Old MATH 230 Exams

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Contents

Chapter 1

General information about these exams

These are the exams I have given between 2010 and 2016 in Calculus 2 courses. Each exam is given here, followed by what I believe are the solutions (there may be some number of computational errors or typos in these answers).

I have edited these exams to remove questions that do not match the current syllabus of MATH 230; so some of them may contain a less than expected number questions.

Note that this calculus course has been revised several times over the years, and what was on “Exam 1” in past years may not match what is on “Exam 1” now. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like “(3.2)”). That means that question can be solved using material from that section (or from earlier sections) in the 2023 edition of my *Calculus 2 Lecture Notes*.

Last, my exam-writing style has evolved over the years; generally speaking, the more recent the exam, the more likely you are to see something similar on one of your tests. In particular, the exams prior to Fall 2012 were given at a college other than Ferris, where the Calculus II course content doesn’t exactly match what is taught at FSU.

Chapter 2

Exams from 2010 to 2011

2.1 Fall 2010 Exam 1

1. (2.4, 2.5) Each of the following integrals can be done with an appropriate u-substitution. Give the substitution which is most appropriate for the integral (a sample answer might be something like "$u = x^2 - 5$"). You do not need to calculate the integrals.

   a) $\displaystyle\int \cos^2 x \sin^5 x \, dx$

   b) $\displaystyle\int x^3 \sec^2(x^4 - 2) \, dx$

   c) $\displaystyle\int \frac{e^x + e^{-x}}{e^x - e^{-x}} \, dx$

2. (2.8) Calculate one of the following two integrals:

$$\int \frac{x+4}{x(x-1)^2} \, dx \qquad\qquad \int \frac{3x^2+1}{x^3+x} \, dx$$

3. (2.6) Calculate one of the following two integrals:

$$\int x^2 e^{5x} \, dx \qquad\qquad \int x\sqrt{6-x} \, dx$$

Solutions

1. a) Since the power on the sine is odd, substitute $u = \cos x$.

   b) $u = x^4 - 2$.

   c) Since the derivative of the denominator is the numerator, substitute $u =$ the denominator, i.e. $u = e^x - e^{-x}$.

2. a) Perform a partial fraction decomposition. First write

$$\frac{x+4}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2},$$

then write all terms with a common denominator. Clearing this common denominator, we obtain

$$x + 4 = A(x-1)^2 + B(x)(x-1) + C(x).$$

Substitute $x = 0$ to obtain $4 = A$; substitute $x = 1$ to obtain $C = 5$, and substitute $x = 2, A = 4, C = 5$ to obtain $6 = 4 + 2B + 10$, i.e. $B = -4$. Therefore

$$\int \frac{x+4}{x(x-1)^2}\,dx = \int \left(\frac{4}{x} - \frac{4}{x-1} + \frac{5}{(x-1)^2}\right)dx = 4\ln|x| - 4\ln|x-1| - \frac{5}{x-1} + C.$$

   b) The slick way to do this is with a $u-$substitution: let $u = x^2 + x$, then $du = (3x^2 + 1)\,dx$ so the integral reduces to $\int \frac{1}{u}\,du$. The answer is then

$$\ln|u| + C = \ln|x^3 + x| + C.$$

Alternatively ,you can perform a partial fraction decomposition. First write

$$\frac{3x^2+1}{x^3+x} = \frac{3x^2+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

then write all terms with a common denominator. Clearing this common denominator, we obtain

$$3x^2 + 1 = A(x^2+1) + (Bx + C)x.$$

Combine the like terms on the right-hand side to obtain

$$3x^2 + 1 = (A+B)x^2 + Cx + A.$$

Now by equating coefficients we see $A + B = 3$, $C = 0$, $A = 1$. Therefore $B = 2$ and

$$\int \frac{3x^2+1}{x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{2x}{x^2+1}\right)dx.$$

The second integral is done with the $u-$substitution $u = x^2 + 1$; the whole answer is

$$\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x| + \ln|x^2+1| + C.$$

Notice that by a logarithm rule, this can be rewritten in the same form as the earlier answer.

