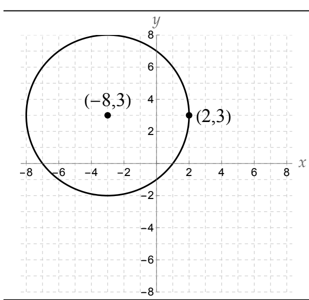
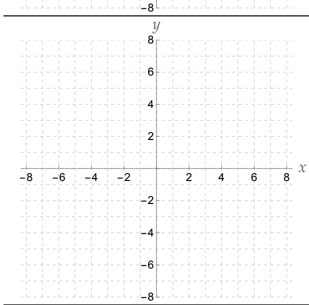


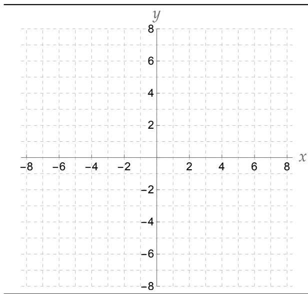
Theorem 1.14 (Equations of circles) *The equation of the circle of radius r centered at (h, k) is*

$$(x - h)^2 + (y - k)^2 = r^2.$$

§1.5 Example 4

Complete the following table:

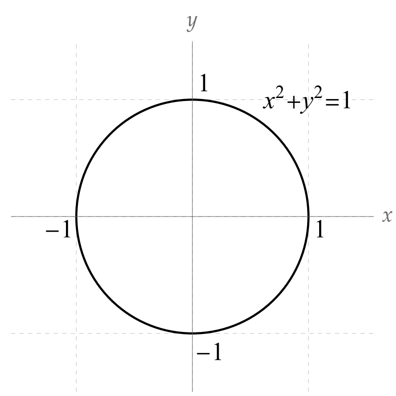
Graph of circle	Equation of circle	Radius Diameter Center
		
	$(x - 3)^2 + (y - 2)^2 = 4$	

Graph of circle	Equation of circle	Radius Diameter Center
		$r = 3$ $(-3, 0)$

The unit circle

Definition 1.15 (Unit circle) *The circle of radius* 1 *centered at* $(0, 0)$ *is called the* **unit circle**. *Its equation is*

$$x^2 + y^2 = 1.$$



1.6 Quick review of unit circle trigonometry

RECALL

The trigonometric functions sine, cosine and tangent are used to convert *rotational measurements* into *distance measurements*:

Definition 1.16 (Unit circle definition of sine, cosine and tangent) *Take a number θ. Starting at the point $(1,0)$, mark off an arc of length θ on the unit circle (go counterclockwise if $\theta > 0$ and clockwise if $\theta < 0$). In other words, mark off an angle of θ radians.*

Call the point on the unit circle where the arc ends (x,y).

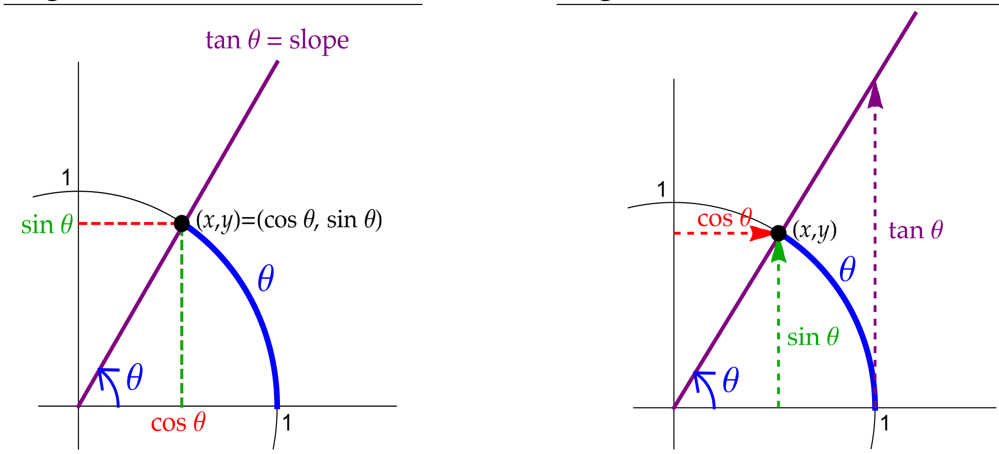
Then define the **sine**, **cosine** *and* **tangent** *of θ to be:*

$$\sin\theta = \sin(\theta) \quad = y$$

$$\cos\theta = \cos(\theta) \quad = x$$

$$\tan\theta = \tan(\theta) \quad = \frac{y}{x} = \frac{\sin\theta}{\cos\theta} = \text{ slope of the terminal side of } \theta$$

Trig functions as coordinates | **Trig functions as directed distances**



In the second picture, we call these "directed" distances because they could be negative (if the arrows point left or down, for example).

§1.5 EXAMPLE 1

Evaluate each quantity:

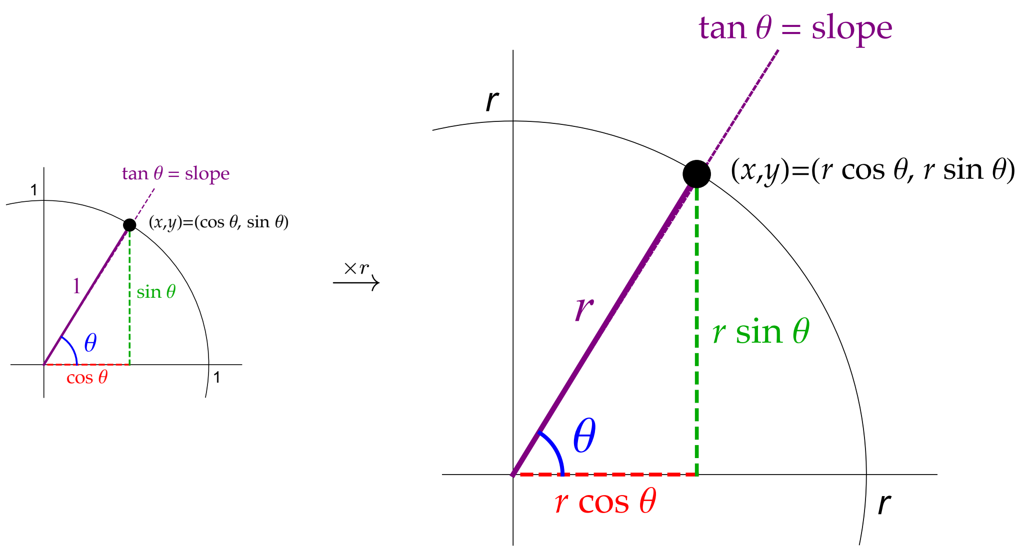
a) $\cos\pi$

b) $\sin\dfrac{3\pi}{2}$

c) $\tan 2\pi$

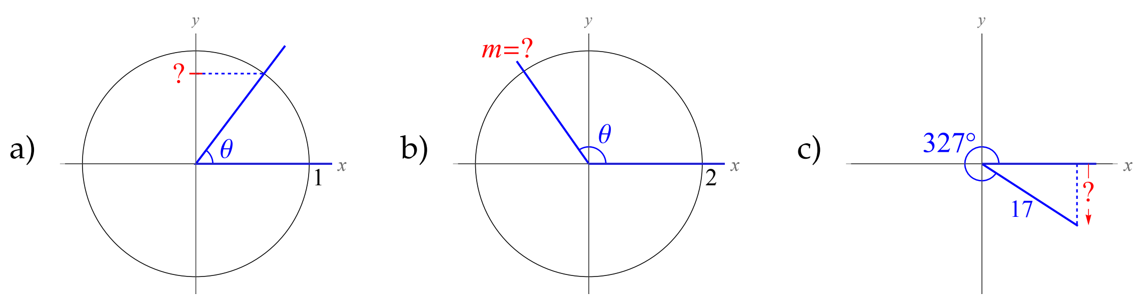
d) $\cos\dfrac{-\pi}{2}$

If we are applying trig to a problem where our circle doesn't have radius 1, we think of "blowing up" the previous pictures by a factor of r, where r is the radius of the circle we're considering:



§1.5 Example 2

In each picture, you are given a length or coordinate marked with a "?". Write a formula for the "?" in terms of the other numbers and variables given in the picture:



Secant, cosecant and cotangent

There are three other trig functions, which are the reciprocals of the functions we defined above:

Definition 1.17 *Let θ be a number or an angle. Define the* **secant**, **cosecant** *and* **cotangent** *of θ to be, respectively,*

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

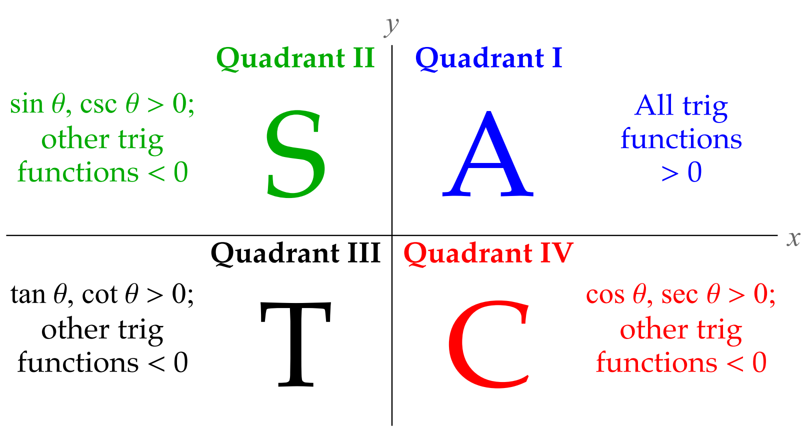
Generally speaking, secant, cosecant and cotangent aren't that useful, but they streamline some computations in calculus, so we'll cover a couple of important things about these functions later.

Signs of the trig functions

Whether or not a trig function of θ is positive or negative depends on the quadrant in which the angle θ lies. We use the phrase

All Scholars Take Calculus

to help remember the signs of the trig functions:



Definition 1.18 *An angle is called* **quadrantal** *if it is a multiple of $\frac{\pi}{2}$ (i.e. is a multiple of $90°$). Such angles include*

$$0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi, \pm\frac{7\pi}{2}, \ldots$$

These angles do not belong to any of the four quadrants I-IV.

Trig functions of special angles

A "special angle" is any multiple of $30°$ or any multiple of $45°$, i.e. angles like

$$0°, 30°, 60°, 90°, 120°, 150°, -30°, -60°, -90°, -120°, -150°, \ldots$$

$$45°, 90°, 135°, 180°, 225°, -45°, -90°, -135°, -180°, -225°, \ldots$$

In radians, these special angles are any multiple of $\frac{\pi}{6}$, any multiple of $\frac{\pi}{4}$, any multiple of $\frac{\pi}{3}$, any multiple of $\frac{\pi}{2}$ or any multiple of π:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-5\pi}{6}, \frac{7\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{3}, \frac{-\pi}{3}, \frac{3\pi}{4}, \frac{-5\pi}{4}, \frac{3\pi}{2}, \frac{-5\pi}{2}, 0, \pi, -2\pi, 3\pi, \ldots$$

These are the angles you get when you divide a right angle into halves or thirds, and are the most commonly used angles in math courses and in real-world situations.

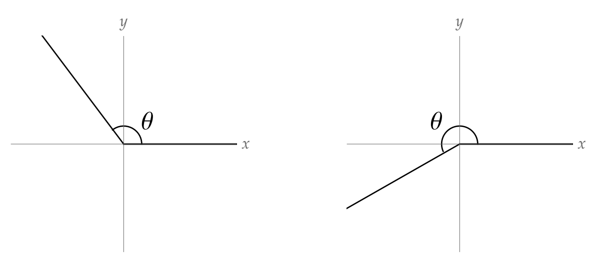
DISCLAIMER: You will pretty much never see degrees used in any math class again. All problems in MATH 130 will be asked in radians, and any problem asking for an angle in MATH 130 is asking for an answer in radians.

That said, most folks are more used to thinking in degrees. If you prefer to think in degrees, there are certain angles (in radians) that you should just "know" how many degrees they correspond to:

$$\pi = 180° \qquad \frac{\pi}{2} = 90° \qquad \frac{\pi}{3} = 60° \qquad \frac{\pi}{4} = 45° \qquad \frac{\pi}{6} = 30°$$

Reference angles

Given angle θ, a reference angle of θ is an angle $\widehat{\theta}$ in Quadrant I which can be obtained from θ by reflecting θ across the x- and/or y-axes:



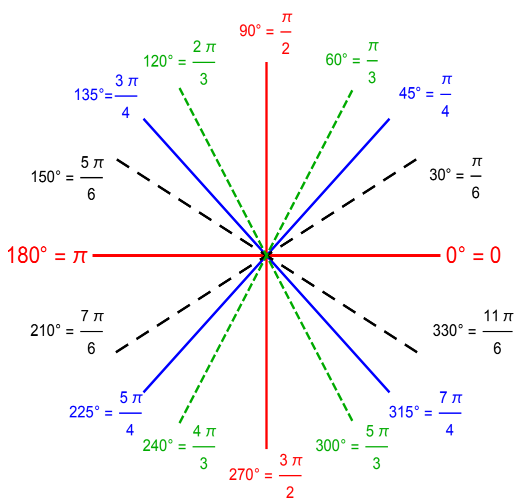
That means that if (x, y) is on the terminal side of $\widehat{\theta}$, then $(\pm x, \pm y)$ is on the terminal side of θ.

Therefore, since the trig functions are computed from such a point, we know:

Theorem 1.19 *If $\widehat{\theta}$ is the reference angle of θ, then*

$$\sin\theta = \pm\sin\widehat{\theta}; \qquad \cos\theta = \pm\cos\widehat{\theta}; \qquad \tan\theta = \pm\tan\widehat{\theta}.$$

It is fairly easy to compute the reference angle of a special angle given in radians:



IF IN LOWEST TERMS:

$(\text{whole \#})\,\pi \longleftrightarrow$

$\dfrac{(\text{odd \#})\,\pi}{2} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{4} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{3} \longleftrightarrow$

$\dfrac{(\text{whole \#})\,\pi}{6} \longleftrightarrow$

Theorem 1.20 *If θ is in radians, expressed as a fraction $\frac{a\pi}{b}$ where $\frac{a}{b}$* ***is in lowest terms*** *and $b = 2, 3, 4$, or 6, then the reference angle of θ is $\widehat{\theta} = \frac{\pi}{b}$.*

How to compute trig functions of special angles in any quadrant

1. Determine whether or not the angle θ is quadrantal (i.e. whether or not θ is a multiple of $90°$, or a multiple of $\frac{\pi}{2}$ or π).

2. **If θ is quadrantal**, determine the point on the unit circle at angle θ. This point will be $(\pm 1, 0)$ or $(0, \pm 1)$. Then:

$$\begin{aligned} \cos\theta &= x & &\longrightarrow \text{ flip over } \cos\theta \text{ to get } \sec\theta \\ \sin\theta &= y & &\longrightarrow \text{ flip over } \sin\theta \text{ to get } \csc\theta \\ \tan\theta &= \text{ slope} & &\longrightarrow \text{ flip over } \cot\theta \text{ to get } \cot\theta \end{aligned}$$

3. **If θ is not quadrantal**,

   a) Determine the reference angle $\widehat{\theta}$ of θ. The ref. angle will be $30°$, $45°$ or $60°$.

   b) Compute $\sin\widehat{\theta}$ or $\cos\widehat{\theta}$ or $\tan\widehat{\theta}$ by remembering the following table (or using a "finger-counting trick"):

Theorem 1.21 (Trig functions of special angles in Quadrant I)

θ *in degrees*	θ *in radians*	$\sin\theta$	$\cos\theta$	$\tan\theta$
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

   c) If asked to compute $\sec\theta$, $\csc\theta$ or $\cot\theta$, flip over the trig function you found in the previous step as needed.

   d) Determine the quadrant θ is in; this will tell you the sign of your final answer based on the "All Scholars Take Calculus" rules.

   e) Your final answer is the $+/-$ sign from step (d) together with the number from steps (b) and (c).

§1.5 EXAMPLE 3

Compute the exact value of each quantity:

a) $\sin \frac{\pi}{6}$

b) $\cos \frac{5\pi}{3}$

c) $\tan \frac{3\pi}{4}$

d) $\cos \frac{-\pi}{3}$

e) $\sin \frac{\pi}{2}$

f) $\sec \frac{2\pi}{3}$

g) $\sin \pi$

h) $\cos 4\pi$

i) $\cos \frac{15\pi}{4}$

j) $\csc \frac{7\pi}{6}$

k) $\tan\dfrac{4\pi}{3}$

Solution:. $\dfrac{4\pi}{3} = 4 \cdot 60^\circ$ is in Quadrant III (making the answer positive);

the reference angle is $\widehat{\theta} = \dfrac{\pi}{3} = 60^\circ$; $\tan\widehat{\theta} = \sqrt{3}$; so $\tan\dfrac{4\pi}{3} = \boxed{\sqrt{3}}$.

l) $\sin\dfrac{-5\pi}{3}$

Solution: $\dfrac{-5\pi}{3} = -5 \cdot 60^\circ$ is in Quadrant I (making the answer positive);

the reference angle is $\widehat{\theta} = \dfrac{\pi}{3} = 60^\circ$; $\sin\widehat{\theta} = \dfrac{\sqrt{3}}{2}$; so $\sin\dfrac{-5\pi}{3} = \boxed{\dfrac{\sqrt{3}}{2}}$.

m) $\cot\dfrac{\pi}{6}$

Solution: $\dfrac{\pi}{6} = 30^\circ$ is in Quadrant I (making the answer positive);

$\tan 30^\circ = \sqrt{3}$; so $\cot 30^\circ = \boxed{\dfrac{1}{\sqrt{3}}}$.

n) $\cos\dfrac{3\pi}{2}$

Solution: $\dfrac{3\pi}{2} = 3 \cdot 90^\circ = 270^\circ$ is quadrantal;

the point on the unit circle is $(0, -1)$; so $\cos\dfrac{3\pi}{2} = \boxed{0}$.

o) $\tan\dfrac{3\pi}{2}$

Solution: $\dfrac{\pi}{2} = 90^\circ$ is quadrantal;

this angle is vertical so its slope is undefined; so $\tan\dfrac{\pi}{2}\,\boxed{\text{DNE}}$.

p) $\sin\dfrac{5\pi}{4}$

Solution:. $\dfrac{5\pi}{4} = 5 \cdot 45^\circ$ is in Quadrant III (making the answer negative);

the reference angle is $\widehat{\theta} = \dfrac{\pi}{4} = 45^\circ$;

$\sin\widehat{\theta} = \sin 45^\circ = \dfrac{\sqrt{2}}{2}$; so $\sin\dfrac{5\pi}{4} = \boxed{-\dfrac{\sqrt{2}}{2}}$.

Chapter 2

Functions

2.1 Introducing functions

§2.1 MOTIVATING EXAMPLE A

Let's create a "function" called "A" that can be used to compute the area of a square, in terms of the side length of the square.

§2.1 MOTIVATING EXAMPLE B

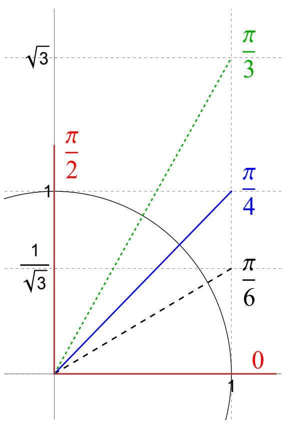
Suppose you buy frozen pizzas at Meijer. Each frozen pizza costs $\$5.00$, with the catch that frozen pizzas are “buy one, get one free”. Create a function called “price” that models this situation:

§2.1 Motivating Example C

Suppose you make $\$20$ per hour (and are paid time-and-a-half for each hour over 40 you work each week). Let's analyze your weekly pay (before taxes and other deductions) in terms of the number of hours you work and create a function called "pay" modeling this:

§2.1 Motivating Example D

Compute the slope of an angle, in terms of the measure (in radians, of course) of the angle. The function that models this is called ________ .



angle (radians)	slope
0	
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	
$\frac{3\pi}{4}$	-1
π	0
2π	0
x	
$\vdots$	$\vdots$

$0 \xrightarrow{\tan}$	$\tan 0$
$\frac{\pi}{6} \xrightarrow{\tan}$	$\tan \frac{\pi}{6}$
$\frac{\pi}{4} \xrightarrow{\tan}$	$\tan \frac{\pi}{4}$
$\frac{\pi}{3} \xrightarrow{\tan}$	$\tan \frac{\pi}{3}$
$\frac{\pi}{2} \xrightarrow{\tan}$	$\tan \frac{\pi}{2}$
$\frac{3\pi}{4} \xrightarrow{\tan} -1$	$\tan \frac{3\pi}{4} = -1$
$\pi \xrightarrow{\tan} 0$	$\tan \pi = 0$
$2\pi \xrightarrow{\tan} 0$	$\tan 2\pi = 0$
$x \xrightarrow{\tan}$	$\vdots$
$\vdots \quad \vdots$	

§2.1 Motivating Example E

Given a human being, determine the year in which they were born; call this process "y":

person	birth year
George Washington	1732
Denzel Washington	1954
Taylor Swift	1989
Babe Ruth	1895
Caligula	12
Woodbridge Ferris	1853
$\vdots$	$\vdots$

G.Washington $\xrightarrow{y}$ 1732	y(G.Washington) $= 1732$
D. Washington $\xrightarrow{y}$ 1954	y(D. Washington) $= 1954$
Taylor Swift $\xrightarrow{y}$ 1989	y(Taylor Swift) $= 1989$
Babe Ruth $\xrightarrow{y}$ 1895	y(Babe Ruth) $= 1895$
Caligula $\xrightarrow{y}$ 12	y(Caligula) $= 12$
W. Ferris $\xrightarrow{y}$ 1853	y(W. Ferris) $= 1853$
$\vdots \quad \vdots$	$\vdots$

Examples A-E above are examples of *functions*. A function is as a *procedure* or a *rule of assignment* that produces outputs from inputs (i.e. assigns outputs to inputs):

- This procedure might be simple, and it might be very complicated.
- The procedure might follow a mathematical rule that is easy to write down (like Motivating Example A);
- the procedure might be impossible to describe without inventing a new name for it (like Motivating Example D);
- the procedure might require different cases, depending on what the input is (like Motivating Examples B or C);
- the procedure might take rather complicated mathematical notation to write (we haven't really seen this in the examples); and
- the procedure might not be "mathematical" at all (like Motivating Example E).

What makes each of these procedures a function is this important principle:

IMPORTANT: for a procedure to be a function, it must be the case that if you perform the procedure twice with the same input each time, you must get the same output each time. In other words,

each input of a function leads to at most one output.

More precisely:

Definition 2.1 *A* **function** *f is a procedure which generates outputs from inputs, in such a way that each input leads to* ***one and only one*** *output.*

The output produced from input x is denoted $f(x)$.

Functions are named by lots of different things:

- most commonly, by a single lowercase letter (f, g, k, etc.);
- capital letters, Greek or Hebrew letters (F, A, α, ψ, $\aleph$, etc.);
- words or phrases ($\sin$, $\cos$, $\sec$, $\log$, $\ln$, etc.);
- symbols ($\sqrt{}$, $\sqrt[3]{}$, $\sqrt[n]{}$, etc.); or
- using "grouping-type" symbols that enclose the input, like $|\ |$ (absolute value) or $\lfloor\ \rfloor$.

Don't misinterpret the parentheses in $f(x)$!

WARNING: All your life you have been told that parenthesis means multiplication, i.e.

$$3(2) = 6 \quad \text{or} \quad a(b + c) = ab + ac.$$

If f is a function, the parenthesis in "$f(x)$" do not mean multiplication. In particular, $f(x)$ does not mean f times x, and $f(a + b)$ is not the same thing as $f(a) + f(b)$ (in general).

$f(x)$ means, literally, this:

"the output of function f when x is the input".

and is better understood through the diagram

$$x \xrightarrow{f} f(x).$$

By the way, this diagram also shows the difference between "f" and "$f(x)$":

- $f(x)$ is the **output** of function f when the input is x; but
- f is the **function** itself. f is neither the input nor the output–it is the name of the **procedure** that produces outputs from inputs.

A useful analogy: at the grocery store, x is an item you buy, $f(x)$ is the price of that item, and f is the cash register (which scans your item and tells you how much you have to pay).

Sometimes the parenthesis in the $f(x)$ is omitted and we just write fx, especially if the function is named after a word or phrase, rather than a single letter.

We usually write $\sin x$ instead of $\sin(x)$, for instance.

But $\sin x$ does NOT mean something called $\sin$ times x.

$\sin x$ refers to the output of the function $\sin$ (which computes y-coordinates from angles), when the input is x.

2.2 The rule of a function

Motivating examples A-E discussed earlier were described in words. Here is another example of a function described in words:

§2.2 EXAMPLE 1

Let g be a function which takes its input, multiplies the input by two less than the input, then takes the cosine of that product, then multiplies the square of that by 2, and finally subtracts 4 times the input cubed.

Questions:

- Was this description of g easy to read?
- Was this description easy to interpret?
- Do you think you could easily perform mathematical operations on this g (combining it with other functions, etc.)?
- Would you enjoy writing descriptions like this of functions you encounter or compute?

We need a way of describing a function that has two attributes:

1. it is **efficient** (meaning it can be written quickly, with a minimal number of symbols and words), and
2. it is **effective** (meaning that the description is useful for performing mathematical operations that solve important applied problems).