3. a) Use integration by parts with $u = x^2$ and $dv = e^{5x}\,dx$; in this case $du = 2x\,dx$ and $v = \frac{1}{5}e^{5x}$ so the integral becomes, by the IBP formula,

$$uv - \int v\,du = \frac{1}{5}x^2e^{5x} - \int \frac{2}{5}xe^{5x}\,dx.$$

For the remaining integral, use integration by parts again with $u = \frac{2}{5}x$ and $dv = e^{5x}\,dx$ so that $du = \frac{2}{5}\,dx$ and $v = \frac{1}{5}e^{5x}$. So by the IBP formula the integral on the right above becomes

$$uv = \int v\,du = \frac{2}{25}xe^{5x} - \int \frac{2}{25}e^{5x}\,dx = \frac{2}{25}xe^{5x} - \frac{2}{125}e^{5x}.$$

Finally the whole answer is

$$\frac{1}{5}x^2e^{5x} - \left(\frac{2}{25}xe^{5x} - \frac{2}{125}e^{5x}\right) + C.$$

b) Use the elementary substitution $u = 6 - x$, $du = -dx$ to obtain

$$\begin{aligned}
\int (6-u)\sqrt{u}(-1)\,dx &= \int \left(-6u^{1/2} + u^{3/2}\right)\,du \\
&= -6(2/3)u^{3/2} + (2/5)u^{5/2} + C \\
&= -4(6-x)^{3/2} + \frac{2}{5}(6-x)^{5/2} + C.
\end{aligned}$$

2.2 Fall 2010 Exam 2

1. a) (3.3) Determine whether or not the following integral converges or diverges (explain your answer):

$$\int_1^\infty \frac{2+e^{-x}}{\sqrt{x}}\,dx$$

   b) (4.4) Write an expression involving one or more integral(s) that gives the length of the curve $y = \tan x$ from $x = 0$ to $x = \pi/4$.

2. Let R be the region enclosed by the graphs of $y = \sqrt{x} + 1$, $y = \frac{1}{3}(x-1)$, and $y = 1 - x$.

   a) (4.1) Write an expression involving one or more integral(s) with respect to x that gives the area of R.

   b) (4.1) Write an expression involving one or more integral(s) with respect to y that gives the area of R.

3. Let R be the region enclosed by the graphs of $y = x^2 + 1$ and $y = 3x + 1$.

   a) (4.2) Let S_1 be the solid formed by revolving R around the $x-$axis. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_1.

   b) (4.2) Let S_2 be the solid formed by revolving R around the line $x = -3$. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_2.

   c) (4.2) Write an expression involving one or more integral(s) with respect to y that gives the volume of the soild S_2 described in part (b).

   d) (4.2) Let S_3 be the solid built over base R where cross-sections to the solid parallel to the $x-$axis are semicircles whose diameters are in R. Write an expression involving one or more integral(s) (with respect to any variable you like) that gives the volume of S_3.

4. (4.5) Consider a wire of length 6 in whose density x inches from the left endpoint of the wire is given by $\delta(x) = 2x + 1$ lb/in. Write an expression involving one or more integral(s) which gives how far from the left endpoint of the wire its center of mass is.

Solutions

1. a) Observe that for any x, $e^{-x} \geq 0$ so $2 + e^{-x} \geq 2$ and therefore

$$\frac{2 + e^{-x}}{\sqrt{x}} \geq \frac{2}{\sqrt{x}}.$$

Now we showed in class that $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, so $\int_1^\infty \frac{2}{\sqrt{x}}\,dx$ diverges as well. Therefore by the Comparison Test, the given integral diverges as it is greater than or equal to a divergent integral.

b) The arc length formula is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$; setting $f(x) = \tan x$ we have $f'(x) = \sec^2 x$ so $L = \int_0^{\pi/4} \sqrt{1 + \sec^4 x}\,dx$.

2. The region R has three vertices; two of the vertices are $(1, 0)$ and $(0, 1)$. To find the third vertex, set $\sqrt{x} + 1 = \frac{1}{3}(x - 1)$. Multiply both sides by 3, then subtract 3 from both sides to obtain $3\sqrt{x} = x - 4$. Now square both sides to obtain $9x = x^2 - 8x + 16$; then subtract $9x$ from both sides and factor to get $0 = (x - 16)(x - 1)$. This gives the solution $x = 16$ (ignore $x = 1$ since it is not a solution to the original equation). Plugging in $x = 16$ to the equation $\sqrt{x} + 1$, we see $y = 5$ so the third vertex is $(16, 5)$.

a) From $x = 0$ to $x = 1$, the top function is $\sqrt{x} + 1$ and the bottom function is $1 - x$. From $x = 1$ to $x = 16$, the top function is $\sqrt{x} + 1$ and the bottom function is $\frac{1}{3}(x - 1)$. So the area is given by

$$A = \int_0^1 \left[(\sqrt{x} + 1) - (1 - x)\right]\,dx + \int_1^{16} \left[(\sqrt{x} + 1) - \frac{1}{3}(x - 1)\right]\,dx.$$

b) Solve all the equations for x to obtain $x = (y - 1)^2$, $x = 3y + 1$ and $x = 1 - y$. From $y = 0$ to $y = 1$, the right-most function is $x = 3y + 1$ and the left-most function is $x = 1 - y$. From $y = 1$ to $y = 16$, the right-most function is $x = 3y + 1$ and the left-most function is $x = (y - 1)^2$. So the area is given by

$$A = \int_0^1 \left[(3y + 1) - (1 - y)\right]\,dy + \int_1^5 \left[(3y + 1) - (y - 1)^2\right]\,dy.$$

3. The vertices of the region R are $(0, 1)$ and $(3, 10)$ (seen by setting $x^2 + 1 = 3x + 1$ and solving for x).

a) Since the direction of integration is parallel to the axis of revolution, we use the washer method to obtain

$$V = \int_a^b A(x)\,dx = \int_0^3 \pi[R^2 - r^2]\,dx = \int_0^3 \pi\left[(3x + 1)^2 - (x^2 + 1)^2\right]\,dx.$$

b) Since the direction of integration is perpendicular to the axis of revolution, we use the shell method to obtain

$$V = \int_a^b SA(x)\,dx = \int_0^3 2\pi rh\,dx = \int_0^3 2\pi(x + 3)[(3x + 1) - (x^2 + 1)]\,dx.$$

c) Since we are integrating with respect to y, we solve each equation for x and get $x = \frac{1}{3}(y-1)$ (the left-most function) and $x = \sqrt{y-1}$ (the right-most function). Since the direction of integration is parallel to the axis of revolution, we use the washer method to obtain

$$V = \int_a^b A(y)\,dy = \int_1^{10} \pi[R^2 - r^2]\,dy = \int_1^{10} \pi\left[\left(\frac{1}{3}(y-1)+3\right)^2 - \left(\sqrt{y-1}+3\right)^2\right]\,dx.$$

d) Since the known cross-sections are parallel to the $x-$axis, they are perpendicular to the $y-$direction so we must integrate with respect to y. At height y, the cross-section is a semicircle with diameter $\frac{1}{3}(y-1) - \sqrt{y-1}$, so it has area $A(y) = \pi r^2 = \frac{\pi}{4}\left[\frac{1}{3}(y-1) - \sqrt{y-1}\right]^2$. Finally,

$$V = \int_a^b A(y)\,dy = \int_1^{10} \frac{\pi}{4}\left[\frac{1}{3}(y-1) - \sqrt{y-1}\right]^2\,dy.$$

4. This is a one-dimensional problem, we set our axis so that the wire runs from $x = 0$ to $x = 6$, then $\delta(x) = 2x + 1$. Now the moment of the wire about the origin is $M_0 = \int_0^6 x\delta(x)\,dx = \int_0^6 x(2x+1)\,dx$ and the mass of the wire is $M = \int_0^6 \delta(x)\,dx = \int_0^6 (2x+1)\,dx$. Finally the center of mass is

$$\overline{x} = \frac{M_0}{M} = \frac{\int_0^6 x(2x+1)\,dx}{\int_0^6 (2x+1)\,dx}.$$

2.3 Fall 2010 Exam 3

1. a) (7.2) State the Alternating Series Test.

   b) (7.3) Define what it means for a series to be *absolutely convergent*.

   c) (7.3) What is the main theoretical reason why we care whether or not a series converges absolutely (as opposed to just knowing that it converges)?