Here's how we do this:

We compute the output of the function if the input is generic and arbitrary.

In math, generic and arbitrary things are represented by variables like x.

So we compute the output of the function if the input is a generic x.
(You can use a letter or symbol other than x. The choice of variable often has to do with what the variable means. For instance, if the input to a function is *time*, we probably will use _____ instead of x.)

This gives us some formula with x in it.

It is valid to say that $f(x)$ **equals** this formula (assuming the function is named f), since the formula gives the output associated to x.

(If we used t as the input, our formula would be $f(t) = ...$, not $f(x) = ...$)

This leads to an equation called a *rule* for the function:

Definition 2.2 *Let f be a function. A* **rule** *for f is an equation of the form*

$$f(x) = \textit{something}$$

where the "something" is the output that comes from input x.

As mentioned, we don't have to use x for our generic input; we can use t or y or θ or some other letter or symbol. These would produce rules that look like

$$f(t) = \text{something of } t \qquad f(y) = \text{something of } y \qquad f(\theta) = \text{something of } \theta$$

Main concept: To describe a function, it is sufficient to write down its rule.

Reason: Think of the x as a placeholder which represents where the input goes. Given a rule for f, to find any output you take whatever input you are given and replace all the xs in the rule with the appropriate input.

§2.2 EXAMPLE 2

Write down a rule for each function:

a) Let h be the function which takes the cube root of its input and then adds 4 to produce the output.

b) Let k be the function that takes the product of the sine of its input and the tangent of twice its input (to produce its output).

c) Let F be the function that squares its input, then takes the cosine of that.

d) Let f be the function that adds 3 to twice the input, takes the seventh power of that, subtracts twice the input and then takes a cube root of that to produce the output.

BACK TO EXAMPLE 1

Let g be a function which takes its input, multiplies the input by two less than the input, then takes the cosine of that product, then multiplies the square of that by 2, and finally subtracts 4 times the input cubed.

What is a rule for g?

§2.2 EXAMPLE 3

Rule of f	Arrow diagram	Table of values	Description of f in words
		x | $f(x)$ -1 | 0 | 2 | 4 |	f takes the input, multiplies it by itself, then adds 4 times the input
$f(x) = 4\cos x$		x | $f(x)$ 0 | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | π |	
	$x \xrightarrow{f} x+2$ $-3 \xrightarrow{f}$ $0 \xrightarrow{f}$ $3 \xrightarrow{f}$ $4 \xrightarrow{f}$ $\vdots$	x | $f(x)$ 0 | 2 | 4 | 10 |	

Why rules of functions are effective

§2.2 EXAMPLE 4

Let $f(x) = x^2 - 3x$. Compute and simplify each quantity:

1. $f(3)$

2. $f(-2)$

3. $f(4) + f(1)$

4. $f(t)$

5. $f(\heartsuit)$

6. $f(\text{peanut})$

7. $f(x - 1)$

8. $f(3x)$

9. $3f(x)$

10. $\dfrac{f(x+h) - f(x)}{h}$

Applying a function to a set

Definition 2.3 *Suppose f is a function and $\{x_1, x_2, x_3, ...\}$ is some set/list of inputs. By*

$$f(\{x_1, x_2, x_3, ...\}),$$

we mean the set/list of outputs

$$\{f(x_1), f(x_2), f(x_3), ...\}.$$

In other words, to apply a function f to a set, apply it to each member of the set.

§2.2 EXAMPLE 5

Let $f(x) = x^2 - 3x$. Compute $f(\{1, 2, 3\})$.

Applying functions to both sides of an equation

In Chapter 1 we talked about equations as being like balanced scales. Since functions only have one output, if you apply a function to two equal inputs you must get equal outputs. In symbols, this means

If $a = b$, then $f(a) = f(b)$ for any function f.

and restated, this is:

> **General principle of manipulating equations:** If you start with any equation and apply the same function to both sides, what you get is still an equation.

§2.2 Example 6

In each part of this example, you are given an equation and a function. Apply the given function to both sides of the equation, simplifying both sides of what you get:

a) $3x + 2 = 9$ $\qquad f(x) = x - 2$

b) $2x - 1 = 5x + 3$ $\qquad f(x) = \frac{1}{2}(x + 1)$

c) $4\cos x = 3x - 7$ $\qquad f(x) = x^2$

Solution:

$$\begin{aligned} 4\cos x &= 3x - 7 \\ (4\cos x)^2 &= (3x - 7)^2 \\ 16\cos^2 x &= (3x - 7)^2 \end{aligned}$$

d) $2x^3 + 5 = 8x^3$ $\qquad f(x) = \sqrt[3]{x}$

2.3 Domain, codomain and range

We can associate to each function three sets, which describe the kinds of inputs and outputs that the function has:

Definition 2.4
The set of inputs to a function f is called the **domain** *of f. This set is denoted $Dom(f)$.*

Any set which contains all the outputs of f is called a **codomain** *of f.*

The set of actual outputs of f is called the **range** *of f. This set is denoted $Range(f)$.*

If the domain of a function f is a subset of some set A and if B is a codomain of f, then we write

$$f : A \to B$$

and say "f is a function from A to B".

§2.3 EXAMPLE 1

In Motivating Example A (area of a square), we described the function $A(x) = x^2$.

$$Dom(A) =$$

$$Range(A) =$$

$$\text{codomain of } A =$$

so it is valid to write

§2.3 EXAMPLE 2

In Motivating Example B (the pizza example), we described a function called price.

$$Dom(\text{price}) =$$

$$Range(\text{price}) =$$

$$\text{codomain of price} =$$

so it is valid to write

Most of the time, the domains and codomains of functions we study in MATH 120 (and in calculus) are subsets of $\mathbb{R}$, the set of real numbers. In other words, we study functions f for which we can write

So if you see "Let $f : \mathbb{R} \to \mathbb{R}$" in some statement or exercise, that just means that f is a function which has as its inputs some set of real numbers, and produces real numbers as outputs.

Interval notation

Notice that the range of the function A in Example 1 was described with an inequality. It is convenient to have some shorthand notation for sets described by inequalities, so that we can write the domains and ranges of functions more quickly.

§2.3 EXAMPLE 3

a) Sketch a picture of the set of real numbers x satisfying $1 < x \leq 5$ (this means $1 < x$ and $x \leq 5$).

b) Sketch a picture of the set of real numbers x satisfying $x \geq -4$.

We adapt the ideas of these examples to describe certain sets with two numbers, where the set goes from the first number to the second.

We use [and] are for weak inequalities $\leq$ and $\geq$ where the endpoint(s) is/are included in the set; we use (and) for strict inequalities $<$ and $>$ where the endpoint(s) isn't/aren't in the set.

We use ∞ (**infinity**) to represent the right edge of a set with no biggest number, and $-\infty$ (**negative infinity**) for the left edge of a set with no smallest number.

Definition 2.5 *Let a and b be numbers.*

This notation ↓	*describes the set of real numbers x satisfying this inequality:*
$[a, b]$	$a \le x \le b$
(a, b)	$a < x < b$
$[a, b)$	$a \le x < b$
$(a, b]$	$a < x \le b$
$[a, \infty)$	$a \le x$
(a, ∞)	$a < x$
$(-\infty, a]$	$x \le a$
$(-\infty, a)$	$x < a$

Any set of any of these eight types is called an **interval**.

NOTE: ∞ and $-\infty$ never have square brackets on them. That's because these sets don't actually include endpoints called "∞" or "$-\infty$".

One ambiguity

In a vacuum, something like $(3, 5)$ might mean one of two different things: it is either the ordered pair $(x.y) = (3, 5)$, or it is the interval of real numbers running from 3 to 5 but including neither endpoint.

Whether $(3, 5)$ is an ordered pair or an interval depends on the context in which it is written.

Equality of functions

Definition 2.6 *Two functions f and g are called* **equal** *if the functions have the same domain, and if they produce the same output for every input in their common domain. In this situation we write $f = g$.*

The difference between "$f = g$" and "$f(x) = g(x)$"

- Writing $f(x) = g(x)$ means the outputs of f and g are the same *for one particular input x* (which you might be trying to solve for).
- Writing $f = g$ means the outputs of f and g are the same *for every input x*.

2.4 Piecewise-defined functions

A common class of functions $f : \mathbb{R} \to \mathbb{R}$ can be described by a procedure like this:

- Take your input x.
- If x is one type of input, do one thing to produce $f(x)$.
- If x is a second type of input, do something else to produce $f(x)$.

A function f that works like this is called a **piecewise-defined function**.

§2.4 EXAMPLE 1

Let $f : \mathbb{R} \to \mathbb{R}$ be the absolute value function $f(x) = |x|$.

a) Give a description of this function f in words.

b) Write a rule for the description given in part (a).

Remark: We don't want to have to write this rule out over and over. This is why we invent the symbol $|x|$ for absolute value.

c) Compute $f(7)$ and $f\left(-\frac{2}{3}\right)$.

§2.4 EXAMPLE 2

Let $g(x) = \begin{cases} -2x+1 & x \leq 3 \\ x-2 & 3 < x < 5 \\ 7 & x \geq 5 \end{cases}$. Compute each quantity:

a) $g(-2)$

b) $g(5)$

c) $g(8)$

Who cares about piecewise-defined functions? Everyone should. Here's why:

§2.4 EXAMPLE 3

Here is a piecewise-defined function that determines the amount $T(x)$ of federal income tax (not counting deductions) owed by an individual who has income x in the year 2023:

$$T(x) = \begin{cases} .1x & x < 11000 \\ .12x - 220 & 11000 \leq x < 44725 \\ .22x - 4692.50 & 44725 \leq x < 95375 \\ .24x - 6600 & 95375 \leq x < 182100 \\ .32x - 21168 & 182100 \leq x < 231250 \\ .35x - 149512 & 231250 \leq x < 578125 \\ .37x - 161074 & x \geq 578125 \end{cases}$$

a) If you make $\$35000$ in 2023, how much do you owe in tax?

b) How much tax would a citizen owe on lottery winnings of $\$250000$?

2.5 Multifunctions

RECALL: MOTIVATING EXAMPLE B FROM §2.1

(This was the function "price" where price(x) is the price of x frozen pizzas purchased at $\$5$ per pizza, where every other pizza was free.)

x	price(x)
0	0
1	5
2	5
3	10
4	10
5	15
$\vdots$	$\vdots$

$$0 \xrightarrow{\text{price}} 0$$
$$1 \xrightarrow{\text{price}} 5$$
$$2 \xrightarrow{\text{price}} 5$$
$$3 \xrightarrow{\text{price}} 10$$
$$\vdots \qquad \vdots$$

Question: Can you go backwards? More precisely, can you tell how many pizzas you bought, based on the amount you spent?

A procedure that produces a list of (maybe more than one) possible output(s) from each single input is called a *multifunction*:

Definition 2.7 *A* **multifunction** f *is a procedure which takes each input* x *(in the domain) and produces outputs (in the codomain) from that input, where* ***each input might lead to more than one output****.*

If f *is a multifunction and* x *is an input with outputs* $y_1, y_2, y_3, ...$ *then we list the associated outputs as a set by writing*

$$f(x) = \{y_1, y_2, y_3, ...\};$$

Equality of multifunctions is the same as equality of functions:

Definition 2.8 *Let f and g be multifunctions. We say f and g are* **equal** *if f and g have the same domain, and if $f(x) = g(x)$ for every input x in their common domain.*

§2.5 EXAMPLE 1

Consider the "plus-minus" multifunction

$$f(x) = \pm x.$$

Compute each quantity:

1. $f(6)$

2. $f(0)$

§2.5 EXAMPLE 2

Let g be the multifunction $g(x) = \pm\sqrt{x}$. Compute each quantity:

1. $g(16)$

2. $g(144)$

3. $g(0)$

4. $g(-9)$

2.6 Inverting a function

In the previous section, we were motivated by trying to *undo* or *reverse* the price function of Motivating Example B (the pizza example).

Undoing/reversing a function is called *inverting* the function. Based on our motivating example, we see that in general, to invert a function, we have to use a multifunction.

Definition 2.9 *Let $f : \mathbb{R} \to \mathbb{R}$. Then the* **inverse** *of f, denoted f^{-1} and pronounced "f inverse", is the multifunction with all these properties:*

- *Outputs of f are inputs of f^{-1} and vice versa (meaning that the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f).*
- *$f^{-1}(y)$ is defined to be the set of all the inputs of f that produces output y.*

WARNING: the -1 in this notation is **not** an exponent.

$$f^{-1}(x) \text{ does not mean } (f(x))^{-1} = \frac{1}{f(x)}.$$

If you have to write $\frac{1}{f(x)}$ as an exponent, write it as $(f(x))^{-1}$.

It is useful to visualize inverses with the following arrow diagram:

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

Another way of looking at this: $f^{-1}(y)$ is a set/list of all the things that would work as "?" in this diagram:

$$? \xrightarrow{f} y$$

Contrast this with $f(x)$, which is at most one item which works as the "?" in this diagram:

$$x \xrightarrow{f} ?$$

§2.6 Example 1

Let price be the function described in Motivating Example B. If you don't remember, price is the function with this table of values:

x	0	1	2	3	4	5	$\cdots$
price(x)	0	5	5	10	10	15	$\cdots$

Compute each quantity:

a) price 3

b) $\text{price}^{-1}(3)$

P.S. The expression in (b) is pronounced "price inverse of 3".

c) $\text{price}^{-1}(10)$

d) $\text{price}^{-1}\,0$

e) $\text{price}^{-1}(\text{price}(1))$

f) $\text{price}(\text{price}^{-1}\,5)$

Parts (e) and (f) of Example 1 illustrate the major concept concept behind inverses: the inverse f^{-1} is a multifunction which "undoes" whatever f does (and vice versa):

Theorem 2.10 *If f is a function with inverse f^{-1}, then*

$$f^{-1}(f(x)) \text{ includes } x \quad \text{and} \quad f(f^{-1}(y)) = y.$$

§2.6 EXAMPLE 2

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. In parts (a)-(e), write an arrow diagram that explains what is being asked for, and then compute the indicated quantity:

a) $f(4)$

b) $f^{-1}(4)$

c) $f^{-1}(-4)$

d) $f^{-1}(0)$

e) $f^{-1}(y)$

f) Is f^{-1} a function $\mathbb{R} \to \mathbb{R}$?

Remark: The work in part (e) can be thought of as finding a rule for f^{-1}.

§2.6 EXAMPLE 3

Let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = 5 - 3x$. In parts (a)-(d), write an arrow diagram that explains what is being asked for, and then compute the indicated quantity:

a) $g(2)$

Solution: $\boxed{2 \xrightarrow{g} ?}$; $\quad g(2) = 5 - 3(2) = \boxed{-1}$.

b) $g^{-1}(2)$

Solution: $\boxed{? \xrightarrow{g} 2}$;

c) $g^{-1}(y)$

d) $g^{-1}(x)$

e) Is g^{-1} a function $\mathbb{R} \to \mathbb{R}$?

§2.6 EXAMPLE 4

Suppose f and g are functions, each with domain $\{-3, -2, -1, 0, 1, 2, 3\}$, described completely by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	-2	0	0	-3	2
$g(x)$	1	3	4	2	0	-3	-2

Compute each quantity:

a) $f(2)$

b) $f^{-1}(2)$

c) $g^{-1}(-3)$

d) $g^{-1}(-1)$

One-to-one functions

In general, the inverse of a function is a multifunction. But sometimes, the inverse of a function is itself a function (like Example 3 of this section)! This happens if...

Definition 2.11 *A function f is called* **one-to-one**, *a.k.a.* **injective**, *a.k.a.* $1-1$ *if* different inputs to f must always lead to different outputs. *Algebraically, this means*

$$x_1 \neq x_2 \textit{ implies } f(x_1) \neq f(x_2).$$

§2.2 EXAMPLE 5

Determine whether or not each given function is one-to-one:

1. The price function coming from Motivating Example B (pizzas $\$5$ each, every other pizza free)

2. A function modeling the price you pay for bags of chips, if bags of chips are $\$3.29$ each

3. The function in Motivating Example E (which gives the birth year as a function of the person in question)

4. The function f described by the table in Example 4 (one page ago)

5. The function g described by the table in Example 4

6. The function $h(x) = |x|$

7. The function $k(x) = \sin x$

8. The function $F(x) = x^3$

Here's why we care whether or not a function is one-to-one:

Theorem 2.12 *The inverse of a one-to-one function is itself a function.*

Using inverses to solve equations

A major reason we care about inverses is that an understanding of inverses is important in solving equations. Here are some basic examples:

§2.2 EXAMPLE 6

a) Consider the equation $x - 13 = 17$.

The reason we add 13 to both sides has to do with functions and inverses.

b) Consider the equation $8x = 48$. To solve this, we divide both sides by 8:

$$\begin{aligned} 8x &= 48 \\ \frac{8x}{8} &= \frac{48}{8} \\ x &= 6 \end{aligned}$$

Why do we divide both sides by 8? Well, if we let $f(x) = 8x$, we can think of the equation $8x = 48$ as

$$f(x) = 48 \qquad \text{i.e.} \qquad x \xrightarrow{f} 48 \text{ or } x \xrightarrow{\times 8} 48 \,.$$

The inverse of the function $f(x) = 8x$ is $f^{-1}(x) = \boxed{\quad\quad}$, so we get the solution as $f^{-1}(48) = \dfrac{48}{8} = \boxed{6}$. As an arrow diagram, this is

$$6 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} 48 \qquad \text{or} \qquad 6 \underset{\div 8}{\overset{\times 8}{\rightleftarrows}} 48 \,.$$

c) Consider the equation $x^2 = 81$.

More on these ideas later, both in Section 2.8 and when we study equations in more detail.

2.7 Order of operations with functions

§2.7 EXAMPLE 1

Let $f(x) = x^2 + 3x$. Compute and simplify the following expressions:

a) $f(2-1)$

b) $f(2) - f(1)$

c) $f(2) - 1$

§2.7 EXAMPLE 2

Let $h(x) = 2\sqrt{x} + 1$. Compute and simplify the following expressions:

a) $h(4 \cdot 9)$

b) $4h(9)$

c) $9h(4)$

d) $h(4)h(9)$

e) $h(4h(9))$

§2.7 EXAMPLE 3

Let $g(t) = 3t^2$. Compute and simplify the following expressions:

1. $g(t+1)$

2. $g(t) + 1$

3. $g(t) + g(1)$

One of the most important things to master in MATH 130 is how to interpret expressions with functions that have "invisible parentheses" in them. In particular, this means learning order of operations with functions.

A new general rule: when reading an expression, whenever you see a function, immediately after the function there is an invisible (.

Everything after the invisible (is grouped until you get to one of three things: addition, subtraction or the name of another function.

§2.3 EXAMPLE 2

Let job $x = 4x - 1$. Compute and simplify each expression:

1. job(x)
2. job x
3. job$(2 \cdot 3)$
4. job $2 \cdot 3$
5. 2 job 3
6. job $(1 + 3)$
7. job $1 + 3$
8. job 1 + job 3
9. job $2 \cdot 4 - 1$
10. job (3 job 1)
11. job 3 job 1

Order of operations with functions and exponents

Definition 2.13 *Let $f:\mathbb{R}\to\mathbb{R}$ be a function. For any $n\neq -1$, the notation $f^n(x)$ (a.k.a. $f^n x$) means $(f(x))^n$.*

REMINDER: $f^{-1}(x)$ is the inverse of f, not $f(x)$ to the -1 power.

One instance where this notation is used is in the following trig identity:

Theorem 2.14 (Pythagorean Identity) *For any real number x,*

$$\cos^2 x + \sin^2 x = 1.$$

§2.7 EXAMPLE 3

Suppose car $x = 3x - 1$. Compute each quantity:

a) car 2^2

b) car^2 2

c) car^3 1

d) car $3^2 \cdot 2$

Solution: car $3^2 \cdot 2 =$ car $(3^2 \cdot 2) =$ car $(9 \cdot 2) =$ car $18 = 3(18) - 1 = \boxed{53}$.

e) 2 car 3^2

Solution: 2 car $3^2 = 2$ car $(3^2) = 2$ car $9 = 2\,[3(9) - 1] = 2\,[26] = \boxed{52}$.

f) 3 car^3 $(2-1)^2 + 5$

g) car^{-1} 1

Substitutions in functional expressions

§2.7 EXAMPLE 4

a) If you know $x = 7$, how does the expression buzz $x + 3$ simplify?
(Put another way, how do you substitute "$x = 7$" into "buzz $x + 3$"?)

b) If you know buzz $x = 7$, how does the expression buzz $x + 3$ simplify?

c) If you know $x = 7$, how does the expression buzz $(x + 1)$ simplify?

d) If you know buzz $x = 7$, how does the expression buzz $(x + 1)$ simplify?

e) If you know $x = 7$, how does the expression buzz ${}^2x$ simplify?

f) If you know buzz $x = 3$, how does the expression buzz ${}^2x$ simplify?

§2.7 EXAMPLE 5

a) Substitute $x = 2$ into moon ${}^2x + 3$ sun ${}^2x = 17$.
Solution: $\boxed{\text{moon }{}^2 2 + 3 \text{ sun }{}^2 2 = 17}$.

b) Substitute moon $x = 2$ into moon ${}^2x + 3$ sun ${}^2x = 17$. Then solve for sun x.

c) Substitute moon ${}^2x = 2$ into moon ${}^2x + 3$ sun${}^2x = 17$. Then solve for sun x.

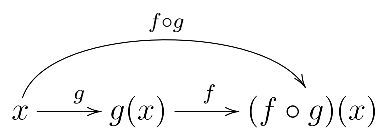
2.8 Composition of functions

Definition 2.15 *Let f and g be functions. The* **composition** *of f and g, denoted $f \circ g$, is the function defined by the rule*

$$(f \circ g)(x) = f(g(x)).$$

Composing functions enables us to build more complicated functions out of easier ones. In particular, composition corresponds to doing functions in sequence, **from right to left**: to evaluate $f \circ g$, you do the procedure g first, then do the procedure f.

An arrow diagram to explain:



§2.8 Example 1

Let $f(x) = x^2$, $g(x) = 3x - 4$ and $h(x) = 2x^2 + 1$. Compute and simplify the rule for each of these functions:

a) $f \circ g$

c) $g \circ g$

b) $g \circ f$

d) $g \circ h \circ f$

WARNING: composition of functions is not commutative (the order matters). In general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

§2.8 EXAMPLE 2

Continuing with the functions $f(x) = x^2$, $g(x) = 3x - 4$ and $h(x) = 2x^2 + 1$ described in Example 1, first draw an arrow diagram indicating what is asked for; then compute each quantity:

a) $(f \circ g)(3)$

b) $(g \circ h)(-2)$

Solution: $\boxed{-2 \xrightarrow{h} \quad \xrightarrow{g} ?}$;

$(g \circ h)(-2) = g(h(-2)) = g\left(2(-2)^2 + 1\right) = g\left(2(4) + 1\right) = g(9) = 3(9) - 4 = \boxed{23}$.

c) $(g \circ g \circ f)(3)$

§2.8 EXAMPLE 3

Suppose f and g are one-to-one functions, each with domain $\{-3, -2, -1, 0, 1, 2, 3\}$, described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	-2	0	3	-3	-1
$g(x)$	1	3	-1	2	0	-3	-2

Compute each quantity:

a) $(f \circ g)(3)$

b) $(f^{-1} \circ g)(-2)$

c) $(g^{-1} \circ f^{-1})(-1)$

d) $(f \circ g \circ f^{-1})(0)$

Composing multifunctions

We compose multifunctions in the same way that we compose functions:

§2.8 EXAMPLE 5

Suppose f and g are multifunctions with $g(1) = \{4, 5, 6\}$, $f(4) = \{2, 3\}$, $f(5) = \{3, 4\}$ and $f(6) = 10$. Compute $(f \circ g)(1)$.

Inverting a composition

Let gf be the "function" representing the procedure of putting your socks on.

Then g^{-1} is the "function" ______________________________ .

Also, let f be the "function" of putting your shoes on.

Then f^{-1} is the "function" ______________________________ .

When you get ready to go outside, usually you perform the function that is written

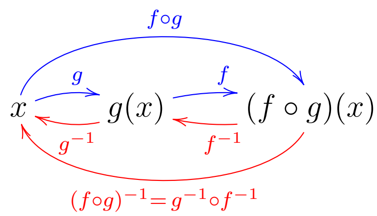
symbolically as ____________ .

Question: When you come back inside, what do you usually do? (There are two ways of writing the answer.)

Theorem 2.16 *Let f and g be functions. Then*

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

An arrow diagram to explain this:



Elementary functions

Every chemical compound is made up of molecules, where each molecule is itself built from atomic *elements*. As an example, water is built from 2 hydrogen atoms and an oxygen atom (water is H_2O).

Suppose you have a (reasonable) mathematical expression with a single variable that only appears once, like

$$\cos \sqrt[5]{3x^2+5} \qquad \sin^2 4x - 8 \qquad \ln(5\arctan(2x^2-3)) \qquad \text{etc.}$$

Every expression like this is the rule for a function, and that function is a composition of functions from a list of functions called **elementary functions**, in the same way that every molecule is made up of atoms from the elements in the periodic table.

On the next page, I give a list of names and rules of the elementary functions, together with how they are represented in arrow diagrams:

List of elementary functions

CLASS	EXAMPLES	LABELLED ARROWS
Absolute value function	$f(x) = \|x\|$	$\xrightarrow{\|\cdot\|}$ or $\xrightarrow{\text{abs}}$
Addition / subtraction by a constant	$f(x) = x + 3$ $f(x) = x - 1.752$ $f(x) = x + \frac{\sqrt{7}}{4}$ etc.	$\xrightarrow{+3}$ $\xrightarrow{-1.752}$ $\xrightarrow{+\sqrt{7}/4}$
Multiplication / division by a constant	$f(x) = 2x$ $f(x) = \frac{2}{5}x$ $f(x) = \pi x$ $f(x) = \frac{x}{\sqrt{3}}$ etc.	$\xrightarrow{\cdot 2}$ or $\xrightarrow{\times 2}$ $\xrightarrow{\cdot 2/5}$ or $\xrightarrow{\times 2/5}$ $\xrightarrow{\cdot \pi}$ or $\xrightarrow{\times \pi}$ $\xrightarrow{\div\sqrt{3}}$
Whole number power functions	$f(x) = x^2$ $f(x) = x^3$ $f(x) = x^n$ etc.	$\xrightarrow{\wedge 2}$ $\xrightarrow{\wedge 3}$ $\xrightarrow{\wedge n}$
Reciprocal function	$f(x) = x^{-1} = \dfrac{1}{x}$	$\xrightarrow{\frac{1}{\cdot}}$ or $\xrightarrow{1/\cdot}$ or $\xrightarrow{\wedge -1}$
Root functions	$f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[n]{x}$ etc.	$\xrightarrow{\sqrt{\ }}$ $\xrightarrow{\sqrt[3]{\ }}$ $\xrightarrow{\sqrt[n]{\ }}$
Elementary trig functions	$f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$	$\xrightarrow{\sin}$ $\xrightarrow{\cos}$ $\xrightarrow{\tan}$
Inverse trig functions	$f(x) = \arcsin x$ $f(x) = \arccos x$ $f(x) = \arctan x$	$\xrightarrow{\arcsin}$ $\xrightarrow{\arccos}$ $\xrightarrow{\arctan}$
Exponential function	$f(x) = e^x$	$\xrightarrow{\exp}$
Natural logarithm function	$f(x) = \ln x$	$\xrightarrow{\ln}$

Functions you may have heard of that are <u>not</u> elementary:

$$5 - x \qquad x^{-2} \qquad x^{3/4} \qquad \csc x \qquad \sec x \qquad \cot x$$

$$\cosh x \qquad 2^x \qquad \log_6 x \qquad \log x$$

Diagramming functions

In this section, we are going to examine the connection between composition of functions and order of operations.

§2.8 EXAMPLE 5

In each part of this problem, you are given a function which is a composition of one or more elementary functions. **Diagram the function**. This means you are to write an arrow diagram which indicates which elementary functions are composed (and in what order) to produce the output of the given function.

a) $f(x) = (x+1)^3$

b) $f(x) = x^3 + 1$

c) $f(x) = x^{-3}$

d) $f(x) = 5x^{2/3}$

e) $f(x) = 1 + \dfrac{7}{|x|}$

f) $f(x) = 4 - \sqrt{x+1}$

g) $f(x) = \sec x$

§2.8 EXAMPLE 6

Diagram these functions:

a) $f(x) = \sin^3 2x$

b) $f(x) = 2\sin^3 x$

c) $f(x) = 2\sin x^3$

d) $f(x) = \sin 2x^3$

Solution: $\boxed{x \xrightarrow{\wedge 3}\xrightarrow{\times 2}\xrightarrow{\sin} f(x)}$

e) $f(x) = \sin(2x)^3$

Solution: $\boxed{x \xrightarrow{\times 2}\xrightarrow{\wedge 3}\xrightarrow{\sin} f(x)}$

§2.8 EXAMPLE 7

In each part, you are given an arrow diagram with elementary functions that indicates how $f(x)$ is produced from x. Write the rule for $f(x)$, simplifying your answer as much as possible. (This is called **reverse diagramming** a function.)

a) $x \xrightarrow{\exp} \xrightarrow{-3} \xrightarrow{\wedge 4} \xrightarrow{1/\cdot} f(x)$

b) $x \xrightarrow{-\pi} \xrightarrow{\sin} \xrightarrow{+4} f(x)$

c) $x \xrightarrow{|\cdot|} \xrightarrow{\exp} \xrightarrow{-1} \xrightarrow{\ln} f(x)$

d) $x \xrightarrow{\times 2} \xrightarrow{\wedge 3} \xrightarrow{1/\cdot} \xrightarrow{\sqrt{\ }} \xrightarrow{\div 4} f(x)$

2.9 Arithmetic operations on functions

In the last section, we talked about combining elementary functions to make more complicated functions using the operation of *composition*.

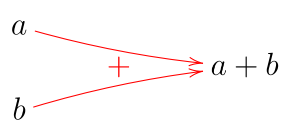
We can also build complicated functions out of elementary ones using other operations that are motivated by basic arithmetic operations we do with numbers.

Adding functions

The process of adding two numbers is a *binary operation*: that is, you add two numbers (i.e. have two inputs) and get one sum (i.e. you have one output). To think of addition as an arrow diagram, consider an example like this:

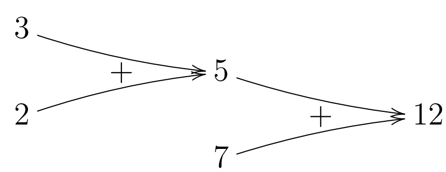
EXAMPLE: Suppose you have 8 red apples and 3 green apples. How many total apples do you have?

More generally, adding two numbers a and b looks like this:



Of course, you can add more than two numbers together, but you have to add them *two at a time*:

$$3 + 2 + 7 = ?$$



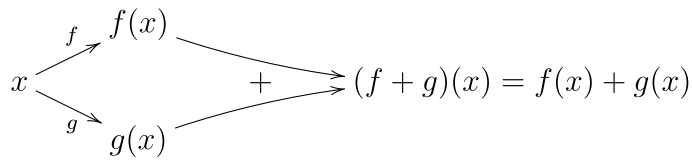
We can also add two functions f and g to get another function called "$f + g$".

Here's how we do this: to define $f + g$, we need to say what its output, given a generic input x. To get the output of $f + g$, we first compute $f(x)$, then compute $g(x)$, and add those together to get the output $(f + g)(x)$. That idea leads to this definition:

Definition 2.17 *Given functions f and g, the* **sum** *of f and g is the function $f + g$ whose rule is*

$$(f + g)(x) = f(x) + g(x).$$

Put another way, $f + g$ is the function with this arrow diagram:

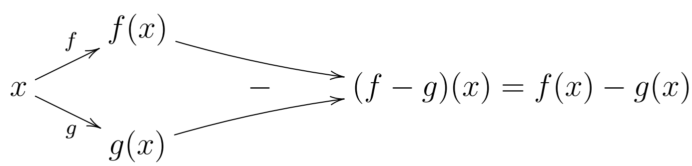


Subtracting functions

Subtracting two functions is similar to adding them: to define function $f - g$ from f and g, we decree that from input x, we first compute $f(x)$ and $g(x)$, then subtract those to get the output of the newly defined $f - g$.

Definition 2.18 *Given functions f and g, the* **difference** *of f and g is the function $f - g$ whose rule is*

$$(f - g)(x) = f(x) - g(x).$$



§2.9 Example 1

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

In each problem, compute the given quantity and in parts (a) and (c), draw an arrow diagram indicating what is being asked for.

a) $(f + g)(-1)$

b) $(f + g^{-1})(0)$

c) $(f - g)(3)$

d) $(f + g \circ f)(2)$

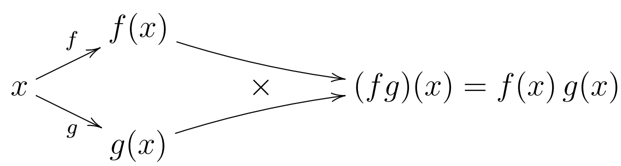
Multiplying and dividing functions

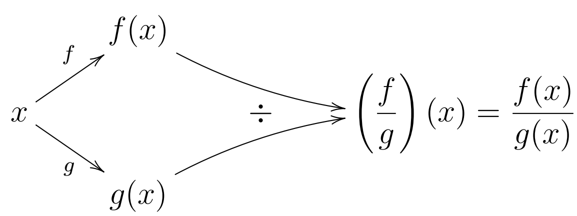
Definition 2.19 *Given functions f and g, the* **product** *of f and g is the function fg whose rule is*

$$(fg)(x) = f(x)g(x).$$

The **quotient** *of f and g is the function $\frac{f}{g}$ whose rule is*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$





§2.9 EXAMPLE 2

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

Compute each quantity:

a) $(fg)(2)$

b) $\left(\frac{f}{g}\right)(-1)$

c) $\left(\frac{g}{f}\right)(-2)$

d) $\left(\frac{fg}{f+g^{-1}}\right)(3-1)$

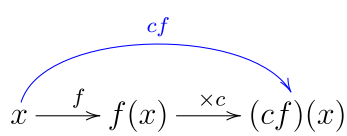
Scalar multiples of functions

There is one other operation we can do on a function. We can take a function and multiply it by a number to produce a new function:

Definition 2.20 *Let f be a function and let c be a number. The* **scalar multiple**, *a.k.a.* **constant multiple** *of f is the function cf whose rule is*

$$(cf)(x) = c\,f(x).$$

Here's the arrow diagram for this, which indicates that cf is a function which first does f, then multiplies the answer by the scalar/constant c:



§2.9 Example 3

Suppose f and g are functions described by the table of values given below:

x	-3	-2	-1	0	1	2	3
$f(x)$	-3	0	-2	4	4	-2	1
$g(x)$	5	-3	2	12	0	-7	-3

Compute each quantity:

a) $(4f)(2)$

b) $(2g)(-2 \cdot -1)$

c) $\left(\frac{g}{3f+g}\right)(-3)$

d) $(4f \circ g)(-1)$

§2.9 EXAMPLE 4

Let $f(x) = 2x - 3$, $g(x) = x^2 + 2$ and $h(x) = 5 - 3x$. Compute the rule for each function:

1. $f - 2g$

2. gh

3. h^2

4. $\dfrac{f}{g - 4h}$

5. $f + f \circ h$

6. $hg \circ f$

Theorem 2.21 (Order of function operations) *When building complicated functions from easier ones, the order of operations is:*

1. *obey any grouping symbols like parentheses or brackets;*
2. *exponents;*
3. *multiplication/division (from left to right);*
4. *composition (from left to right);*
5. *addition/subtraction (from left to right).*

You could shorthand the content of this theorem as "PEMDCAS", which is really

$$\text{P} - \text{E} - \begin{pmatrix} \text{M} \\ \text{D} \end{pmatrix} - \text{C} - \begin{pmatrix} \text{A} \\ \text{S} \end{pmatrix}.$$

More diagramming of functions

§2.9 EXAMPLE 5

Diagram each function:

a) $f(x) = x^4 + \cos x$

b) $f(x) = x^3 \sin \sqrt{x}$

c) $f(x) = \dfrac{8x}{x^{3/4} - 1} + 3x \tan x$

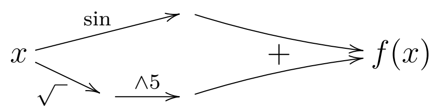
d) $f(x) = (x^5 - 3)^3 (7 \tan x - 3)^4 - |\arctan x|$

e) $f(x) = (x + 3x^2 \cos x)^4 - \log x$

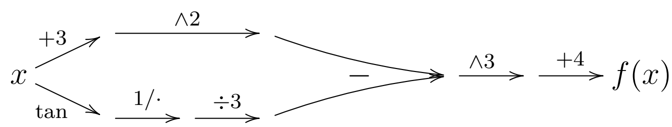
§2.9 EXAMPLE 6

Reverse-diagram each of these pictures (and simplify your answers):

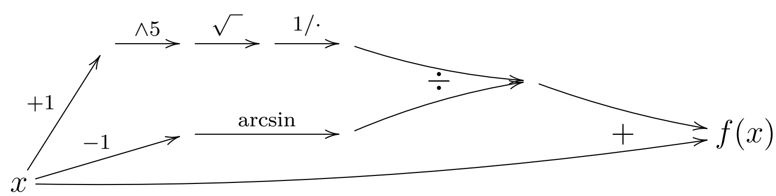
a)



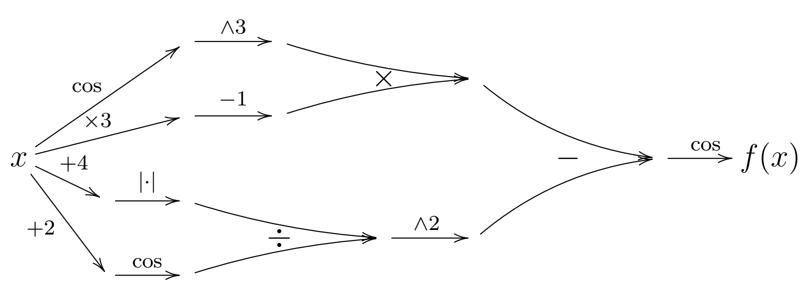
b)



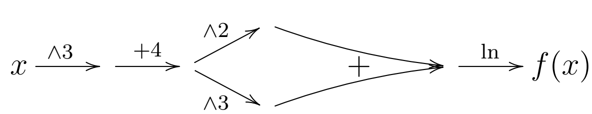
c)



d)



e)

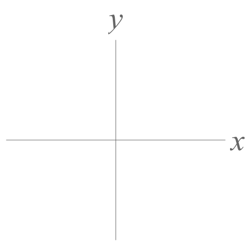


2.10 Graphs of functions

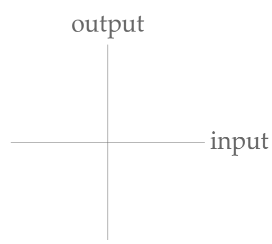
It is often useful to draw a picture representing a function. Such a picture is called a *graph*:

> **Definition 2.22** *The* **graph** *of a function (or multifunction) $f : \mathbb{R} \to \mathbb{R}$ is (a picture of) the set of all points (x, y) such that $y = f(x)$.*

Put another way, this means that we think of the horizontal coordinate of any point on the graph as being an input to the function, and the vertical coordinate is the corresponding output:



should really be thought of as

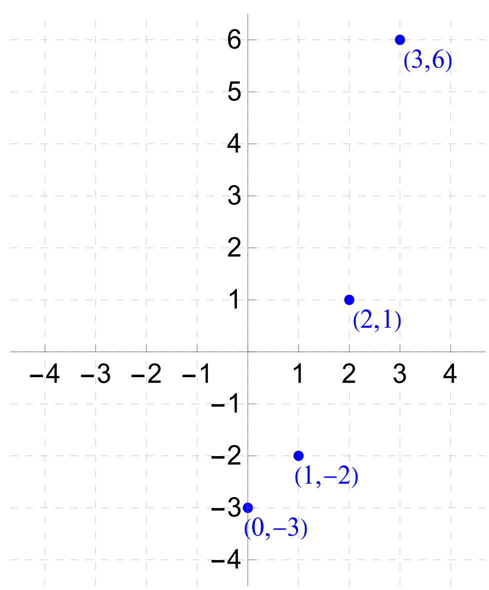


We can make a reasonable guess as to what the graph of a function looks like by plotting some points coming from a table of values:

§2.10 Example 1

Let $f(x) = x^2 - 3$. To produce a picture of the graph of f, let's pick some inputs and find the corresponding outputs:

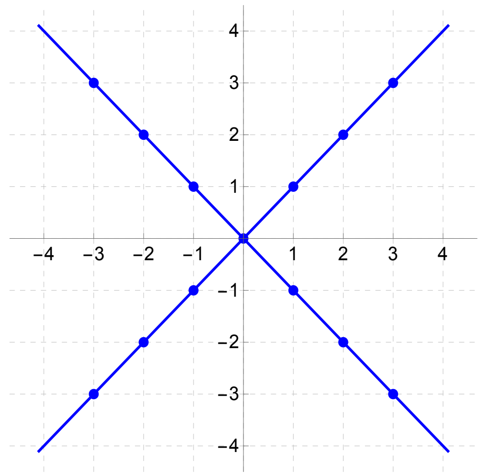
input x	output $f(x)$	ordered pair
-3		
-2		
-1		
0		
1		
2		
3		



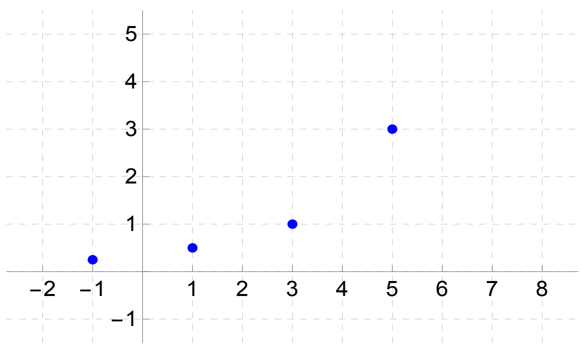
§2.10 EXAMPLE 2

Sketch a graph of the multifunction $f(x) = \pm x$.

input x	output(s) $f(x)$	ordered pair(s)
-3	$\pm(-3) = \{-3, 3\}$	$(-3,-3)$ and $(-3,3)$
-2	$\pm(-2) = \{-2, 2\}$	$(-2,-2)$ and $(-2,2)$
-1	$\pm(-1) = \{-1, 1\}$	$(-1,-1)$ and $(-1,1)$
0	0	$(0,0)$
1	$\pm 1 = \{-1, 1\}$	$(1,-1)$ and $(1,1)$
2	$\pm 2 = \{-2, 2\}$	$(2,-2)$ and $(2,2)$
3	$\pm 3 = \{-3, 3\}$	$(3,-3)$ and $(3,3)$



Potential problem(s) with producing a graph from table of values:



Ways to get around this:

1. **Use technology:** http://www.desmos.com or a graphing calculator (or software like *Mathematica* that you might learn how to use in a future course).

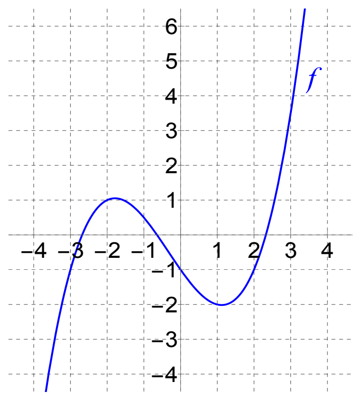
2. **Use stuff we'll learn in Chapter 3:** we'll learn some general theory about classes of functions (closely related to the classes of elementary functions). This will tell us what a lot of graphs basically look like without having to construct tables.

3. **Take calculus:** where you learn some more powerful theory and computational techniques that tell you basically what any graph looks like without having to construct tables.

Strengths and weaknesses of graphs

Graphs are good for:

- allowing you to *estimate* values of the function:

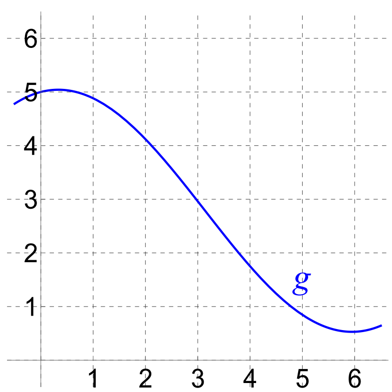


$f(3) \approx$

$f(-2) \approx$

$f^{-1}(4) \approx$

- identifying *relationships* between outputs and inputs, i.e. showing how the output *changes* in response to a change in input:

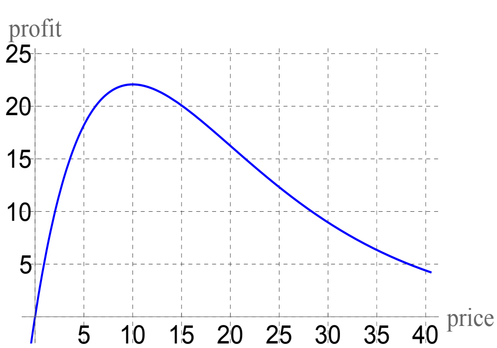


As x increases from 2 to 4,

$g(x)$ ________________

from ____ to ____ .

- identifying *optimal* situations related to the function:



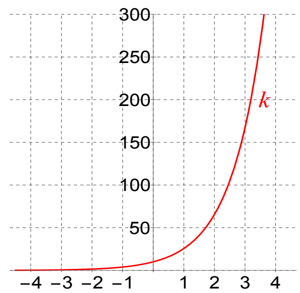
The maximum possible profit is

____ , which occurs when

the product is priced at ____ .

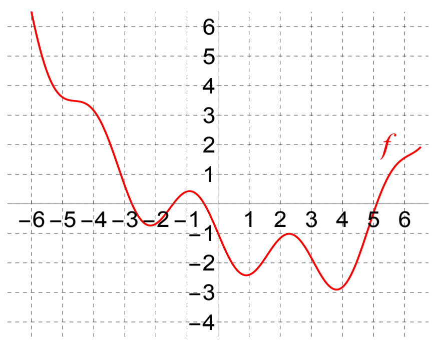
Graphs are not good for:

- determining *exact* values of functions:



$k(3) = ?$

- performing *algebra* on functions:



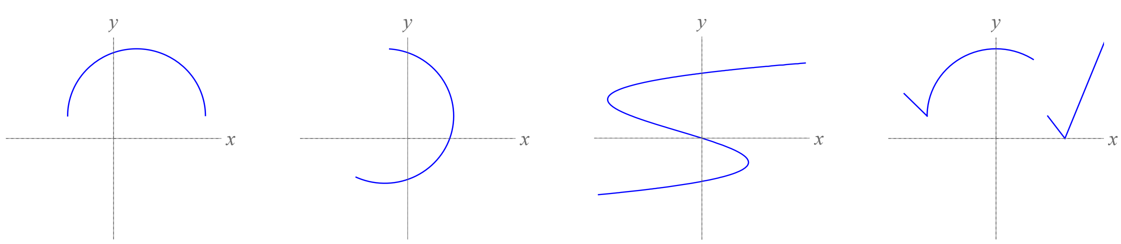
What is the graph of

$f \circ (f^2 + 3f)$?

Vertical and horizontal line tests

§2.10 Example 3

The graphs of several multifunctions are given below. Determine whether each graph is the graph of a function $y = f(x)$:



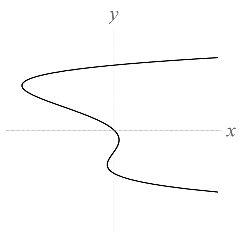
Theorem 2.23 (Vertical Line Test) *A graph in the (x, y)-plane is the graph of a function $y = f(x)$ if and only if the graph intersects every vertical line in at most one point.*

Enrichment: A graph that fails the vertical line test isn't a function $y = f(x)$, BUT it can be the graph of a different kind of function (not $y = f(x)$). More on this in Calculus 3 (MATH 320).