2. a) (6.2) Find the exact sum of the infinite series:

$$\sum_{n=2}^{\infty} \frac{3 \cdot 2^{3n}}{10^n}$$

   b) (6.2) Find the exact sum of the infinite series:

$$8 - 4 + 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + ...$$

3. (7.4) Determine whether or not the following infinite series converge absolutely, converge conditionally, or diverge. Justify your reasoning.

   a) $\displaystyle\sum_{n=4}^{\infty} \frac{6 + n^5}{n^8}$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^7 - 3n^5}{n^7 + 11n^4 + 6}$

4. (8.3) Determine the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{(2n)!}.$$

5. Let $f(x) = 3x\cos(2x^8)$.

   a) (8.1) Write the Taylor series (centered at 0) of $f(x)$.

   b) (8.2) Find $f^{(17)}(0)$, the 17^{th} derivative of f at zero.

   c) (8.2) Find $f^{(2010)}(0)$, the 2010^{th} derivative of f at zero.

Solutions

1. a) The **Alternating Series Test** says: Let $\sum_{n=1}^{\infty}$ be an infinite series. If the series is alternating, if $|a_n| \geq |a_{n+1}|$ for all n, and if $\lim_{n\to\infty} a_n = 0$, then the series converges.

   b) An infinite series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

   c) The terms of an absolutely convergent series can be rearranged and/or regrouped arbitrarily without affecting the sum of the series. This is not the case for conditionally convergent series.

2. a) This is a geometric series, so we write it in the standard form $a_0 \sum_{n=0}^{\infty} r^n$; once written this way the sum is $a_0 \left(\frac{1}{1-r}\right)$ provided $|r| < 1$. Here, we have

$$\sum_{n=2}^{\infty} \frac{3 \cdot 2^{3n}}{10^n} = 3\sum_{n=2}^{\infty} \frac{8^n}{10^n} = 3\left(\frac{8}{10}\right)^2 \sum_{n=0}^{\infty} \left(\frac{8}{10}\right)^n = 3\left(\frac{64}{100}\right)\left(\frac{1}{1-\frac{8}{10}}\right) = \frac{3 \cdot 64}{20} = \frac{48}{5}.$$

   The reason why the second and third terms above are equal is that, for any r, we have:

$$\sum_{n=2}^{\infty} r^n = r^2 + r^3 + r^4 + \ldots = r^2(1 + r + r^2 + \ldots) = r^2 \sum_{n=0}^{\infty} r^n.$$

   b) Working in a similar fashion to part (a):

$$8-4+2-1+\ldots = 8\left[1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \ldots\right] = 8\left(\frac{1}{1-\left(\frac{-1}{2}\right)}\right) = \frac{16}{3}.$$

3. a)

$$\sum_{n=4}^{\infty} \frac{6+n^5}{n^8} = \sum_{n=4}^{\infty} \left(\frac{6}{n^8} + \frac{n^5}{n^8}\right) = \sum_{n=4}^{\infty} \left(\frac{6}{n^8} + \frac{1}{n^3}\right).$$

   Now $\sum_{n=4}^{\infty} \frac{6}{n^8}$ and $\sum_{n=4}^{\infty} \frac{1}{n^3}$ both converge (the first is a constant times a $p-$series where $p = 8$; the second is a $p-$series with $p = 3$), so their sum, which is the original series, converges as well. Since the series given in the problem is positive, it converges absolutely.

   b) This series diverges by the n^{th} Term Test since

$$\lim_{n\to\infty} \frac{n^7 - 3n^5}{n^7 + 11n^4 + 6} = 1.$$

4. Using Abel's Formula,

$$R = \lim_{n\to\infty} \frac{\left|\frac{1}{(2n)!}\right|}{\left|\frac{1}{(2(n+1))!}\right|} = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} = \lim_{n\to\infty} (2n+2)(2n+1) = \infty.$$

   Since $R = \infty$, this series converges absolutely for all x by the Cauchy-Hadamard Theorem.