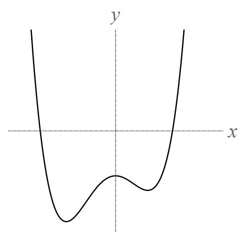
There is also a Horizontal Line Test, which tells us from a graph whether or not a function is one-to-one:

Theorem 2.24 (Horizontal Line Test) *A function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one (a.k.a. injective) if and only if every horizontal line strikes the graph of f in at most one point.*

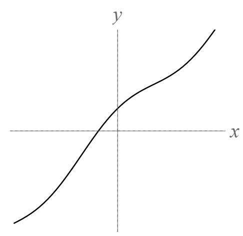
A graph of a multifunction that is **not** a function $y = f(x)$



A graph of a function $y = f(x)$ that is **not** one-to-one



A graph of a one-to-one function $y = f(x)$

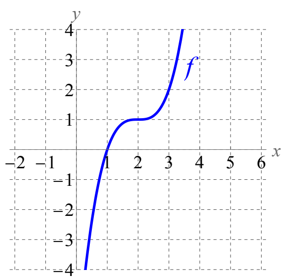


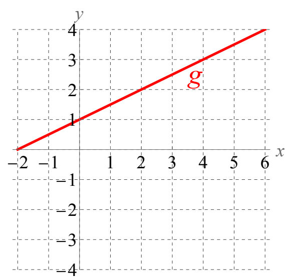
Graphs of piecewise-defined functions

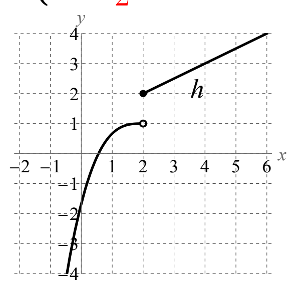
§2.10 EXAMPLE 4

Consider the graphs of these three functions:

$$f(x) = \frac{1}{3}(x-2)^3 + 1 \qquad g(x) = \frac{1}{2}x + 1 \qquad h(x) = \begin{cases} \frac{1}{3}(x-2)^3 + 1 & x < 2 \\ \frac{1}{2}x + 1 & x \geq 2 \end{cases}$$

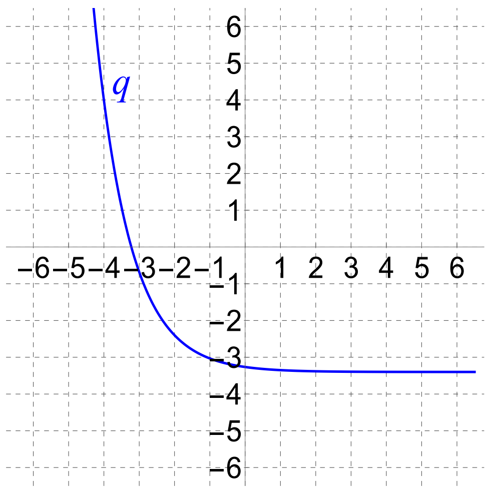


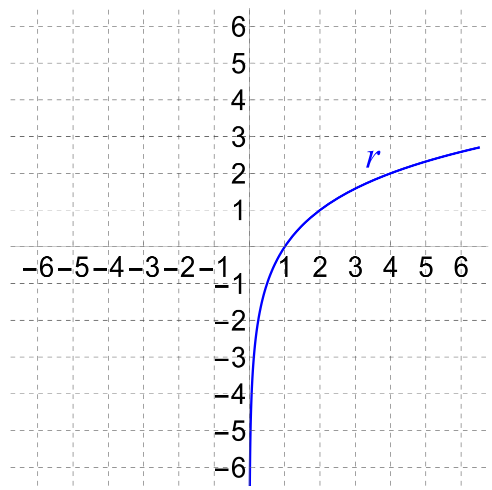




§2.10 EXAMPLE 5

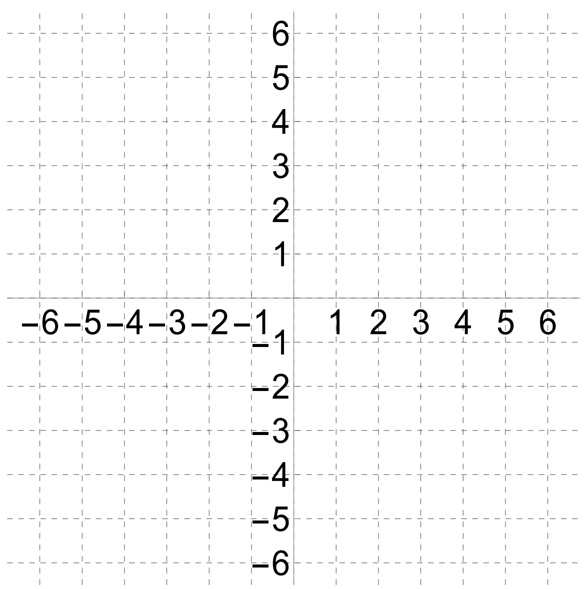
The graphs of functions $q : \mathbb{R} \to \mathbb{R}$ and $r : \mathbb{R} \to \mathbb{R}$ are shown below:





Use these graphs to sketch the graph of

$$f(x) = \begin{cases} q(x) & x < -1 \\ 3 & -1 \leq x \leq 2 \\ 4 & x = 2 \\ r(x) & x > 2 \end{cases} .$$



x- and y-intercepts

Definition 2.25 *Let $f : \mathbb{R} \to \mathbb{R}$.*

- *The **x-intercept(s)** of a function are points on the graph of f that have y-coordinate 0. These are the points where the graph of f intercepts the x-axis.*
- *The **y-intercept** of a function is the point on the graph of f that has x-coordinate 0. These is the point where the graph of f intercepts the y-axis.*

Note: a function cannot have more than one y-intercept (otherwise the VLT would be violated), but it can have any number of x-intercepts.

Note: some functions may not have x- and/or y-intercepts.

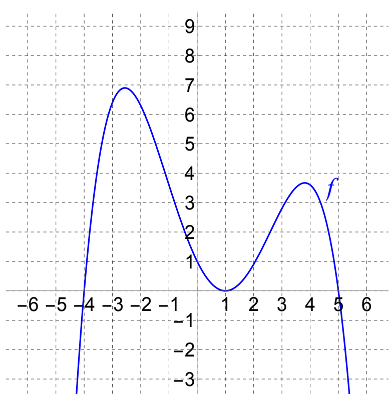
To find x-intercepts, set $f(x) = 0$ (i.e. $y = 0$) and solve for x.

To find the y-intercept, compute $f(0)$ (i.e. set $x = 0$).

Keep in mind: x- and y-intercepts are *points* (ordered pairs), not numbers.

§2.10 Example 6

Use the graph of f given below to estimate all x- and y-intercepts of f.



§2.10 Example 7

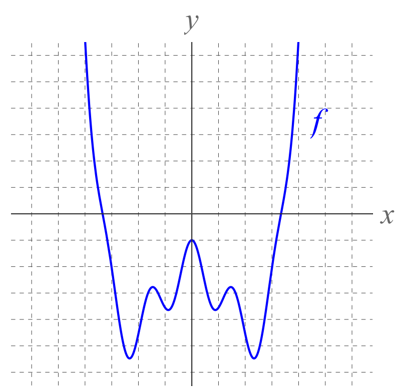
Compute the x- and y-intercepts of the function $g(x) = \dfrac{2}{3}x - \dfrac{8}{3}$.

Symmetry

It is useful to take note of functions whose graphs have *symmetry*, because certain symmetries in a graph can be interpreted in terms of algebra.

§2.10 EXAMPLE 8

Consider the function f that has this graph:



a) Describe the symmetry of this graph.

b) Write down an algebraic formula that corresponds to this symmetry.

Definition 2.26 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called* **even** *if its graph is symmetric across the* x*-axis. Algebraically, this means* $f(-x) = f(x)$*.*

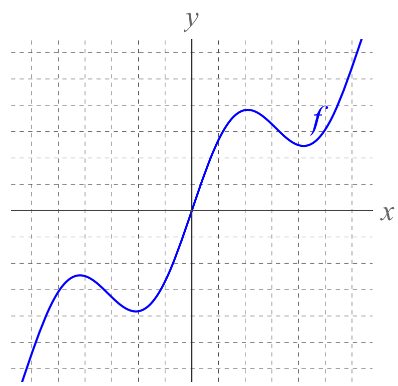
We use the word "even" because any function made up only of even powers of x is even, including things like:

$$f(x) = 3x^2 \qquad g(x) = 5x^8 - 3x^{-2} \qquad h(x) = \sqrt{5}\,x^{100} + 3x^{46} - 2x^2 + 3$$

In trigonometry, we encounter two functions that are even: ____________ and ____________ .

§2.10 EXAMPLE 9

Consider the function f that has this graph:



a) Describe the symmetry of this graph.

b) Write down an algebraic formula that corresponds to this symmetry.

Definition 2.27 *A function $f : \mathbb{R} \to \mathbb{R}$ is called* **odd** *if its graph is unchanged if you rotate the entire plane by 180° about the origin. Algebraically, this means $f(-x) = -f(x)$.*

We use the word "odd" because any function whose terms are odd powers of x will be odd, including things like:

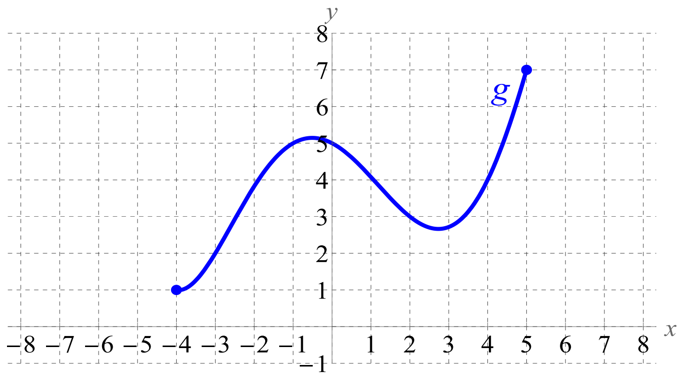
$$f(x) = 3x^4 \qquad g(x) = 7x - 8x^{-3} \qquad h(x) = \frac{2}{3}x^{107} - 4.873x^5 + \frac{5}{\sqrt{7}x^5}$$

In trigonometry, we encounter four functions that are odd: __________ , __________ , __________ and __________ .

2.11 Reading graphs

§2.11 Example 1

The graph of some unknown function g is shown below.



Use the graph of g to answer these questions/estimate these quantities:

a) What is the domain of g?

b) What is the range of g?

c) $g(2)$

d) $g(-1) + 1$

e) $g(2 \cdot 2)$

f) $4g(1) - \frac{1}{2}g(-3)$

g) $g^{-1}(3)$

h) $g^{-1}(2)$

i) Near $x = 2$, would $g(x)$ increase or decrease as x increases?

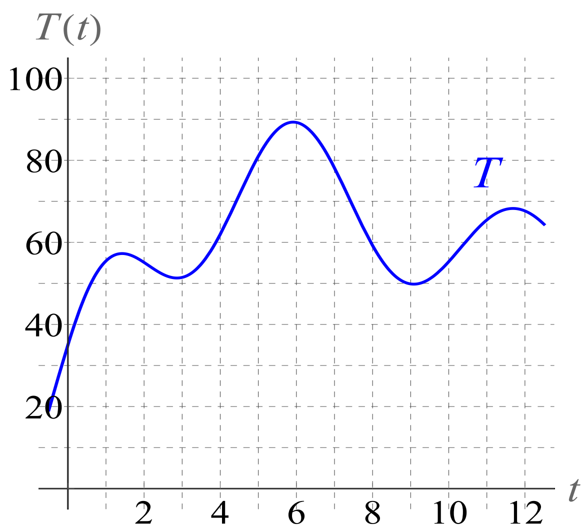
j) Near $x = -3$, would $g(x)$ increase or decrease as x increases?

k) Which is larger, $g(-1)$ or $g(3)$?

l) What is the maximum value achieved by g?

§2.11 EXAMPLE 2

Suppose that the temperature T, in $°C$, of a filament at time t (in minutes) is given by a function whose graph is shown below:



Use this graph to estimate answers to the following questions:

a) What is the temperature of the filament at time 5?

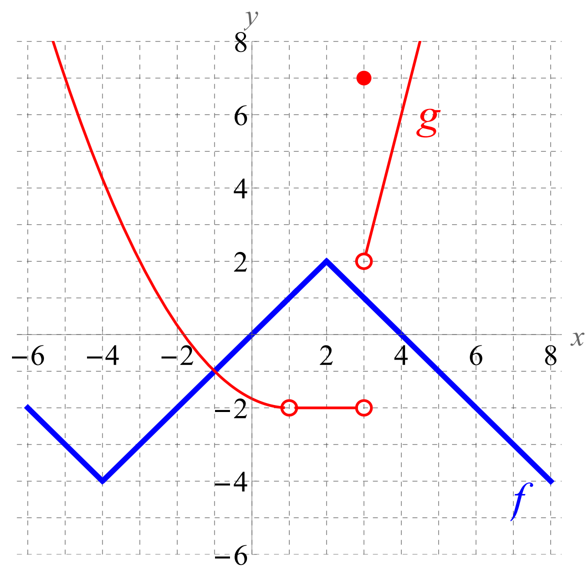
b) At what time(s) is the temperature of the filament equal to $70°$?

c) At time 10.5, is the filament getting hotter or colder?

d) At what time is the temperature of the filament greatest?

§2.11 EXAMPLE 3

The graphs of f and g are given in the picture below (f is the thick graph; g is the thin graph).



Use these graphs to estimate each quantity:

a) $g(3)$

e) $(g \circ f)(0)$

b) $g(1)$

f) $f^3(-1)$

c) $(f+g)(5)$

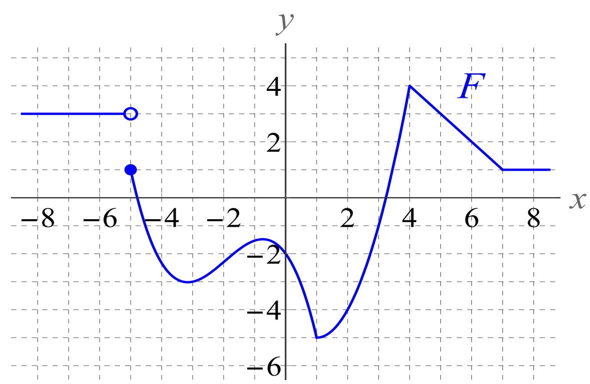
g) $\left(\frac{f}{g} \circ f\right)(2)$

d) $(fg)(4)+1$

h) $\dfrac{f(4)-2}{4-g(2)}$

§2.11 EXAMPLE 4

Let F be the function whose graph is given here:



Also, suppose G is the function given by the table of values

x	-3	-2	-1	0	1	2	3
$G(x)$	4	8	3	-5	3	1	7

and last, suppose $H : \mathbb{R} \to \mathbb{R}$ is the function $H(x) = \begin{cases} 3 - x & x < 1 \\ x^2 + 2 & x \geq 1 \end{cases}$.

Use all this information to compute (or at least estimate) each quantity:

a) $2 + F(-6)$

b) $(FG)(0)$

c) $(H \circ F)(3 + 1)$

d) $(FH^2)(2)$

e) $(3F - G + 2H)(1)$

f) $G(H(0) - F(5))$

g) $(F + H \circ G)(1)$

h) $3(2F \circ G^{-1})(1)$

2.12 Graphs, equations and inequalities

Every equation in one variable has a left-hand side and a right-hand side. If we set $f(x)$ equal to the left-hand side and $g(x)$ equal to the right-hand side, we see that

Every equation in one variable can be thought of as $f(x) = g(x)$.

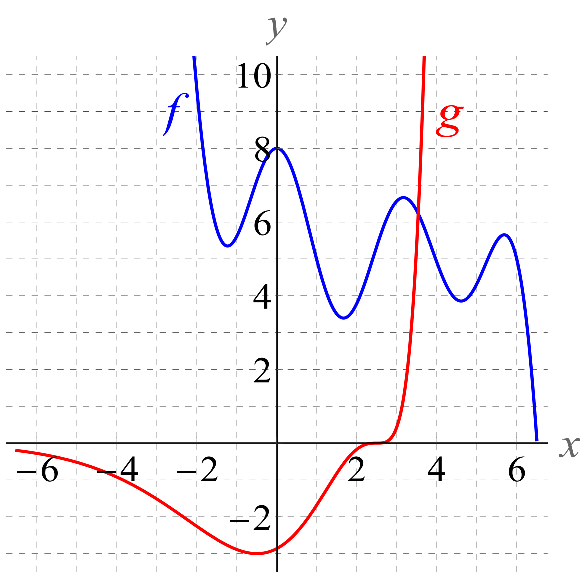
This allows us to use graphs to approach equations. Suppose, for example, we have the equation

$$2\cos 2x - \frac{x^3(x-5)^2}{80} + 6 = 10\left(x - \frac{5}{2}\right)^3 \arctan(e^{x-4}). \qquad (2.1)$$

We can let $f(x) = 2\cos 2x - \frac{x^3(x-5)^2}{80} + 6$ and $g(x) = 10\left(x - \frac{5}{2}\right)^3 \arctan(e^{x-4})$; then this equation becomes

$$f(x) = g(x).$$

To see how graphs help us understand this equation, let's look at the graphs of f and g on the same axes;



Based on these graphs:

1. How many solutions does equation (2.1) above have?

2. What is/are the solution(s) of equation (2.1)?

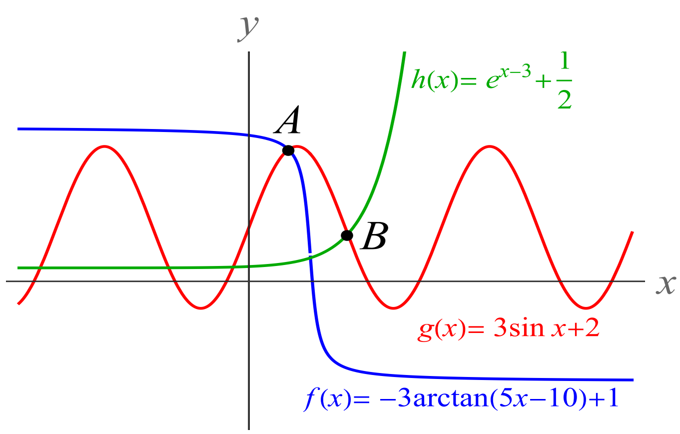
Theorem 2.28 *The solution(s) of the equation $f(x) = g(x)$ are the x-coordinates of the point(s) where the graphs of f and g intersect.*

§2.12 EXAMPLE 1

Find the intersection point of the graphs of $f(x) = 3 - 4x$ and $g(x) = 5x + 8$.

§2.12 EXAMPLE 2

Consider these graphs:



a) What equation would you solve to find point A?

b) What equation would you solve to find point B?

c) When you solve the equation you wrote down in part (b), how many solutions would you get?

d) How would you know which of those solutions is the x-coordinate of point B?

§2.12 EXAMPLE 3

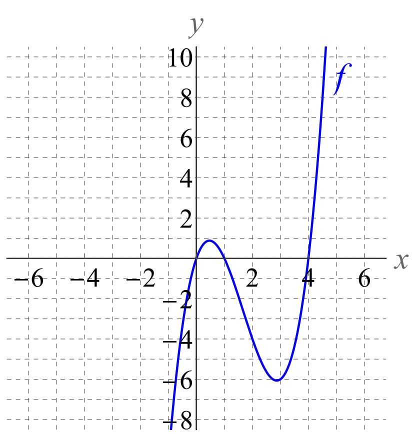
If a company produces x units of its product, then the company incurs costs $C(x) = 30x^{2/3} + 25$ and has revenue $R(x) = xe^x$. Describe a method to determine the number of units the company needs to produce in order to break even.

An important special case

We've seen that every equation in one variable is $f(x) = g(x)$. However, if we move all the non-constant terms of one side of this equation to the other side, we can rewrite the equation as $f(x) = b$ (**not** the same f we started with in $f(x) = g(x)$), where b is a constant. For example, given the equation

$$x^3 + 3x - 5 = 5x^2 - x \tag{2.2}$$

Here's a graph of $f(x) = x^3 - 5x^2 + 4x$:



Based on this graph:

a) How many solutions does equation (2.2) above have?

b) What is/are the solution(s) of equation (2.2)?

Theorem 2.29 *The solutions of the equation $f(x) = b$ are the x-coordinates of points where the graph of f has height b.*

What about inequalities?

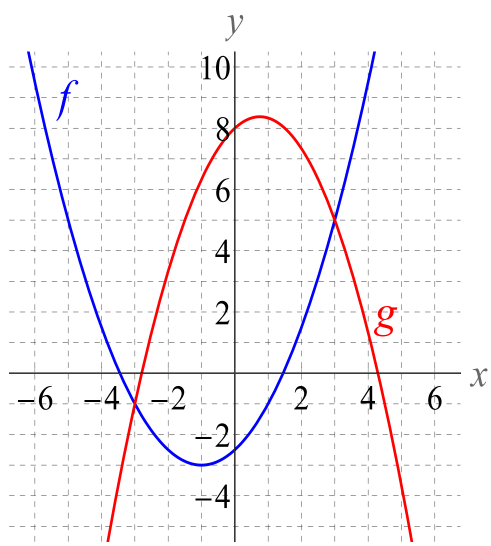
In the same vein that every equation in one variable is $f(x) = g(x)$, every inequality in one variable is either

$$f(x) > g(x) \qquad \text{(a.k.a. } g(x) < f(x)\text{)}$$

or

$$f(x) \geq g(x) \qquad \text{(a.k.a. } g(x) \leq f(x)\text{).}$$

Suppose we have functions f and g with the following graphs:



Based on these graphs,

1. What is the solution set of the inequality $g(x) \geq f(x)$?

2. What is the solution set of the inequality $g(x) > f(x)$?

Theorem 2.30 *The solution set of $f(x) > g(x)$ is the set of x-coordinates where the graph of f lies above the graph of g.*

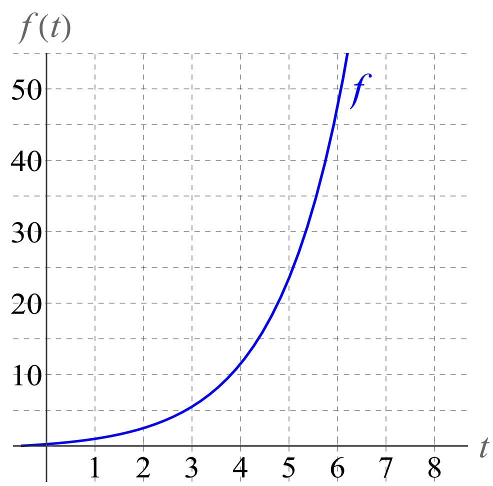
The solution set of $f(x) > b$ is the set of x-coordinates for which the graph of f lies above height b.

The solution set of $f(x) < b$ is the set of x-coordinates for which the graph of f lies below height b.

2.13 Average and instantaneous rate of change

§2.13 EXAMPLE 1

Here is the graph of a function f, where $f(t)$ represents the distance (in meters) an object has travelled after t seconds:



a) How far has the object travelled 1 second after its departure?

b) How far has the object travelled 2 seconds after its departure?

c) How far did the object travel between times 1 and 2?

d) How much time elapsed between times 1 and 2?

e) What is the object's average velocity between times 1 and 2?

f) How far has the object travelled 5 seconds after its departure?

g) What is the object's average velocity between times 1 and 5?

h) What is the object's average velocity between times 3 and 3.0001?

i) What is the object's instantaneous velocity at time 3?

Definition 2.31 *Let $f : \mathbb{R} \to \mathbb{R}$, and suppose $a < b$.*

- *The* **net change** *in f from $x = a$ to $x = b$ is the change in the output, i.e.*

$$\textit{net change } = \triangle y = f(b) - f(a).$$

- *The* **average rate of change** *of f from $x = a$ to $x = b$ is the change in the output per unit change in the input, i.e.*

$$\textit{average rate of change } = \frac{f(b) - f(a)}{b - a}.$$

- *The* **instantaneous rate of change** *of f at $x = a$ is the slope of the line passing through $(a, f(a))$ which best[a] approximates the curve f. This line is called the* **tangent line to** f **at** a.

[a]Exactly which line "best" approximates f is discussed in calculus.

§2.13 EXAMPLE 2

Let $f(x) = \frac{1}{2}x(x + 1) + 3$.

a) Compute the net change of f from $x = 2$ to $x = 5$.

b) Compute the average rate of change of f from $x = 2$ to $x = 5$.

§2.13 EXAMPLE 3

The temperature of my grill is initially $70°$ F when I turn it on. Three minutes later, its temperature is $330°$, and seven minutes after I turn it on its temperature is $490°$. Determine the average rate of change in my grill's temperature between times $t = 3$ and $t = 7$.

Chapter 3

A library of algebraic functions

BIG PICTURE

In the next two chapters, we are going to study several classes of functions. Collectively, these functions:

- model lots of real-world phenomena by themselves;
- are combined (with $+$, $-$, $\times$, $\div$ and $\circ$) to make pretty much every function that models anything;
- are needed to solve equations coming from typical applications; and/or
- are the easiest examples necessary to illustrate calculus concepts.

Each class of function we study has a handful of important properties you need to internalize. These properties are ultimately laid out in the tables at the end of Chapter 4.

The functions we study will break into two types: the first type, consisting of those coming from the operations $+$, $-$, $\times$, $\div$, powers and roots, are functions called **algebraic**. Algebraic functions are the subject matter of Chapter 3.

The second type of functions, which include trig functions, exponentials and logarithms, are called **transcendental** and will be discussed in Chapter 4.

3.1 Linear functions

Definition 3.1 *A* **linear function** *is a function $f : \mathbb{R} \to \mathbb{R}$ with a constant rate of change.*

In this definition, "rate of change" means both *average* rate of change and *instantaneous* rate of change: for a linear function, both of these are constant.

Remark: If you ever take MATH 322 (Linear Algebra), you will see a different definition of the word "linear" that doesn't match this one.

Linear functions are by far the most important class of functions to be familiar with. This is because they have three very important properties:

1. linear functions are **ubiquitous**, meaning that they occur in lots of different fields (biology, economics, physics, engineering, health, etc.);
2. linear functions are (relatively) **easy** to describe and analyze mathematically;
3. linear functions can be used to **approximate** harder functions (more on this in calculus).

§3.1 Example 1

a) Suppose the balance on your credit card statement is given in the table below. Is the balance a linear function of the number of months that have passed?

time (months)	0	1	2	3	4	5	6
balance ($)	800	750	700	650	500	200	150

b) The number of miles you have gone on a trip at various times is given in the table below. Is the distance travelled a linear function of the amount of time that has passed?

time (hrs)	0	1	2	3	4	5	6
distance (mi)	0	45	90	135	180	225	270

c) Suppose that the grade you will make on an exam depends on the number of hours you study, as given in the chart below. Is your grade a linear function of the amount of time you study?

time (hrs)	0	1	2	3	4	5	6
grade (pts)	25	50	75	85	90	92	93

d) In DNA molecules, the GC content (this is the % of guanine and cytosine in the molecule) and its melting point (the temperature at which the DNA molecule will denature) is given in the table below. Is the melting point a linear function of the GC content?

GC content (%)	0	10	20	30	40	50	60
melting point (°C)	80	85	90	95	100	105	110

Slope

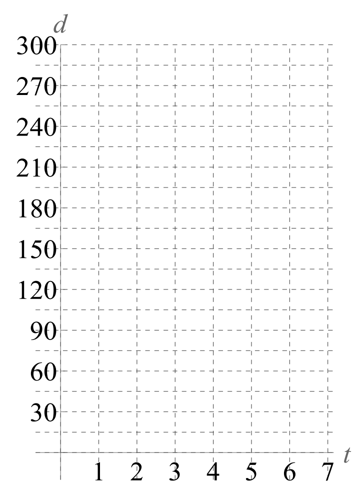
§3.1 EXAMPLE 2

Below, you are given tables of values for linear functions. For each table of values,

i. graph the function,
ii. compute the rate of change of the function (since the function is linear, this rate will be constant), and
iii. identify what the rate of change of the function has to do with the graph.

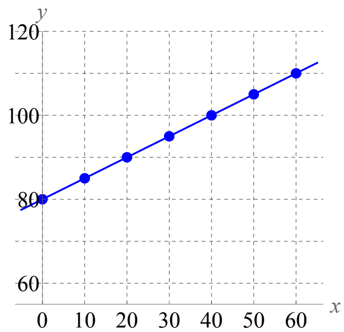
a) (Example 1 (b) earlier)

time t (hrs)	0	1	2	3	4	5	6
distance $d(t)$ (mi)	0	45	90	135	180	225	270



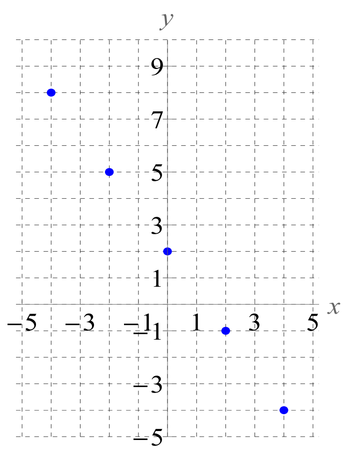
b) (Example 1 (d) earlier)

GC content x (%)	0	10	20	30	40	50	60
melting point $f(x)$ (°C)	80	85	90	95	100	105	110



c)

x	-4	-2	0	2	4
$f(x)$	8	5	2	-1	-4



Definition 3.2 *The* **slope** *of a linear function is its constant rate of change. This is a number, usually denoted by m, that represents how much the graph of the function goes up/down for each unit x moves to the right.*

§3.1 EXAMPLE 3

a) Suppose a linear function f has slope 4. If $f(5) = 2$, what is $f(6)$?

b) Suppose a linear function f has slope -2. If $f(1) = 14$, what is $f(7)$?

c) Suppose a linear function f has slope 3. If $f(3) = 11$, what is $f(-2)$?

§3.1 EXAMPLE 4

Complete the following table:

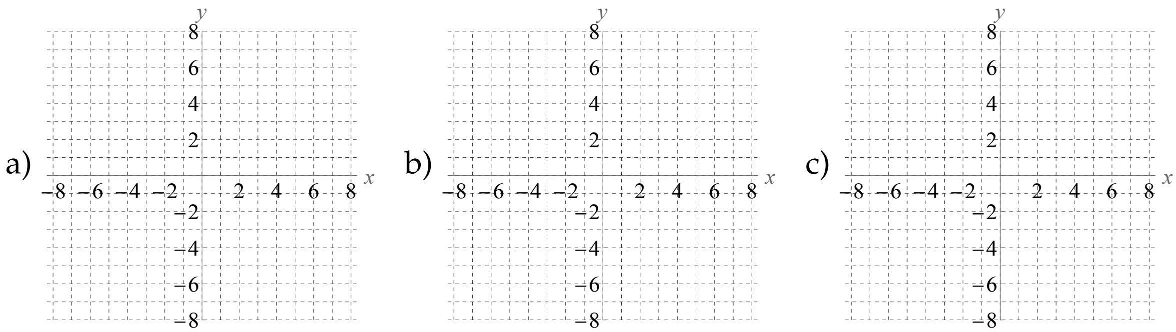
slope of linear function	amount the input changes	amount the output changes
7	4	
	12	2
$\frac{3}{5}$	-8	
$-\frac{2}{3}$		$-\frac{3}{4}$

§3.1 EXAMPLE 5

a) Graph the linear function g which passes through $(-4, 3)$ and has slope 2.

b) Graph the linear function h that passes through $(2, 0)$ with slope $\frac{1}{2}$.

c) Graph the linear function h that passes through $(-1, 5)$ with slope 0.



These graphs explain why we call linear functions *linear*:

Theorem 3.3 *The graph of a linear function is a line.*

P.S. All lines are straight; all straight things are lines.
If a graph isn't straight, it would be called a *curve* or a *path*, not a line.

Since the slope of a line is its rate of change, we can adapt the rate of change formula we learned in Chapter 2 to produce a formula for the slope:

Theorem 3.4 (Slope formula) *If f is a linear function so that (x_0, y_0) and (x_1, y_1) are on the graph of f, then the slope of f is*

$$m = \frac{\textit{rise}}{\textit{run}} = \frac{\triangle y}{\triangle x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

§3.1 EXAMPLE 6

Compute the slope of the line that passes through each pair of points:

a) $(3, -2)$ and $(-5, -6)$

b) $(5, -3)$ and $(-2, 4)$

c) $(8, 7)$ and $(-12, 1)$

Solution: $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{1-7}{-12-8} = \dfrac{-6}{-20} = \boxed{\dfrac{3}{10}}$.

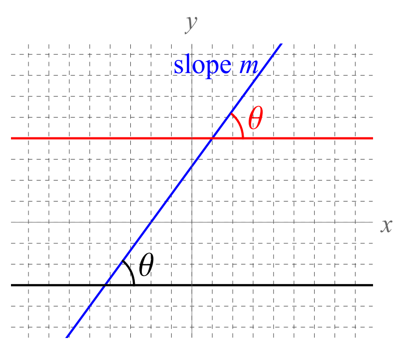
d) $(4, -6)$ and $(4, 2)$

Solution: $m = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{2-(-6)}{4-4} = \dfrac{8}{0}$

Slope and angle

Slope is very strongly connected with the trig function *tangent*:

Theorem 3.5 *If f is a linear function with slope m and if θ is the angle the graph of f makes with the horizontal, then $m = \tan\theta$.*



§3.1 EXAMPLE 7

a) Find the slope of a line which makes an angle of $\dfrac{\pi}{6}$ with the horizontal.

Solution: The slope is $m = \tan\dfrac{\pi}{6} = \boxed{\dfrac{1}{\sqrt{3}}}$.

b) Find the slope of any vertical line.

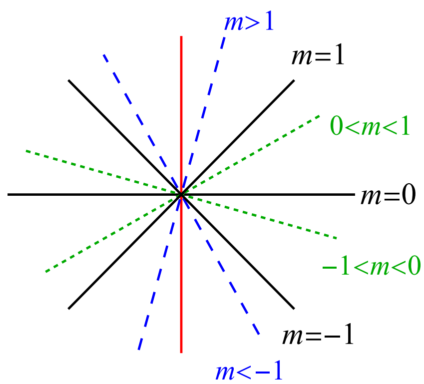
Solution: Vertical lines have a slope of _____ = _____ with the horizontal,

so their slope is $m = \tan \boxed{}$ which

Note: this computation reinforces Example 6 (d) above.

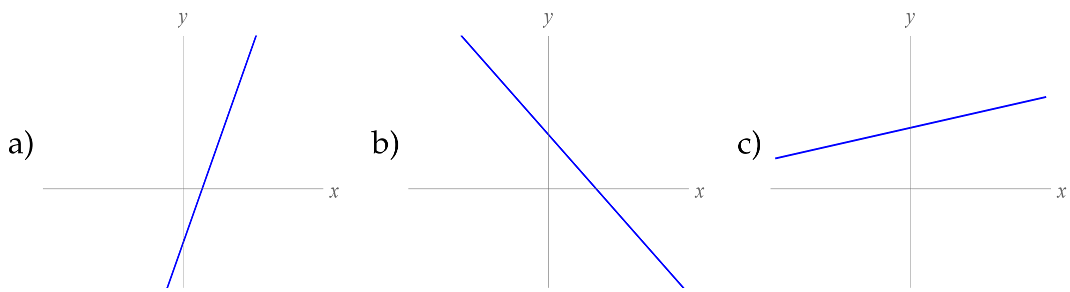
Estimating slope

Assuming the scales on the x- and y-axes are the same, we can estimate the slope of a line by looking at it:



§3.1 Example 8

Estimate the slope of each linear function by looking at its graph (assume the scales on the x- and y-axes are the same):



3.2 More on linear functions

The rule of a linear function

§3.1 EXAMPLE 3(B) (EARLIER)

Suppose a linear function f has slope 3. If $f(1) = 4$, what is $f(7)$?

Solution: $f(7) = f(1) + m(7-1) = 4 + 3(7-1) = 4 + 3 \cdot 6 = \boxed{22}$.

We want to generalize this example, to give a *general* rule for all linear functions. Suppose a linear function f passes through the point (x_0, y_0) and has slope m. Then

We have derived this important fact:

Theorem 3.6 (Point-slope formula) *The linear function passing through the point* (x_0, y_0) *with slope* m *has rule*

$$f(x) = y_0 + m(x - x_0)$$

a.k.a.

$$y = y_0 + m(x - x_0).$$

§3.2 EXAMPLE 1

Write an equation (a.k.a. rule) of each line with the given properties:

a) the line has slope $\dfrac{-3}{4}$ and passes through $\left(\dfrac{1}{2}, \dfrac{-7}{2}\right)$

b) the line passes through $(-5, 3)$ and $(4, -15)$

c) the line passes through $(6, 2)$ and makes an angle of $\frac{2\pi}{3}$ with the horizontal

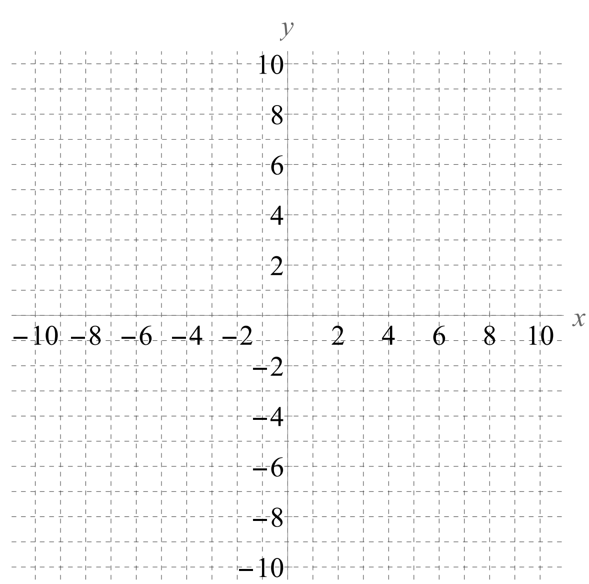
§3.2 EXAMPLE 2

Are the points $(-5, -8)$, $(1, -4)$ and $(10, 1)$ on the same line? Why or why not?

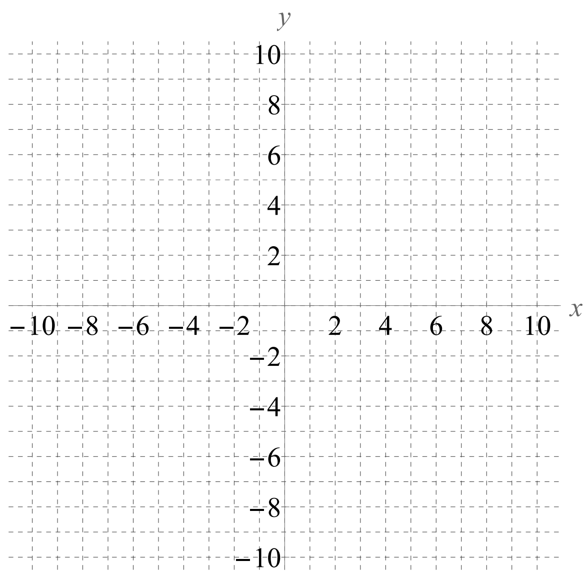
§3.2 EXAMPLE 3

Graph each line:

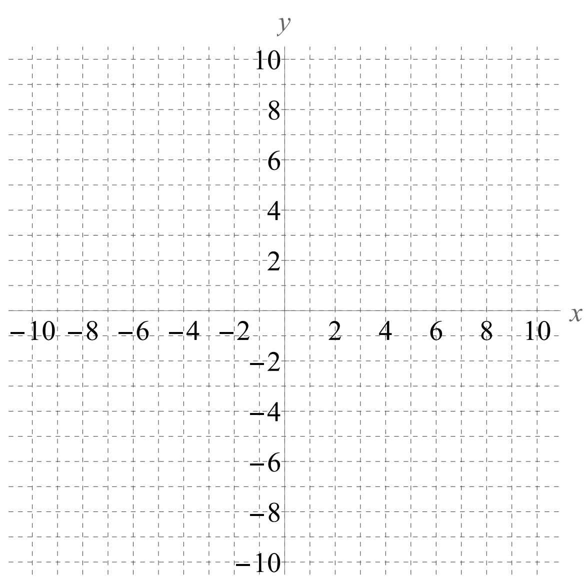
a) $y = 3 - 2(x - 4)$



b) $y = 2 + \frac{3}{4}(x+5)$



c) $y = 4(x-3)$



Other algebraic representations of lines

As we have seen, the line with slope m passing through (x_0, y_0) has equation

There are other ways to rewrite this equation with algebra. For example:

$$y = y_0 + m(x - x_0)$$

Theorem 3.7 (Slope-intercept formula) *The linear function with slope m and y-intercept $(0, b)$ has rule*

$$f(x) = mx + b \qquad \textit{a.k.a.} \qquad y = mx + b.$$

WARNING: The slope-intercept formula is highly overrated, because if you started with a point-slope equation, converting it to $y = mx + b$ *obscures the point (x_0, y_0) you used to write the equation* (which you usually don't want to do).

Here is another way to rewrite the equation of a line:

$$\begin{aligned} y &= y_0 + m(x - x_0) \\ y &= y_0 + mx - mx_0 \\ y - mx &= y_0 - mx_0 \\ -mx + y &= y_0 - mx_0 \\ -mBx + By &= B(y_0 - mx_0) \end{aligned}$$

This gives us the equation on the next page:

Definition 3.8 (Standard equation) *The* **standard equation** *of a line is*

$$Ax + By = C$$

where A, B and C are constants.

Advantages of using the standard equation of a line:

- it allows you to write the equation of vertical lines, which are lines but not linear functions (next page);
- it is useful for working with systems of linear equations (coming soon).

A major disadvantage of the standard equation is that y isn't solved for, so it doesn't immediately give you a rule for the linear function you're thinking of. The slope also isn't readily apparent from looking at the standard equation.

§3.2 EXAMPLE 4

Consider the line whose standard equation is $2x - 5y = 17$.

a) Find a rule for this linear function.

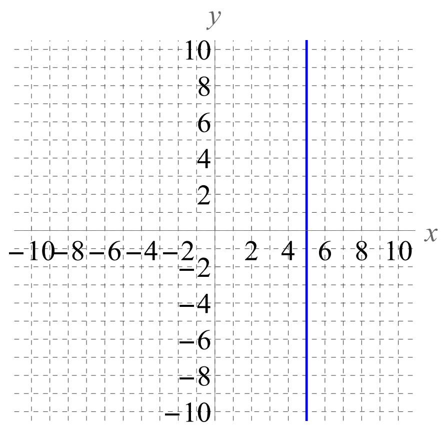
b) Determine the slope of this line.

Solution: From the rule $f(x) = \frac{2}{5}x - \frac{17}{5}$ we found in part (a), we know that the coefficient on the x-term is the slope . Therefore $m = \boxed{\frac{2}{5}}$.

Vertical lines

§3.2 EXAMPLE 5

Here is an example of a vertical line:



All the points on this line have the same ______________________, so an equation that describes this line is ___________ .

Is this line a function $y = f(x)$?

We have seen earlier that vertical lines do not have a defined slope.

You should avoid using the phase "no slope". The reason is that this can be interpreted two ways:

To summarize:

Theorem 3.9 (Vertical lines) *Vertical lines have undefined slope. They have equation $x = c$, where c is a constant. Vertical lines are lines, but not linear functions $y = f(x)$.*

Real-world interpretation of slope; units

Suppose we have some real-world quantities that are related by an equation whose graph is a line. This means that in a linear setting,

the amount the output changes is proportional to the input changes

and the slope of the line is the ______________________________ .

Since the slope is obtained by dividing the rise (obtained by subtracting outputs) by the run (obtained by subtracting inputs), the units of slope are always

units of slope =

§3.2 Example 6

a) Hooke's Law says that the force y needed to keep a spring stretched x units beyond its natural length is directly proportional to x. (Obviously, no force is required to stretch the spring 0 units beyond its natural length.)

   If it takes 9 N of force to stretch a particular spring 35 cm beyond its natural length, find the slope of the line relating y and x (with correct units), and interpret the meaning of the slope.

b) At a grocery store, a 90 ounce jug of orange juice costs $\$6.50$, but a 60 ounce jug of orange juice costs $\$5.00$. Assume a linear relationship between the amount of orange juice purchased and the cost of said amount.

   i. Find the slope of this linear relationship and interpret the slope.
   ii. How much does 190 ounces of orange juice cost?

Various methods of graphing lines

The most efficient method of sketching the graph of a line depends on the equation you are given:

1. **If the line is vertical** (has only x in the equation), **or** if the line is **horizontal** (has only y in the equation), sketch the line by drawing one point on the line and then sketching a vertical or horizontal line as appropriate. "Just do it."

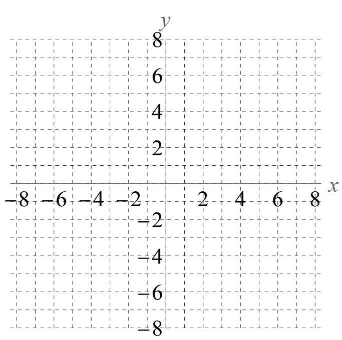
2. If the line isn't vertical or horizontal:

   - **if the line is in slope-intercept form** $f(x) = mx + b$, plot the y-intercept $(0, b)$, use the slope to get a second point, then connect the points;
   - **if the line is in point-slope form** $f(x) = y_0 + m(x - x_0)$, plot the point (x_0, y_0), use the slope to get a second point, then connect the points;
   - **if you are given the standard equation** $Ax+By = C$, find the x-intercept (by setting $y = 0$ and solving for x) and the y-intercept (by setting $x = 0$ and solving for y); connect these points.

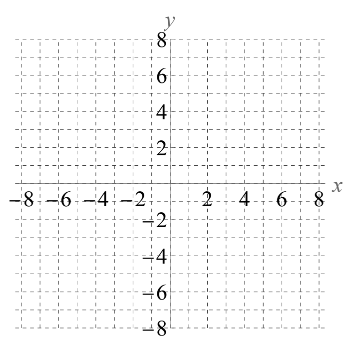
§3.2 EXAMPLE 8

Sketch the graph of each line:

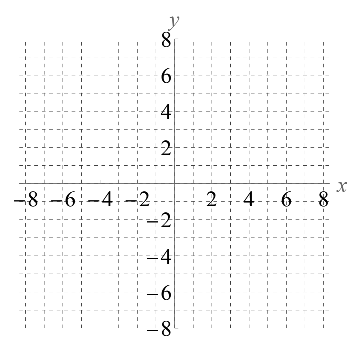
a) $y = -2x + 5$



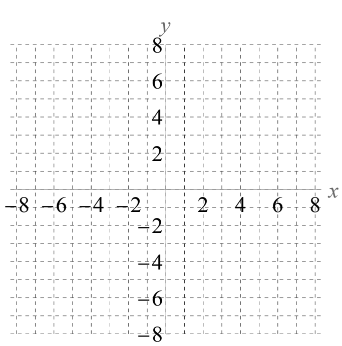
c) $x = -3$



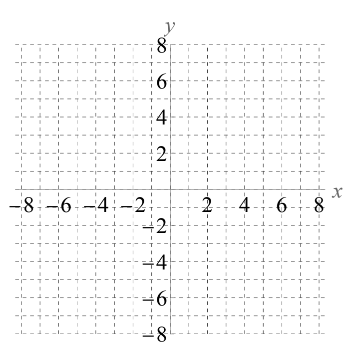
e) $y = 0$



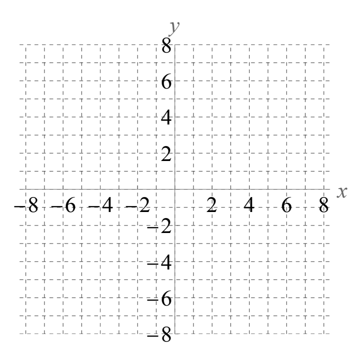
b) $3x + 4y = 24$



d) $y = -5 + \frac{2}{3}(x - 4)$

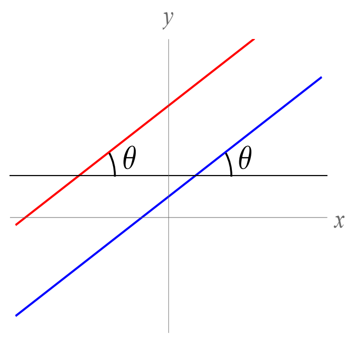


f) $x = 3y + 6$



Parallel and perpendicular lines

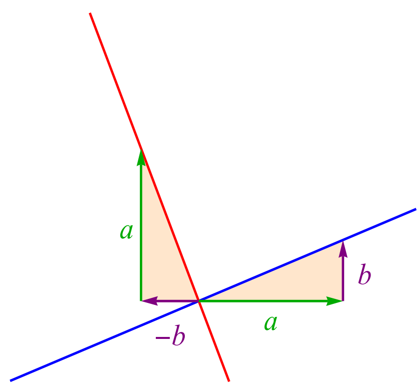
Two parallel lines make the same angle with the horizontal:



Since the slopes of these lines are the tangent of the same angle θ, we see

Theorem 3.10 *Parallel lines have the same slope.*

Two perpendicular lines look like this:



We see from the congruent triangles in this picture that if the first line has slope $m_1 = \frac{b}{a}$, then the second line has slope $m_2 = \frac{a}{-b}$. That means

Put another way,

Theorem 3.11 *Perpendicular lines have slopes that multiply to* -1*.*
Put another way, perpendicular lines have slopes that are negative reciprocals.

§3.2 EXAMPLE 9

a) Write the equation of the line parallel to $y = 2 + 7(x - 1)$ passing through the point $(5, -3)$.

b) Write the equation of the line perpendicular to $2x - 7y = 11$ passing through the point $(-4, 0)$.

Other useful theoretical facts about linear functions

Inverses

Theorem 3.12 *Any linear function whose slope is nonzero is one-to-one, and the inverse of such a function is linear.*

REASON A line that isn't horizontal will pass the HLT, so it must be one-to-one.

For the inverse, if f is linear then $f(x) = mx + b$. To undo this f, we would first subtract b and then divide by m, so

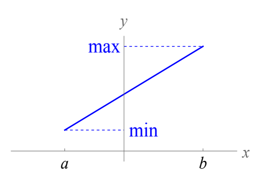
$$f^{-1}(y) = \frac{y-b}{m} = \frac{1}{m}y - \frac{b}{m},$$

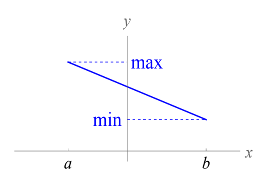
which is linear.

Maximum and minimum values

Theorem 3.13 *When the inputs are restricted to a closed interval $[a, b]$, the maximum and minimum values of a linear function must occur at endpoints, meaning when the input is a and/or when the input is b.*

REASON If the slope is positive, the maximum value is at the right-most point (when $x = b$) and the minimum is at the left-most point (when $x = a$). If the slope is negative, the reverse holds.





Closure properties

Theorem 3.14 *Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are linear, and let r be a constant. Then, the following functions are also linear:*

$$f+g \qquad f-g \qquad rf \qquad f\circ g$$

REASON If f and g are linear, then $f(x) = mx + b$ and $g(x) = nx + c$. Then:

$$\begin{aligned}
(f+g)(x) &= (mx+b)+(nx+c) = (m+n)x+(b+c)\\
(f-g)(x) &= (mx+b)-(nx+c) = (m-n)x+(b-c)\\
(rf)(x) &= r(mx+b) = (rm)x+rb\\
(f\circ g)(x) &= f(nx+c) = m(nx+c)+b = (mn)x+(mc+b).
\end{aligned}$$

All these rules are of the form (constant)$x+$ (constant), so they are all linear.

3.3 Solving linear equations

Suppose you have an equation in one variable where both sides of the equation are linear functions.

That means the equation has constant terms and x-terms, but nothing else (no x^2, no e^x, no $\sin x$, no $\sqrt{x}$, no $|x|$, etc.)

> **Note:** most linear equations have 1 solution; there are some "stupid" linear equations with infinitely many solutions ($0x = 0$) or no solution ($0x = 1$).

Such an equation is called a **linear equation** and can be solved by moving all the x-terms to one side and all the constant terms on the other side:

§3.3 Example 1

Solve for x:

a) $7x + 9 = 65$

Solution:

$$\begin{aligned} 7x + 9 &= 65 \\ 7x &= 56 \\ x &= \boxed{8} \end{aligned}$$

b) $3(x-1) + 5 = 4(3x-2) + x$

Solution:

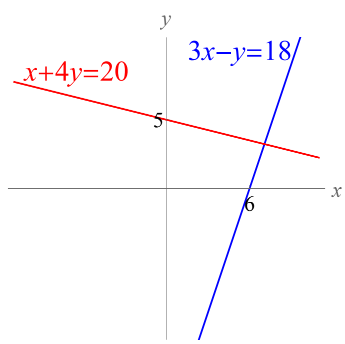
$$\begin{aligned} 3(x-1) + 5 &= 4(3x-2) + x \\ 3x - 3 + 5 &= 12x - 8 + x \\ 3x + 2 &= 13x - 8 \\ 10 &= 10x \\ \boxed{1} &= x \end{aligned}$$

c) $\frac{1}{2}x + \frac{1}{3} = \frac{3}{4}x - \frac{1}{5}$

Systems of linear equations

To find the intersection points of two lines (say the graphs of $y = f(x)$ and $y = g(x)$, based on what we learned in Chapter 2, we would solve the equation

However, sometimes the equation of a line isn't given in $y = f(x)$ form, so we need to know some other techniques for solving for intersection points. For example, suppose want to find the intersection of the lines $3x - y = 18$ and $x + 4y = 20$.



To solve for this point algebraically, we need to find (x, y) which is a solution of the **system of equations**

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases},$$

meaning we need to find (x, y) that works in *both equations*.

Methods of solving a system of equations

METHOD 1: Substitution. Solve for one variable in one equation and substitute into the other equation (usually easier if at least one line is given in slope-intercept form).

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases}$$

METHOD 2: Addition/elimination. Multiply through each equation by a constant, then add the equations to eliminate one variable (usually easier if both equations are in standard form).

$$\begin{cases} 3x - y = 18 \\ x + 4y = 19 \end{cases}$$

§3.3 EXAMPLE 2

Find the point of intersection of the two lines $y = -2x - 1$ and $8x + 5y = -1$.

§3.3 EXAMPLE 3

Find the point of intersection of the two lines $y = 3x - 7$ and $6x - 2y = 5$.

3.4 Introducing transformations

§3.4 EXAMPLE 1

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of the other function:

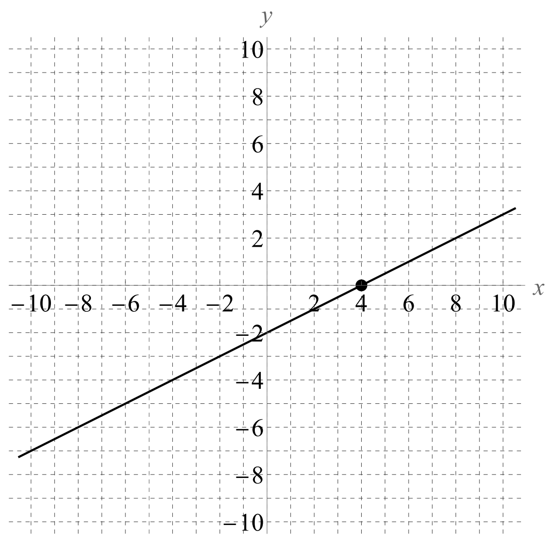
$$\boxed{f(x) = \frac{1}{2}(x-4)}$$

$$f_{-2}(x) = -2 + \frac{1}{2}(x-4)$$

$$f_{-6}(x) = -6 + \frac{1}{2}(x-4)$$

$$f_{3}(x) = 3 + \frac{1}{2}(x-4)$$

$$f_{8}(x) = 8 + \frac{1}{2}(x-4)$$



From these graphs, we see:

- when we add a constant c to the rule of f, the graph ______________________________

  ____ .

- when we subtract a constant c from the rule of f, the graph __________________

  ____ .

This works not just in the example we just did, but for any function:

Theorem 3.15 (Vertical shifts) *Let $c > 0$ be a constant.*

- *The graph of the function $f(x) + c$ is the same as the graph of f, shifted up by c units.*

$$x \xrightarrow{f} \xrightarrow{+c} f(x) + c$$

- *The graph of the function $f(x) - c$ is the same as the graph of f, shifted down by c units.*

$$x \xrightarrow{f} \xrightarrow{-c} f(x) - c$$

§3.4 EXAMPLE 2

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of each given function:

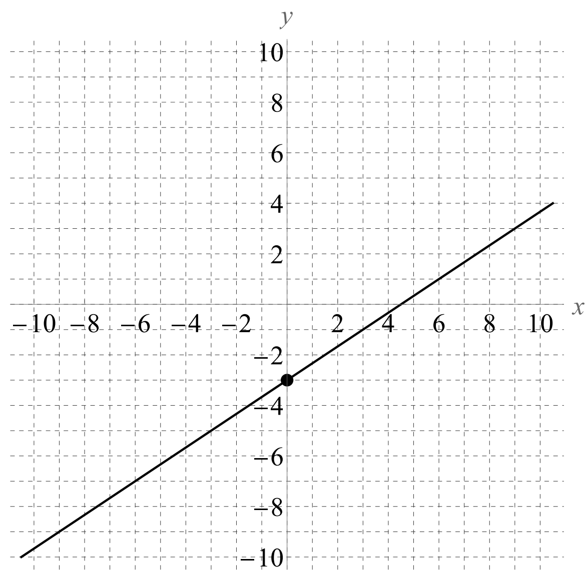
$$\boxed{g(x) = -3 + \frac{2}{3}x}$$

$$g_2(x) = -3 + \frac{2}{3}(x-2)$$

$$g_5(x) = -3 + \frac{2}{3}(x-5)$$

$$g_{-3}(x) = -3 + \frac{2}{3}(x+3)$$

$$g_{-7}(x) = -3 + \frac{2}{3}(x+7)$$



From these graphs, we see:

- when we replace x with $(x-c)$ in the rule of f, the graph ____________________ ____ .

- when we replace x with $(x+c)$ in the rule of f, the graph ____________________ ____ .

This works not just in the example we just did, but for any function:

Theorem 3.16 (Horizontal shifts) *Let $c > 0$ be a constant.*

- *The graph of the function $f(x-c)$ is the same as the graph of f, shifted right by c units.*
$$x \xrightarrow{-c} \xrightarrow{f} f(x-c)$$

- *The graph of the function $f(x+c)$ is the same as the graph of f, shifted left by c units.*
$$x \xrightarrow{+c} \xrightarrow{f} f(x+c)$$

§3.4 EXAMPLE 3

The graph of the linear function in the box is provided for you. On the same axes, sketch the graph of each given function:

$$\boxed{h(x) = x - 2}$$

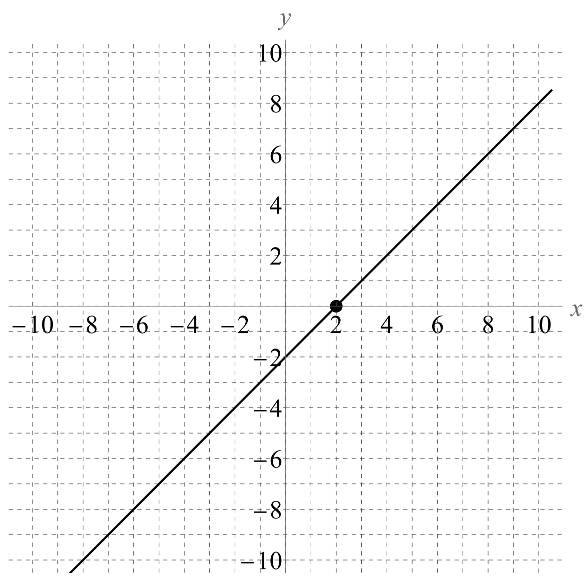
$$h_3(x) = 3(x-2)$$

$$h_5(x) = 5(x-2)$$

$$h_{-2}(x) = -2(x-2)$$

$$h_{-1/2}(x) = -\frac{1}{2}(x-2)$$

$$h_{1/4}(x) = \frac{1}{4}(x-2)$$



From these graphs, we see:

- when we multiply the rule for h by a constant c, the x-intercept of the graph stays the same, and the graph is stretched/compressed vertically by a factor of c.
- if $c < 0$, the graph of h is reflected across the x-axis (flipped vertically).

Theorem 3.17 (Vertical stretch/compression) *Let c be a nonzero constant.*

- *The graph of the function $cf(x)$ is the same as the graph of f, stretched vertically by a factor of c units.*

$$x \xrightarrow{f} \xrightarrow{\times c} cf(x)$$

- *If $|c| > 1$, the graph is stretched; if $|c| < 1$ the graph is compressed.*
- *The x-intercept(s) of $cf(x)$ are the same as the x-intercept(s) of f.*
- *If $c < 0$, the graph of the function $cf(x)$ is the graph of f, stretched vertically by a factor of c and reflected across the x-axis.*

What a transformation is

Definition 3.18 *Let $f : \mathbb{R} \to \mathbb{R}$. A* **transformation** *of f is any other function whose rule comes from the rule of f by including some extra constant(s) via addition, subtraction, multiplication or division.*

EXAMPLES

- Transformations of the function $f(x) = x^2$ include

$$g(x) = x^2 + 2, \quad h(x) = \frac{-1}{4}x^2, \quad k(x) = (x-2)^2, \quad l(x) = (3x)^2 - 1, \quad \text{etc.}$$

- Transformations of the function $F(x) = \sin x$ include

$$G(x) = 2\sin x, \qquad H(x) = \sin(x+3), \qquad K(x) = \frac{2}{3}\sin 2x - 1, \qquad \text{etc.}$$

- Transformations of the function $p(x) = e^x$ include

$$q(x) = \frac{3}{4}e^x, \qquad r(x) = -e^{x-4}, \qquad s(x) = e^{7x} - 3, \qquad u(x) = -8e^{4x}, \quad \text{etc.}$$

The graph of a transformation of f is obtained from the graph of f by doing something relatively "simple". For example, we've seen

TRANSFORMATION OF f	EFFECT ON GRAPH OF f
$f(x) + c$	shifts up c units
$f(x) - c$	shifts down c units
$f(x - c)$	shifts right c units
$f(x + c)$	shifts left c units
$cf(x)$	stretched vertically by factor of c (flipped across the x-axis if $c < 0$)

Note: all these transformations have graphs that have the same general shape as the graph of f (just shifted, stretched or reflected).

Key theme with transformations: when you transform f in a way that takes place after f in an arrow diagram, the graph is affected vertically:

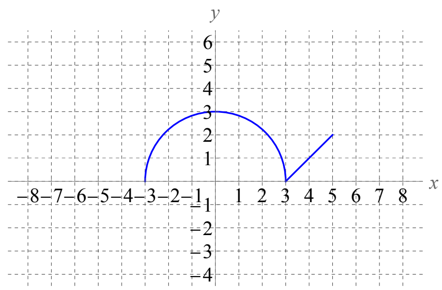
$$x \xrightarrow{f} \xrightarrow{+c} g(x) \qquad\qquad x \xrightarrow{f} \xrightarrow{\times c} g(x)$$

But when you transform f in a way that takes place before f in an arrow diagram, the graph is affected horizontally:

$$x \xrightarrow{+c} \xrightarrow{f} g(x) \qquad\qquad x \xrightarrow{\times c} \xrightarrow{f} g(x)$$

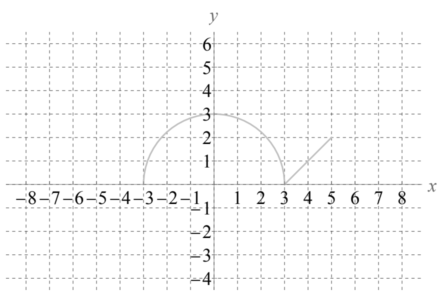
§3.4 EXAMPLE 4

The graph of some unknown function f is given below:

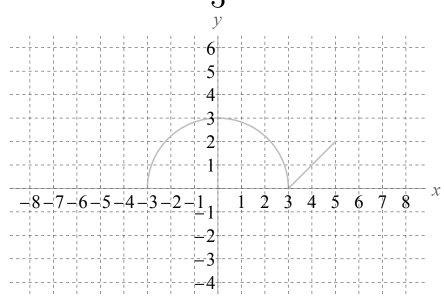


Use this graph to sketch the graph of each function:

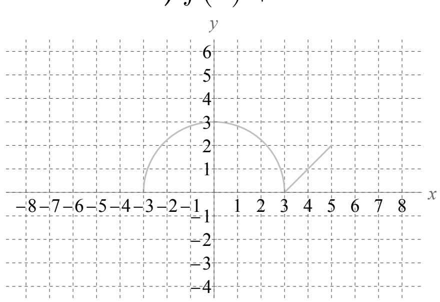
a) $f(x-2)$



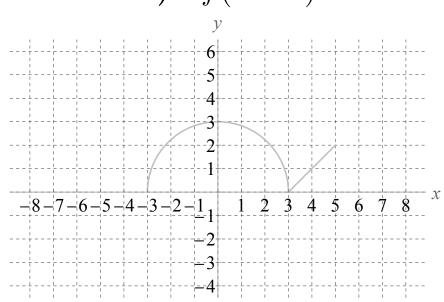
d) $\frac{1}{3}f(x)$



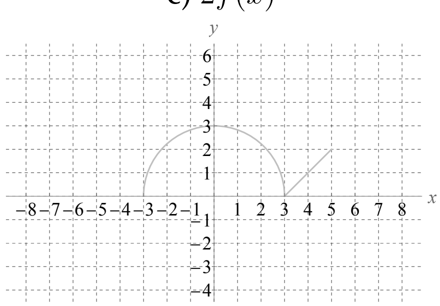
b) $f(x)+1$



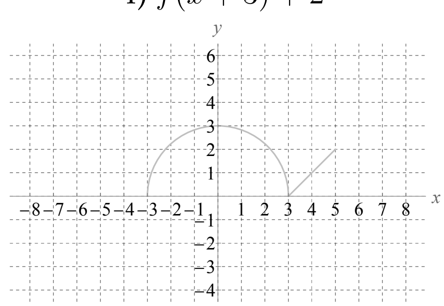
e) $-f(x-1)$



c) $2f(x)$



f) $f(x+3)+2$



3.5 Quadratic functions

CONCEPT

We saw in Sections 3.1 and 3.2 that linear functions are those with a constant rate of change.

Question: What does a function look like if it has a *linear* rate of change?

§3.5 EXAMPLE 1

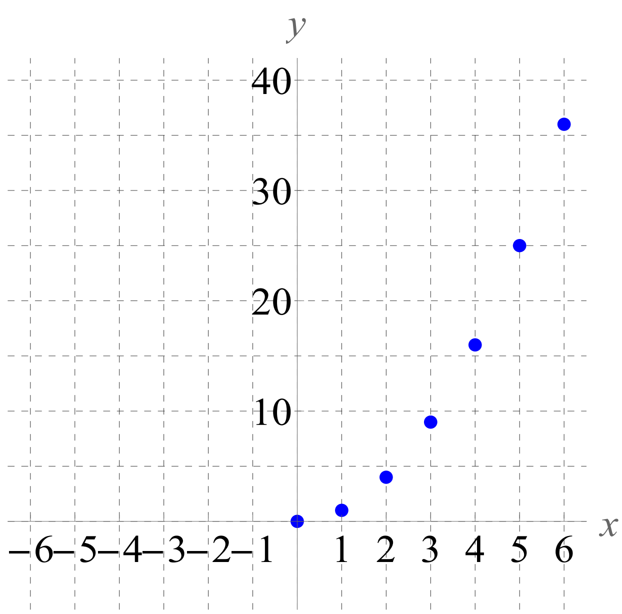
Let f be the function which satisfies

- $f(0) = 0$; and
- the average rate of change of f from x to $x + 1$ is $2x + 1$.

The second bullet point above means

Using this fact, we can produce a table of values for x, which will lead us to a rule for f and a graph of f:

x	$f(x)$
-1	
0	0
1	
2	
3	
4	
5	
6	
$\vdots$	$\vdots$



In general, if the rate of change of a function f is linear, then f itself will be *quadratic*:

Definition 3.19 *A* **quadratic function** *(or just* **quadratic***) is a function $f : \mathbb{R} \to \mathbb{R}$ whose rule is*

$$f(x) = ax^2 + bx + c$$

for constants a, b and c (where $a \neq 0$).

Theorem 3.20 *Let $f : \mathbb{R} \to \mathbb{R}$. f is quadratic if and only if the rate of change of f is linear.*

(Proving this theorem rigorously requires calculus.)

§3.5 EXAMPLE 2

A beanbag is tossed out of a window so that its height (in meters) at time t, in seconds, is given by $h(t) = -5t^2 + 15t + 40$.

1. Compute the average rate of change of the height of the beanbag between times $t = 0$ and $t = 1$.

2. Compute the average rate of change of the height of the beanbag between times $t = 1$ and $t = 2$.

3. Compute the average rate of change of the height of the beanbag between times $t = 2$ and $t = 3$.

   Solution:

$$\begin{aligned} \frac{h(3) - h(2)}{3 - 2} &= \frac{[-5(3^2) + 15(3) + 40] - [-5(2^2) + 15(2) + 40]}{1} \\ &= [-45 + 45 + 40] - [-20 + 30 + 40] \\ &= 40 - 50 = \boxed{-10 \text{ m/sec}}. \end{aligned}$$

4. Compute the average rate of change of the height of the beanbag between times $t = 3$ and $t = 4$.

   Solution: Mimicking what was done in Questions 1-3, we get

$$\frac{h(4) - h(3)}{4 - 3} = \frac{20 - 40}{1} = \boxed{-20 \text{ m/sec}}.$$

5. Explain why h is **not** a linear function.

6. Verify the average rate of change of the height across a 1 second interval of time *is* a linear function.

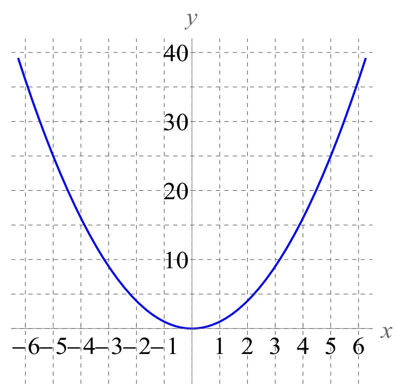
7. Verify the average rate of change of the height across an s second interval of time is also a linear function.

   Solution: Similar to what was done in Question 6, we have

$$\begin{aligned}
\frac{f(t+s) - f(t)}{(t+s) - t} &= \frac{[-5(t+s)^2 + 15(t+s) + 40] - [-5t^2 + 15t + 40]}{s} \\
&= \frac{[-5(t^2 + 2st + s^2) + 15t + 15s + 40] + 5t^2 - 15t - 40}{s} \\
&= \frac{-5t^2 - 10st - 5s^2 + 15t + 15s + 40] + 5t^2 - 15t - 40}{s} \\
&= \frac{-10st - 5s^2 + 15s}{s} \\
&= \frac{s(-10t - 5s + 15)}{s} = \boxed{-10t + (-5s + 15)}.
\end{aligned}$$

The two forms of a quadratic function

We've seen that the graph of $f(x) = x^2$ is a parabola:



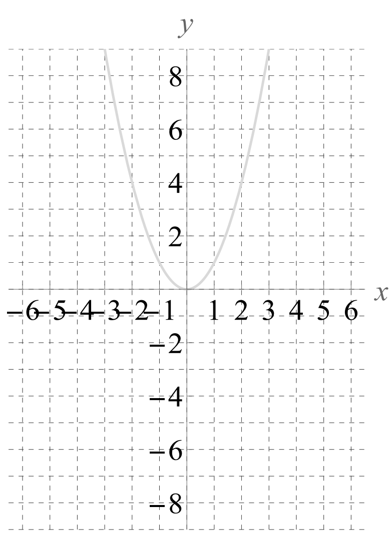
Our next task is to see why the graph of *any* quadratic is a parabola. To do this, we need to find a second version of a rule that describes quadratic function. And to do this, we'll use transformations.

§3.5 EXAMPLE 3

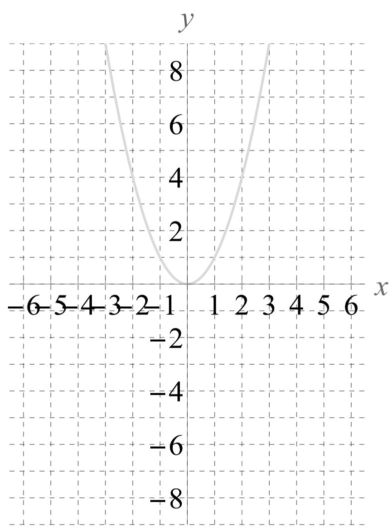
For each given function:

1. Sketch the graph of the function by transforming the graph of $f(x) = x^2$.
2. Write the rule for the function in the form $ax^2 + bx + c$.

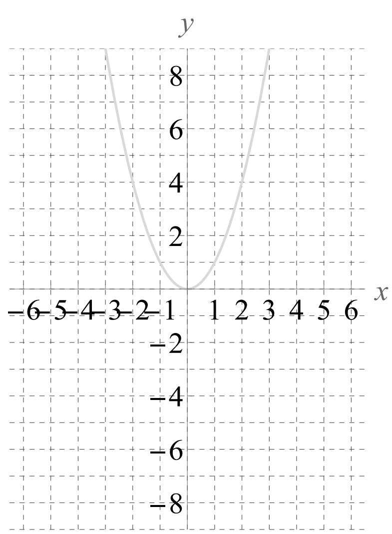
a) $g(x) = (x-3)^2 + 1$



b) $h(x) = 2(x+3)^2$



c) $k(x) = -\dfrac{1}{2}(x-1)^2 + 5$



Observations about the graph of $f(x) = a(x-h)^2 + k$:

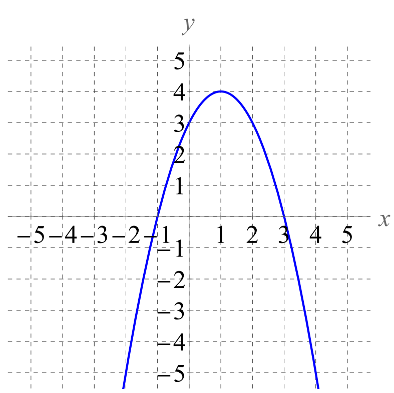
- All these graphs are parabolas;
- the vertical line $x = h$ is an axis of symmetry for the parabola;
- the parabola "turns around" at the point (h, k);
- the domain of f is the set $\mathbb{R}$ of all real numbers;
- if $a > 0$, then:
  - the parabola opens upward,
  - the function f has a minimum value of k, and
  - the range of f is $[k, \infty)$;
- if $a < 0$, then:
  - the parabola opens downward,
  - the function f has a maximum value of k, and
  - the range of f is $(-\infty, k]$.

The vertex of a parabola

Definition 3.21 *The point where a parabola "turns around" is called the* **vertex** *of the parabola. This point is usually denoted* (h, k).

§3.5 EXAMPLE 4

Identify the x-intercepts, the y-intercept and the vertex of this parabola:



x-ints:

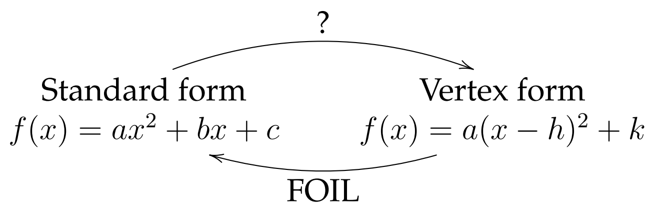
y-int:

vertex:

Completing the square

QUESTION

Can you write *any* quadratic function in the form $a(x-h)^2+k$?



Answer: Yes; to do this you use a technique called **completing the square**.

§3.5 EXAMPLE 4

Write each quadratic function in vertex form:

$$f(x) = 2x^2 + 16x - 10 \qquad\qquad f(x) = ax^2 + bx + c$$

Theorem 3.22 (Completing the square) *Given a quadratic expression ax^2+bx+c, we can rewrite this expression as*

$$ax^2 + bx + c = a\left(x - h\right)^2 + k$$

where $h = -\dfrac{b}{2a}$ and $k = ah^2 + bh + c$.
As a consequence, the graph of every quadratic function is a parabola.

Definition 3.23 *The formula $f(x) = a(x-h)^2 + k$ is called the* **vertex form** *of the quadratic.*

Theorem 3.24 *The vertex of $f(x) = ax^2 + bx + c$ has x-coordinate $h = -\dfrac{b}{2a}$.*

§3.5 EXAMPLE 5

Find the coordinates of the vertex of $f(x) = 2x^2 - 12x - 7$.

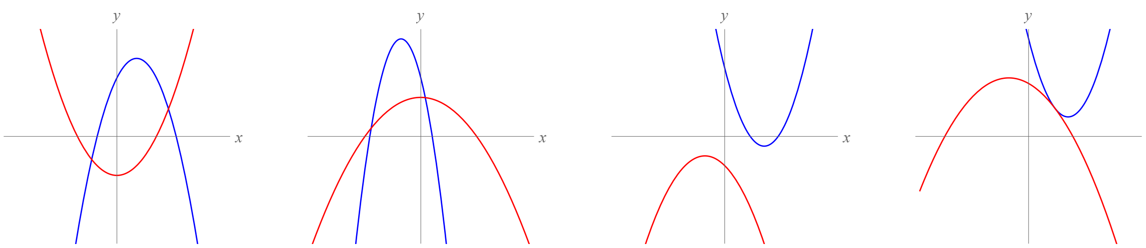
Solving quadratic equations

A quadratic equation is an equation where the LHS and RHS are both quadratic. The most general equation of this type is

which can be thought of as trying to find the (x-coordinates of) intersection points of two parabolas.

Number of solutions

Here are some pictures which show the ways two parabolas can intersect:



Theorem 3.25 *A quadratic equation has* 0, 1 *or* 2 *solutions.*

Solving quadratic equations by factoring

To solve a quadratic equation by factoring, we

1. move all the terms to one side of the equation (i.e. make one side zero), then
2. factor the non-zero side, then
3. set each factor equal to zero and solve for the variable.

This method is based on the following algebraic concept:

If two (or more) terms multiply to make zero,
at least one of the terms must itself be zero.

§3.5 Example 6

Solve for x in each equation:

a) $x^2 - 5x - 14 = 0$

b) $x^2 = 18 + 3x$

c) $2x^2 - 28x + 98 = 0$

WARNING: Factoring is only a useful technique to solve equations if one side of the equation is zero:

$$(\square)(\triangle) = 0 \qquad\qquad (\square)(\triangle) = 4$$

Solving quadratic equations with no x term

If a quadratic equation has x^2 terms but no x terms, we can solve it without factoring. To do this,

1. isolate the x^2 term by itself, then
2. take $\pm\sqrt{\ }$ of both sides.

IMPORTANT: Don't forget the $\pm$ here!

This method takes advantage of the following fact:

Theorem 3.26 (Inverse of a quadratic) *If $f(x) = x^2$, then $f^{-1}(x) = \pm\sqrt{x}$.*

As an arrow diagram, this concept is

§3.5 EXAMPLE 6

Solve for x in each equation:

a) $x^2 - 25 = 0$

b) $2x^2 + 7 = 5(x^2 - 2)$

b) $2x^2 + 7 = 0$

More completing the square

We saw earlier how completing the square could be used to find the vertex of a parabola. The method of completing the square is also useful for solving quadratic equations:

§3.5 EXAMPLE 7

Solve for x in each equation by completing the square:

a) $3x^2 + 12x - 7 = 0$

b) $x^2 + 10x - 3 = 0$

Solution: We have

$$h = -\frac{b}{2a} = -\frac{10}{2(1)} = -5$$

and

$$k = (-5)^2 + 10(-5) + 3 = 25 - 50 + 3 = -22,$$

so by completing the square on the left-hand side we can rewrite the equation as

$$\begin{aligned} (x+5)^2 - 22 &= 0 \\ (x+5)^2 &= 22 \\ x + 5 &= \pm\sqrt{22} \\ x &= \boxed{-5 \pm \sqrt{22}}. \end{aligned}$$

The quadratic formula

What do you do with a quadratic equation if you can't factor it and you don't want to complete the square?

Theorem 3.27 (Quadratic Formula)

$$\textit{If } ax^2 + bx + c = 0, \qquad \textit{then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular:

- *If* $b^2 - 4ac > 0$, *then the quadratic equation* $ax^2 + bx + c = 0$ *has two solutions.*
- *If* $b^2 - 4ac = 0$, *then the quadratic equation* $ax^2 + bx + c = 0$ *has one solution.*
- *If* $b^2 - 4ac < 0$, *then the quadratic equation* $ax^2 + bx + c = 0$ *has no solution.*

NOTE: The quadratic formula only works on *quadratic* equations, where *one side of the equation is zero*.

WHERE THE QUADRATIC FORMULA COMES FROM

Suppose you complete the square on an arbitrary quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a(x-h)^2 + k &= 0 \\ a(x-h)^2 &= -k \\ (x-h)^2 &= -\frac{k}{a} \\ x - h &= \pm\sqrt{-\frac{k}{a}} \\ x &= h \pm \sqrt{-\frac{k}{a}} \end{aligned}$$

Now, we know $h = -\dfrac{b}{2a}$ and $k = ah^2 + bh + c$. If you plug these formulas in for h and k and do algebra to simplify this answer, you get the quadratic formula.

§3.5 EXAMPLE 8

Solve for x in each equation:

a) $2x^2 + 3x + 1 = 6x$

b) $3x^2 - 4x = -11$

c) $-x^2 + 5x + 3 = 0$

Graphing parabolas

The most efficient method of sketching the graph of a quadratic depends on the equation you are given:

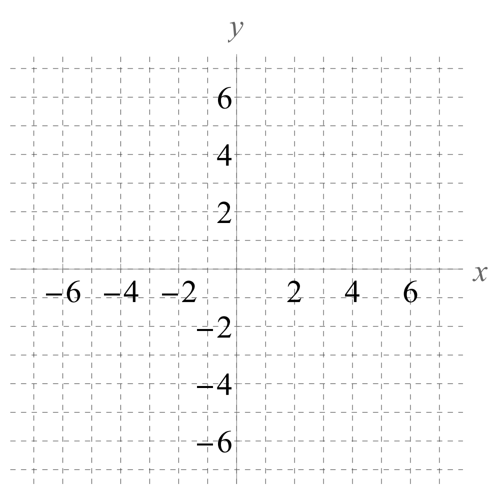
1. **If you are given the standard form** $f(x) = ax^2 + bx + c$, find the x-intercept(s) and the y-intercept and sketch the parabola (it opens up if $a > 0$ and opens down if $a < 0$). The vertex will be halfway between the x-intercepts, by the way.

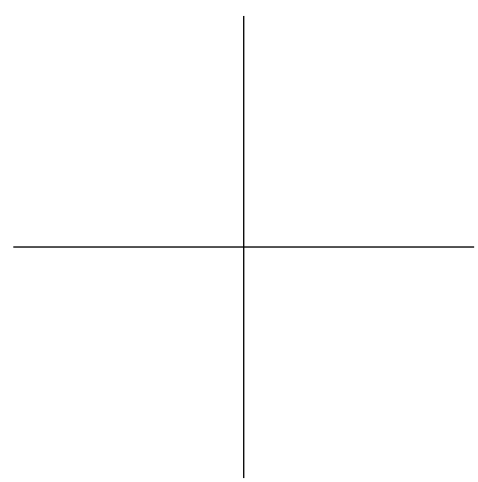
2. **If you are given the vertex form** $f(x) = a(x - h)^2 + k$, plot the vertex (h, k) and maybe also the y-intercept, and sketch the parabola (it opens up if $a > 0$ and opens down if $a < 0$). (Alternatively, just shift the parabola $y = x^2$ using transformation methods.)

§3.5 EXAMPLE 9

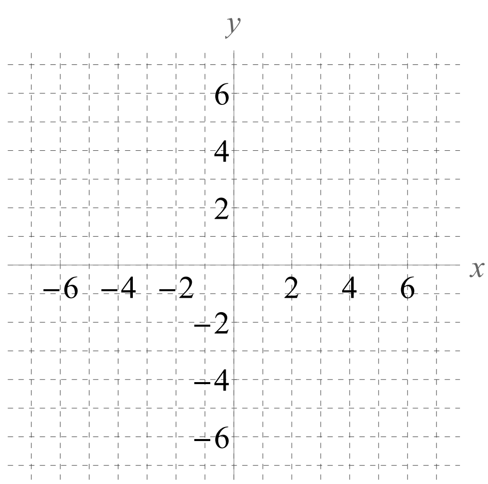
Sketch the graph of each function:

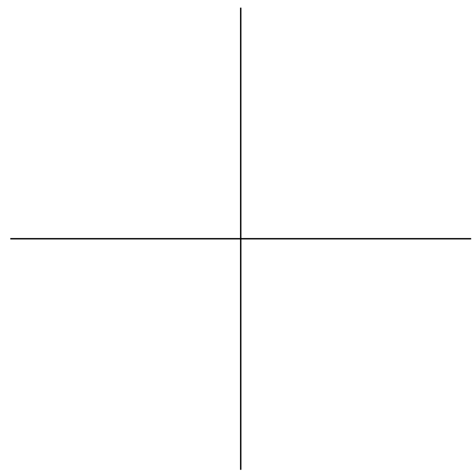
a) $f(x) = 2(x + 3)^2 - 5$



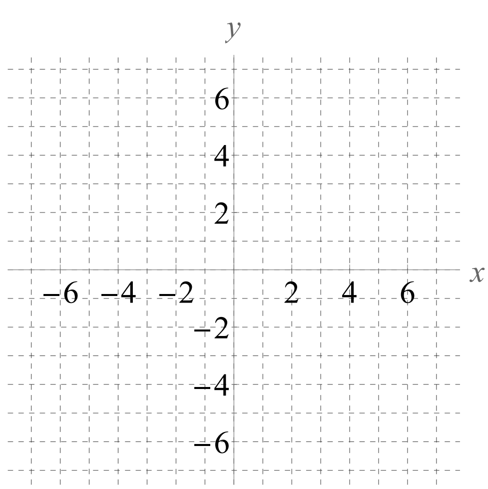


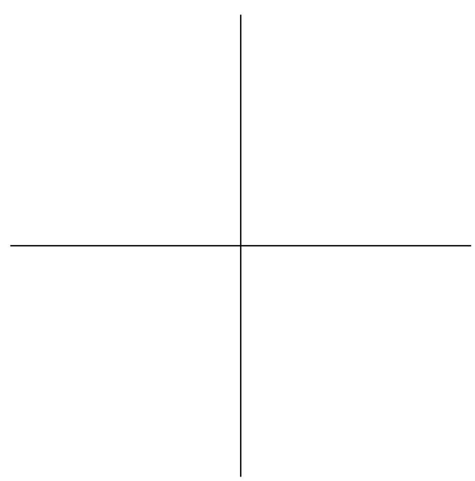
b) $f(x) = -x^2 + 7x - 6$





c) $f(x) = 2x^2 + x - 15$





3.6 Polynomial functions

CONCEPT

IF THE RATE OF CHANGE OF f IS	THEN f IS	AND f HAS THIS RULE
zero	constant	$f(x) = c$
constant	linear	$f(x) = mx + b$
linear	quadratic	$f(x) = ax^2 + bx + c$
quadratic		

If you continue with this chart indefinitely, you get a collection of functions called *polynomials*:

Definition 3.28 *A* **polynomial** *is a function* $f : \mathbb{R} \to \mathbb{R}$ *whose rule can be written as*

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_d x^d$$

for constants $a_0, a_1, ..., a_d$*, where* $a_d \neq 0$.

d *is called the* **degree** *of the polynomial.*

a_d *is called the* **leading coefficient** *of the polynomial.*

Note: Constants (which have degree 0), linear functions (degree 1), and quadratic functions (degree 2) are all polynomials.

Note: Only non-negative powers are allowed in polynomials: $\sqrt{x} = x^{1/2}$, $\frac{1}{x} = x^{-1}$, $\sin x$, etc. are not polynomials.

§3.6 EXAMPLE 1

Determine whether the given function is a polynomial. If it is, identify its degree and leading coefficient.

a) $f(x) = 7x^3 + 2x^6 - 6$

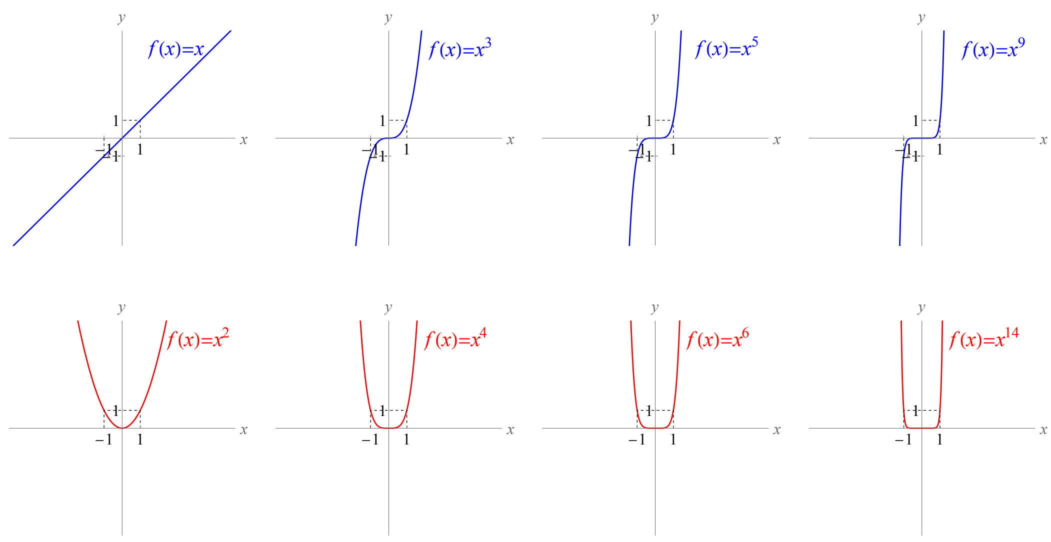
b) $h(x) = 2x^5 + 3x^{3/2} - 4$

c) $r(x) = (4x - 3)(2x^2 + 1)$

d) $f(x) = 4^x - x^4$

Power functions

The simplest polynomials are called **power functions**. These have one term and a leading coefficient of 1, so they have the form $f(x) = x^d$. Here are their graphs:



Theorem 3.29 (Properties of power functions) *Let* $f(x) = x^d$.

Domain: *the domain of every power function is* $\mathbb{R}$.

Range: *if* d *is even, then the range of* f *is* $[0, \infty)$;
if d *is odd, then the range of* f *is* $\mathbb{R}$.

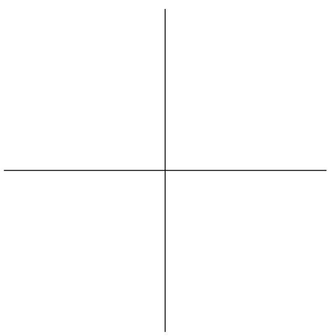
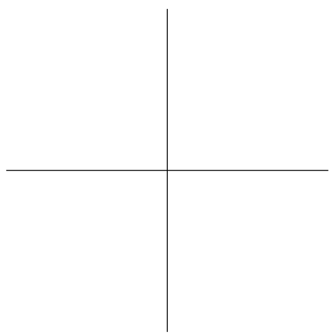
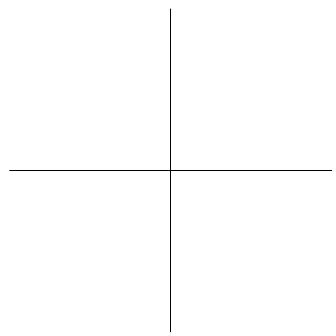
Symmetry: *if* d *is even, then the function* f *is even;*
if d *is odd, then the function* f *is odd.*

Minimum values: *if* d *is even, then* f *has a minimum value of* 0;
if d *is odd then* f *has no maximum or minimum value.*

§3.6 EXAMPLE 2

Sketch a crude graph of each function:

a) $f(x) = -x^3$ b) $g(x) = (x-2)^4$ c) $h(x) = 2x^5 + 3$

Some theory of polynomials

Closure properties

Theorem 3.30 *If* f *and* g *are polynomials, then so are* $f+g$, $f-g$, fg *and* $f \circ g$.

§3.6 EXAMPLE 3

Let $f(x) = x^2 + 4x + 3$ and $g(x) = x^3 + 2$. Compute and simplify $(f \circ g)(x)$.

Continuity and smoothness

Theorem 3.31 *The graph of any polynomial is* **continuous** *(meaning the entire graph can be drawn without lifting your writing instrument from the paper) and* **smooth** *(meaning that the graph has no sharp corners or cusps).*

Turning points

A **turning point** of a polynomial f is an x-coordinate where the graph of f either changes from increasing to decreasing, or decreasing to increasing.

Theorem 3.32 *A polynomial of degree d has at most $d - 1$ turning points.*

Fundamental Theorem of Algebra

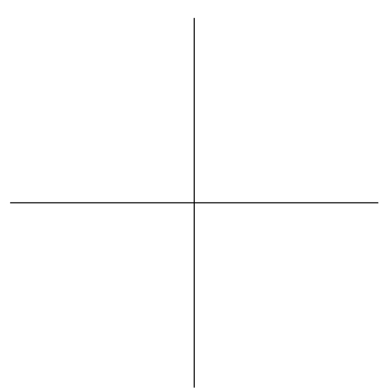
Theorem 3.33 (Fundamental Theorem of Algebra) *If f is a polynomial with degree d, then the equation $f(x) = 0$ has at most d solutions.*

Tail behavior

Earlier, we saw the graphs of $f(x) = ax^d$ (constants multiplied by power functions). As these graphs go to the left and right, the "tails" of these graphs either point upward or downward.

What about the tail behavior of a polynomial? Let's consider an example: let

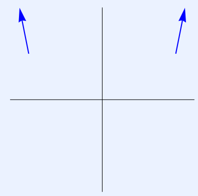
$$f(x) = -4x^3 + 3x^2 + 5x - 1.$$



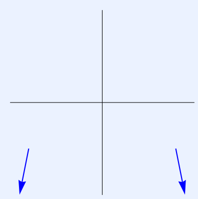
More generally, any polynomial has the same tail behavior as its highest-power term. This means:

Theorem 3.34 (Tail behavior of polynomials) *Suppose f is a polynomial with degree $d \geq 1$ and leading coefficient a_d. Then:*

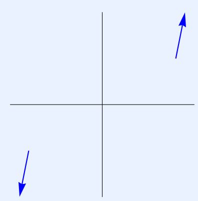
- *if d is even and $a_d > 0$, then both tails of the polynomial point upward, meaning the graph of f looks like this:*



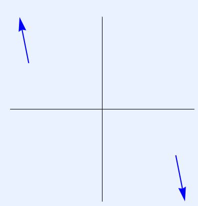
- *if d is even and $a_d < 0$, then both tails of the polynomial point downward, meaning the graph of f looks like this:*



- *if d is odd and $a_d > 0$, then the left tail of the polynomial points downward but the right tail points upward, meaning the graph of f looks like this:*



- *if d is odd and $a_d < 0$, then the left tail of the polynomial points upward but the right tail points downward, meaning the graph of f looks like this:*

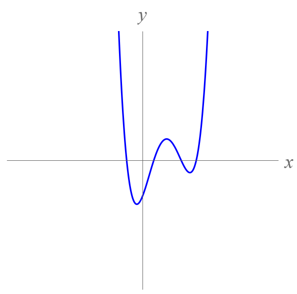


In particular, the tails of a (nonconstant) polynomial function **cannot** point sideways. As you go to the extreme left or right of a polynomial graph, the graph must point upwards or downwards.

§3.6 EXAMPLE 4

For each given graph, answer the following questions:

i. Is this the graph of a polynomial?
ii. If it is a polynomial, is its degree even or odd?
iii. If it is a polynomial, is its leading coefficient positive or negative?
iv. If it is a polynomial, what is its smallest possible degree?

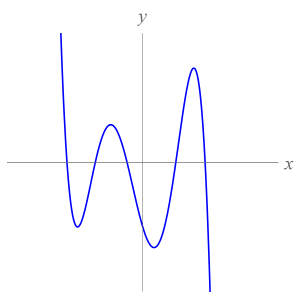
a) 

i. Polynomial?

ii. Degree even or odd?

iii. LC positive or negative?

iv. Smallest possible degree?

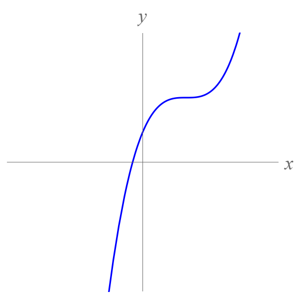
b) 

i. Polynomial?

ii. Degree even or odd?

iii. LC positive or negative?

iv. Smallest possible degree?

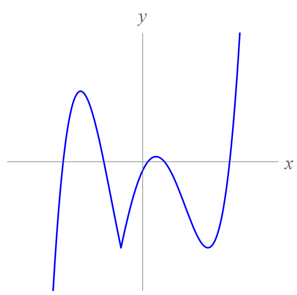
c) 

i. Polynomial?

ii. Degree even or odd?

iii. LC positive or negative?

iv. Smallest possible degree?

d) 

i. Polynomial?

ii. Degree even or odd?

iii. LC positive or negative?

iv. Smallest possible degree?

3.7 Root functions

Review of fractional exponents

Definition of root function

Graphs of root functions

Properties of root functions

Power and radical rules

Inverse properties

3.8 Rational functions

RECALL

Negative exponents are treated as [   ]:

$x^{-3} =$ $\qquad\qquad$ $3x + x^{-2} =$

A function made up of integer exponents is called a *rational function*. Any such function can be combined as in the second example above to be put in a standard form:

Definition 3.35 *A* **rational function** $f : \mathbb{R} \to \mathbb{R}$ *is any function whose rule is the quotient of two polynomials.*

In other words, f is rational if it has a rule of the form

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + ... + a_m x^m}{b_0 + b_1 x + b_2 x^2 + ... + b_n x^n}.$$

EXAMPLES: $f(x) = \dfrac{x^3 - 3x + 4}{2x^5 - 7x^2 - 3}$ $\qquad$ $g(x) = x^{-3}$ $\qquad$ $h(x) = \dfrac{\pi x^8 + 7x^3 - \sqrt{17}x}{x + 4}$

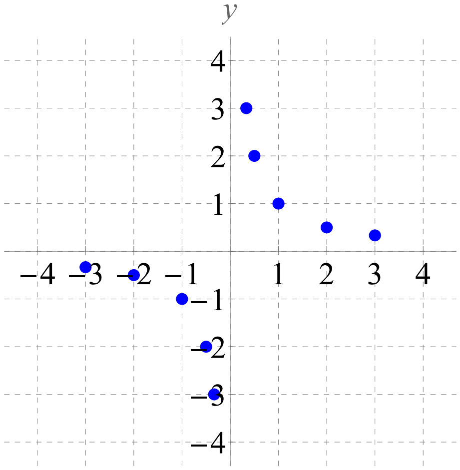
The reciprocal function

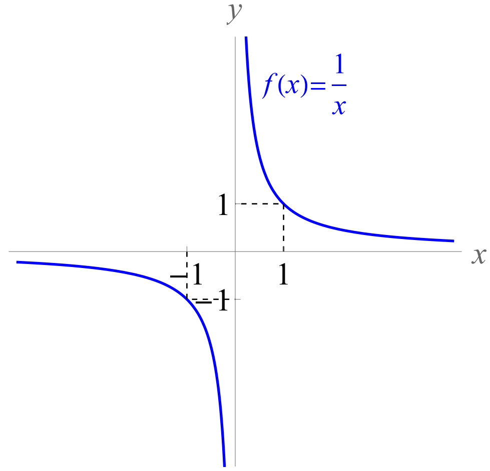
Here is the most important rational function:

Definition 3.36 *The* **reciprocal function** *is the function* $f(x) = \dfrac{1}{x} = x^{-1}$.

Let's graph this function using a table of values:

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3
$f(x) = \frac{1}{x}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	DNE	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$





The shape of this graph, which has two disconnected pieces, is called a **hyperbola**.

Theorem 3.37 (Properties of the reciprocal function) *Let* $f(x) = \dfrac{1}{x}$. *Then:*

Domain *The domain of* f *is* $\mathbb{R} - \{0\}$.

Range: *The range of* f *is* $\mathbb{R} - \{0\}$.

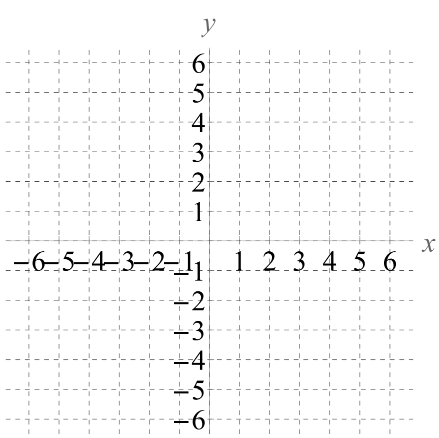
Symmetry: f *is odd.*

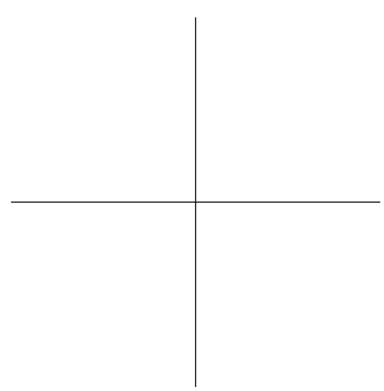
Inverse: f *is one-to-one, and the inverse of* $f(x) = \dfrac{1}{x}$ *is itself:* $f^{-1}(x) = \dfrac{1}{x}$.

§3.8 Example 1

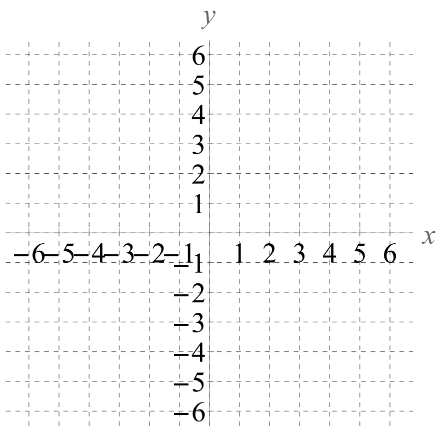
Diagram each function, then sketch its graph:

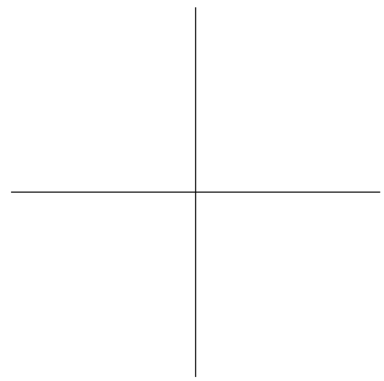
a) $f(x) = \dfrac{1}{x} + 4$



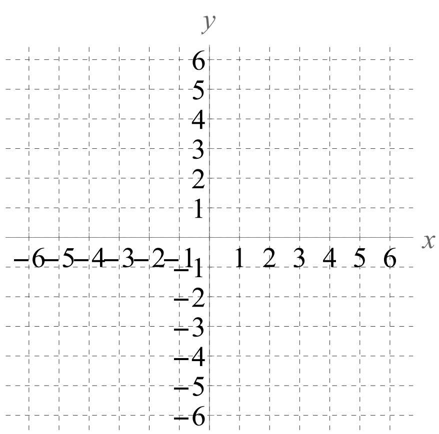


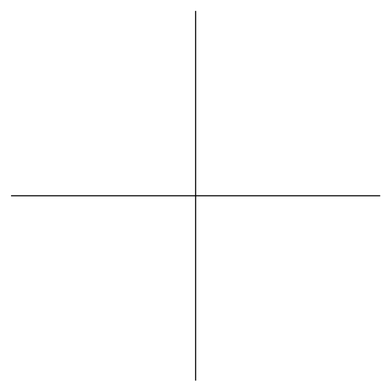
b) $g(x) = \dfrac{1}{x-5}$





c) $k(x) = -\dfrac{1}{x+2} + 3$

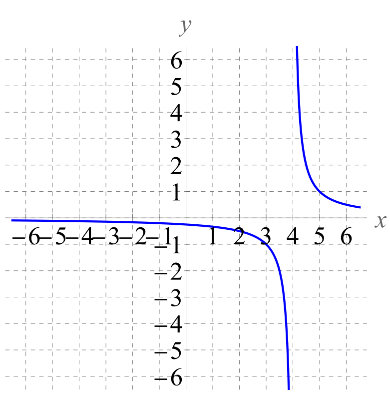




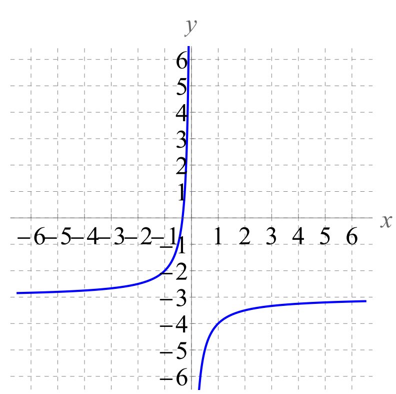
§3.8 Example 2

Write a rule for each function whose graph is shown here:

a)

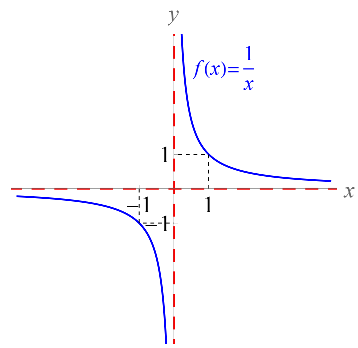


b)



Asymptotes

Let's take another look at the graph of $f(x) = \dfrac{1}{x}$:

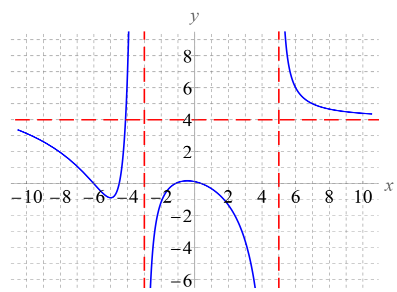


Notice that there are vertical and horizontal lines that the graph of f appears to "merge" into. These lines are called **vertical asymptotes (VA)** and **horizontal asymptotes (HA)** of f. The graph of f *never actually touches the vertical asymptotes*, but gets closer and closer to them.

In MATH 130, we want to be able to write equations of horizontal and vertical asymptotes given a graph:

§3.8 EXAMPLE 3

Write the equations of any horizontal and/or vertical asymptotes of this function:



Computing the asymptotes of a function from its formula is a task you learn how precisely to do in Calculus 1, but here are some general rules for rational functions that, if you remember them, can help you shortcut some calculus computations:

Theorem 3.38 (VA of rational functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function. The VA of f are lines of the form $x = c$, where c makes the denominator of f zero but makes the numerator of f nonzero.*

A graph of a rational function ***never*** *touches/crosses any of its VA.*

Theorem 3.39 (HA of rational functions) *Let $f : \mathbb{R} \to \mathbb{R}$ be a rational function, i.e. has form*

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + ... + a_2x^2 + a_1x + a_0}{b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + ... + b_2x^2 + b_1x + b_0}.$$

Then:

1. *If $m < n$ (i.e. largest power in numerator < largest power in denominator), then $y = 0$ is the HA of f.*
2. *If $m = n$ (i.e. largest powers in numerator and denominator are equal), then $y = \frac{a_m}{b_n}$ is the HA of f.*
3. *If $m > n$ (i.e. largest power in numerator > largest power in denominator), then f has no HA.*

In the first and second situations, the graph of f will appear to merge into the HA at the extreme left and extreme right edges of the graph.

It is possible for the graph of a rational function to cross its HA (but not its VA(s)) one or more times.

§3.8 Example 4

Compute the horizontal and vertical asymptotes of each function:

a) $f(x) = \dfrac{x-3}{x^2 - 5x + 4}$

b) $g(x) = \dfrac{2x^2 - 18}{x^2 - 5x - 24}$

Closure properties

If you build more complicated functions out of rational functions, you will end up getting a rational function:

Theorem 3.40 *If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both rational functions, and if r is any constant, then the following functions are all rational:*

$$f+g \qquad f-g \qquad fg \qquad \frac{f}{g} \qquad rf \qquad f\circ g.$$

To see why this is, we need to learn how to manipulate rational expressions. These manipulations are based on how we manipulate fractions.

Manipulation of rational expressions

To **simplify** a rational expression, factor the top and bottom and cancel factors.

$$\frac{A}{C}\cdot\frac{B}{A} = \frac{B}{C}.$$

WARNING: In a fraction or rational function, you can only cancel **factors** (things being multiplied), **not terms** (things being added). *You must completely factor both the top and bottom of a rational expression before cancelling anything.*

ALWAYS FACTOR BEFORE YOU CANCEL!

Here are two typical examples of illegal "cancelling":

$$\frac{x^2+3}{x^2} = \qquad\qquad \frac{x^2+3}{x^2} =$$

§3.8 EXAMPLE 4

Simplify each expression:

a) $\dfrac{x^2-3x-10}{x^2-12x+35}$

b) $\dfrac{x^3+10x^2+24}{x^4+x^3-12x^2}$

To **multiply** rational expressions, multiply the numerators and multiply the denominators:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}.$$

It is often easiest to factor A, B, C and D first and cancel, to make the multiplication easier.

§3.8 EXAMPLE 5

a) Compute and simplify $\frac{5x+15}{x-2} \cdot \frac{x^2-4}{10x-20}$.

b) Suppose $f(x) = \frac{x+1}{x-3}$ and $g(x) = \frac{x-5}{2x+7}$. Compute and simplify $(fg)(x)$.

To **divide** rational expressions, flip the divisor over and multiply:

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}$$

WARNING: $\frac{A/B}{C} = \frac{\frac{A}{B}}{C}$ means $\frac{A}{B} \div C = \frac{A}{B} \cdot \frac{1}{C}$;
$\frac{A}{B/C} = \frac{A}{\frac{B}{C}}$ means $A \div \frac{B}{C} = \frac{A}{1} \cdot \frac{C}{B}$.

So you should **never** write this:

§3.8 EXAMPLE 6

Compute and simplify $\dfrac{x^2-9}{x+4} \div \dfrac{x^2+5x+4}{x^2+x-6}$.

To **add** or **subtract** rational expressions, find a common denominator (by creatively multiplying each term by 1) and then add the numerators:

$$\frac{A}{B} \pm \frac{C}{B} = \frac{A \pm C}{B} \qquad \frac{A}{B} + \frac{C}{D} = \frac{AD}{BD} + \frac{CB}{BD} = \frac{AD+CB}{BD}.$$

Factoring helps find the smallest common denominator.

§3.8 EXAMPLE 7

a) Compute and simplify $\dfrac{x}{x+4} + \dfrac{3}{x-1}$.

b) Let $f(x) = \dfrac{1}{x^2+2x+1}$ and $g(x) = \dfrac{2}{x^2-1}$. Compute and simplify $(f-g)(x)$.

Compound fractions

Definition 3.41 *A* **compound fraction** *is a fraction whose numerator and/or denominator themselves contain fractions.*

EXAMPLES: $\dfrac{\frac{3}{5}-\frac{2}{3}}{5-\frac{1}{9}}$ $\dfrac{\frac{4}{x+2}+\frac{3x-1}{x+5}}{\frac{x^2+3}{x-5}+\frac{x}{x^2+x-3}}$ $\dfrac{\frac{3(x+h)}{3(x+h+2)}-\frac{3x}{3x+2}}{h}$

To **simplify a compound fraction**, creatively multiply through by 1, where the "1" is a fraction containing all the "small denominators" of the compound fraction.

§3.8 EXAMPLE 8

a) Simplify $\dfrac{\frac{3}{x}-2}{3+\frac{1}{x+1}}$.

b) Simplify $\dfrac{\frac{2}{x+h} - \frac{2}{x}}{h}$.

c) Let $f(x) = \dfrac{x+1}{x-3}$ and $g(x) = \dfrac{x+2}{x-1}$. Compute and simplify $(g \circ f)(x)$.

d) Let $F(x) = (x+4)^{-1}$ and $H(x) = x+3$. Compute and simplify $(F \circ H \circ F)(x)$.

Solution: Notice first that $F(x) = (x+4)^{-1} = \dfrac{1}{x+4}$. Therefore

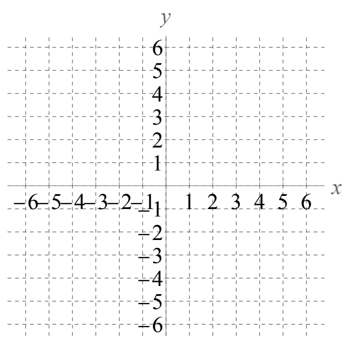
$$\begin{aligned}
(F \circ H \circ F)(x) &= F(H(F(x))) \\
&= F\left(H\left(\frac{1}{x+4}\right)\right) \\
&= F\left(\frac{1}{x+4} + 3\right) \\
&= \frac{1}{\frac{1}{x+4} + 3 + 4} \\
&= \frac{1}{\frac{1}{x+4} + 7} \\
&= \frac{1}{\left(\frac{1}{x+4} + 7\right)} \cdot \frac{(x+4)}{(x+4)} \\
&= \frac{x+4}{1 + 7(x+4)} \\
&= \frac{x+4}{1 + 7x + 28} \\
&= \boxed{\frac{x+4}{7x+29}}.
\end{aligned}$$

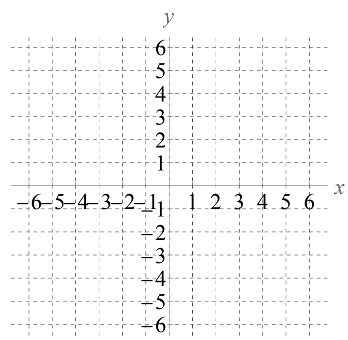
3.9 Absolute value function

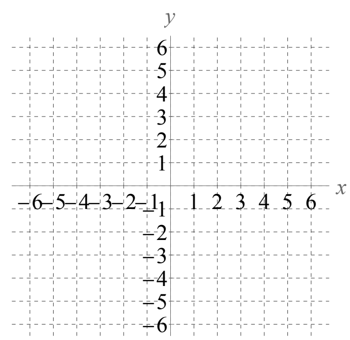
Definition 3.42 *The* **absolute value** *function is the function* $\mathbb{R} \to \mathbb{R}$ *whose rule is*

$$|x| = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0 \end{cases}$$

We can sketch the graph of $|x|$ from its piecewise definition:



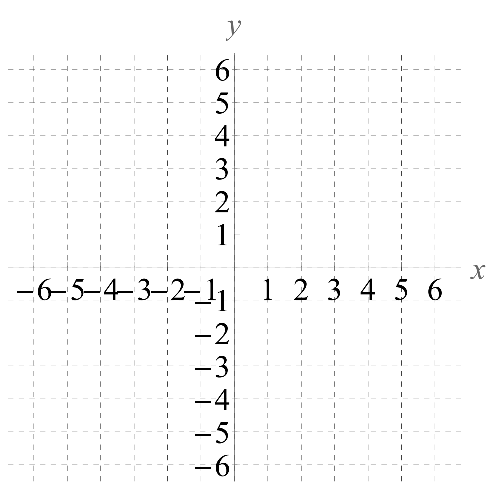


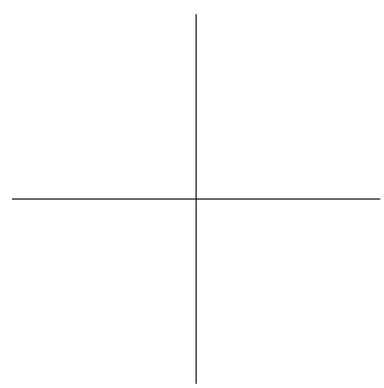


§3.9 EXAMPLE 1

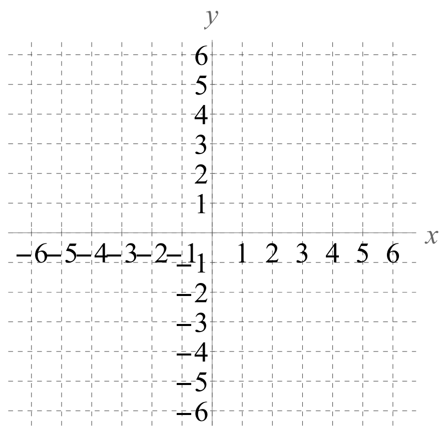
Diagram each function, then sketch its graph:

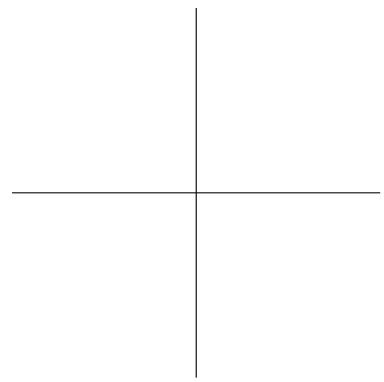
a) $F(t) = |t - 4|$



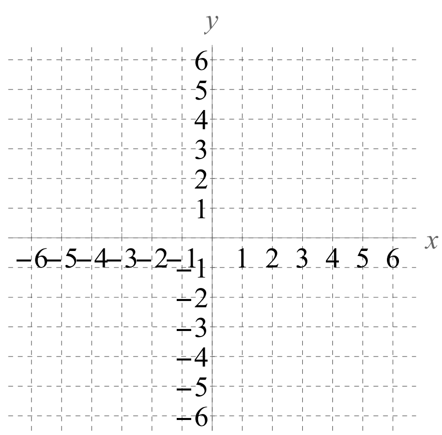


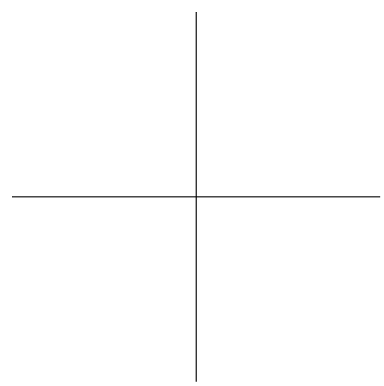
b) $g(x) = |x + 3| + 1$





c) $K(x) = -2|x|$





Theorem 3.43 (Properties of absolute value) *Let* $f(x) = |x|$.

Domain *The domain of* f *is* $\mathbb{R}$.

Range: *The range of* f *is* $[0, \infty)$.

Symmetry: f *is even.*

Inverse: *The inverse of* $f(x) = |x|$ *is the multifunction* $f^{-1}(x) = \pm x$.

Multiplicative: $|xy| = |x|\,|y|$.

Relationship with square and square root: $|x| = \sqrt{x^2}$.

Distance computation: $|x| =$ *the distance from* x *to* 0 *on a number line.*

Signum function

Here is another function used to illustrate some calculus concepts:

Definition 3.44 *The* **signum function** *is the function* $f(x) = \dfrac{|x|}{x}$.

OBSERVE

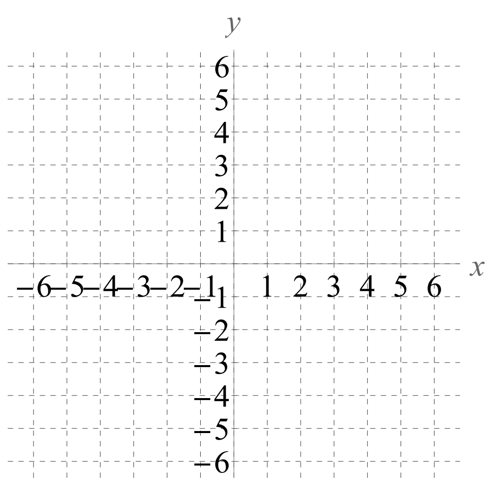
Let f be the signum function.

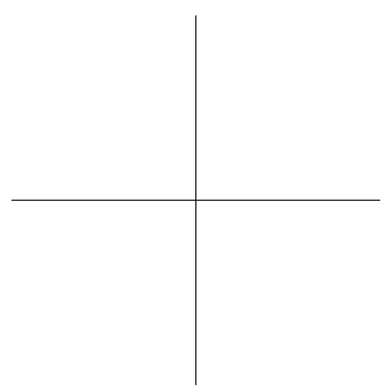
- if $x > 0$, $f(x) = \dfrac{|x|}{x} =$
- if $x < 0$, $f(x) = \dfrac{|x|}{x} =$

so the signum function also has the piecewise definition

$$\frac{|x|}{x} = \begin{cases} & \text{if } x < 0 \\ & \text{if } x > 0 \end{cases}$$

which means its graph looks like this:





3.10 Semicircles

OBSERVE

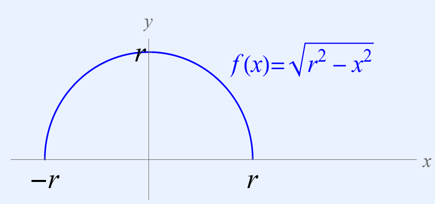
The equation of a circle of radius r centered at $(0, 0)$ is

QUESTION

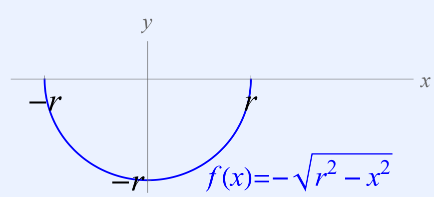
Is a circle the graph of a function $y = f(x)$? Why or why not?

To summarize:

Theorem 3.45 (Semicircles) *The graph of $f(x) = \sqrt{r^2 - x^2}$ is the top half of a circle of radius r centered at the origin:*



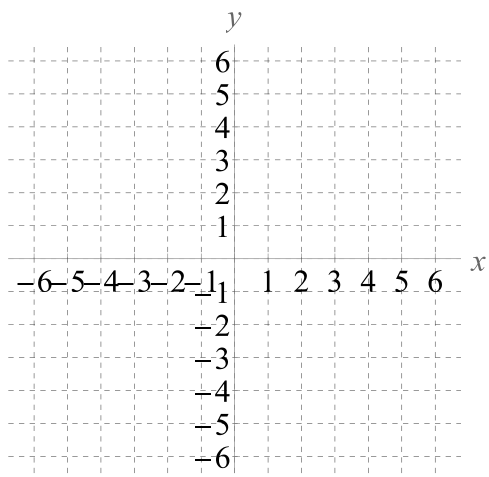
The graph of $f(x) = -\sqrt{r^2 - x^2}$ is the bottom half of the same circle:

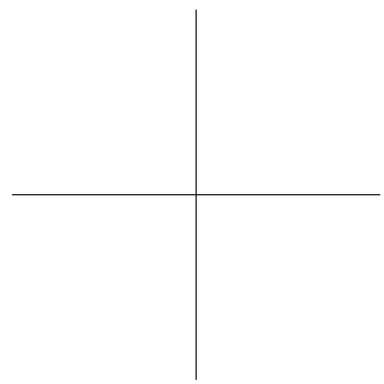


§3.10 EXAMPLE 1

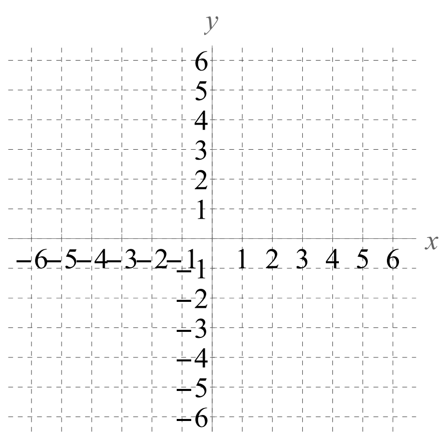
Sketch a graph of each function:

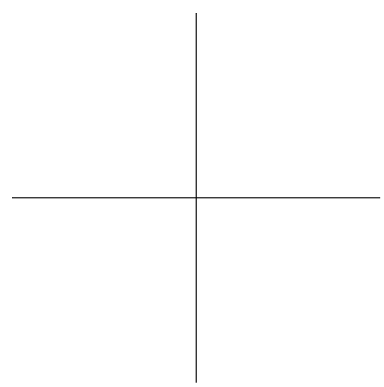
a) $f(x) = \sqrt{25 - x^2}$



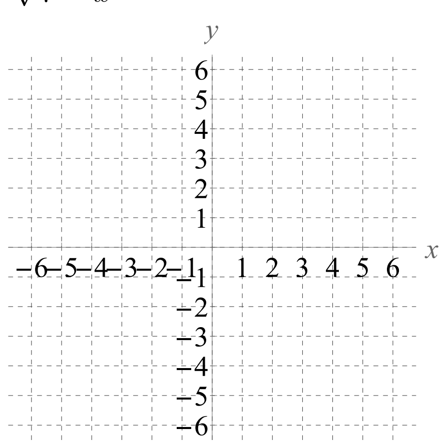


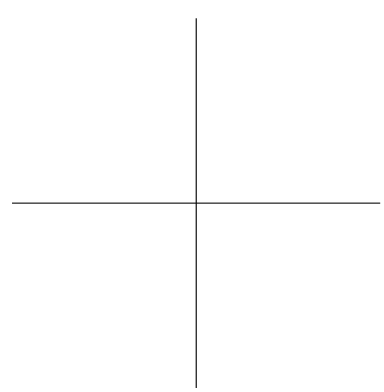
b) $g(x) = -\sqrt{9 - x^2}$



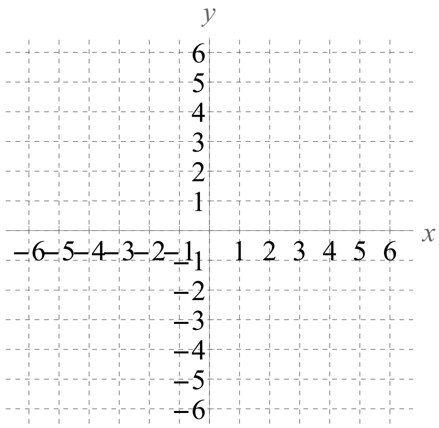


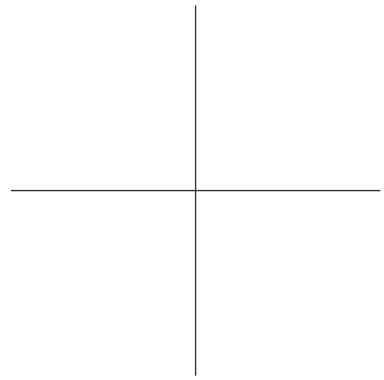
c) $h(x) = \sqrt{7 - x^2}$



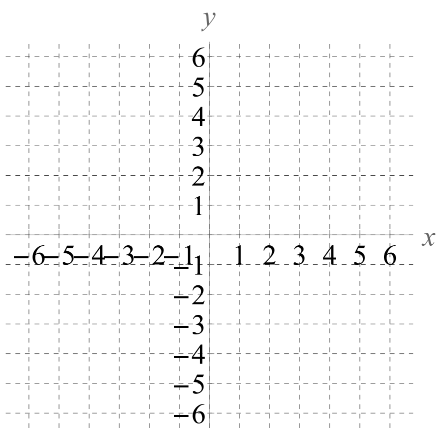


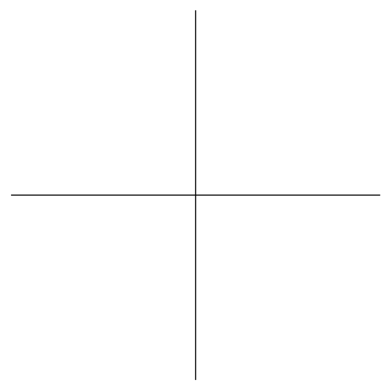
d) $j(x) = \sqrt{16 - (x-2)^2}$





e) $j(x) = \sqrt{4 - x^2} - 3$

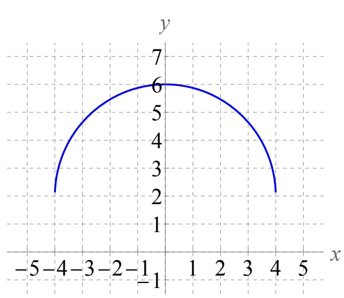




§3.10 EXAMPLE 2

Write a rule for each function whose graph is shown here:

a)



b)