5. Let $f(x) = 3x\cos(2x^8)$.

   a) We know the Taylor series of $\cos x$ is

   $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!};$$

   replacing x with $2x^8$ on both sides of this equation we obtain

   $$\cos(2x^8) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^8)^{2n}}{(2n)!};$$

   then multiplying through by $3x$ we obtain

   $$f(x) = 3x\cos(2x^8) = \sum_{n=0}^{\infty} \frac{(-1)^n 3x(2x^8)^{2n}}{(2n)!}.$$

   This can be rewritten

   $$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 2^{2n}}{(2n)!}\, x^{16n+1} = \sum_{n=0}^{\infty} \frac{3 \cdot (-4)^n}{(2n)!}\, x^{16n+1}.$$

   b) We need to find a_{17}, the 17^{th} coefficient of the Taylor series of $f(x)$. Looking at the last line of the solution to problem (a), we see that the 17^{th} term of the Taylor series of $f(x)$ corresponds to $n = 1$, because $n = 1$ produces the term with $x^{16(1)+1} = x^{17}$ in it. So $a_{17} = \frac{3\cdot(-4)^1}{(2\cdot 1)!} = -6$. Finally, by uniqueness of power series, it must be that

   $$a_{17} = \frac{f^{(17)}(0)}{17!},$$

   multiplying through by 17! on both sides we see that

   $$f^{17}(0) = -6 \cdot 17!.$$

   c) As in part (b), to find this we would need to find a_{2010}, the 2010^{th} coefficient of the Taylor series of $f(x)$. But looking at the formula found at the end of part (a), we see that the Taylor series of $f(x)$ contains only odd powers of x. So $a_{2010} = 0$ and also $f^{(2010)}(0) = 0$.

2.4 Fall 2010 Final Exam

1. a) (6.4) Give an example of an infinite series for which the Ratio Test, by itself, does not tell you whether or not the series converges.

   b) (8.1) Explain what is meant by the *uniqueness of power series*.

2. Evaluate the following integrals:

   a) (2.6) $\int 3x \ln x \, dx$ b) (2.5) $\int_0^2 \frac{x}{x+3} \, dx$

3. Let R be the region enclosed by the graphs of $y = x^3$ and $y = x^2$ between $x = 0$ and $x = 1$.

   a) (4.1) Find the exact area of R. Simplify your answer.

   b) (4.2) Let S_1 be the solid formed by revolving R around the $y-$axis. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_1.

   c) (4.2) Let S_2 be the solid formed by revolving R around the line $y = 5$. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_2.

4. (2.8) Find the partial fraction decomposition of the following expression:

$$\frac{7x - 2}{x^3 - 2x^2 + x}$$

5. (7.4) Determine whether or not the following infinite series converge absolutely, converge conditionally, or diverge. Justify your reasoning.

   a) $\sum_{n=1}^{\infty} \frac{5n - n\sin(3n)}{n^2}$ b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

6. Find the exact sum of each of the following series (simplify your answers):

   a) (6.2) $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \frac{4}{3^5} + \ldots$

   b) (8.2) $4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \ldots$

7. (8.2) Approximate $\cos(.25)$ by using the third Taylor polynomial for an appropriately chosen function. Simplify your answer.

8. The two parts of this problem are unrelated to one another.

a) (8.2) Let $f(x) = x\ln(x^4+1)$. Write the Taylor series (centered at 0) of $f(x)$, and use this series to estimate the integral $\int_0^1 f(x)\,dx$ by replacing the integrand with its ninth Taylor polynomial. Simplify your answer to the integral.

b) (8.2) Evaluate the following limit, and simplify your answer:

$$\lim_{x\to 0} \frac{x^3 + \ln(1-x^3)}{xe^{2x^5} - x}$$

Solutions

1. a) The Ratio Test does not work for the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ ($\rho = 1$ here). Nor does it work for any series where the terms are rational functions of n.

   b) If two power series centered at the same number represent the same function, and both of the power series have positive radius of convergence, then they must have the same coefficients. This is because $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ implies that the coefficients must satisfy the formula $a_n = \frac{f^{(n)}(a)}{n!}$ for all n.

2. a) Use integration by parts with $u = \ln x$ and $dv = 3x\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{3}{2}x^2$, and we have

$$\begin{aligned}\int 3x\ln x\,dx = \int u\,dv = uv - \int v\,du &= \frac{3}{2}x^2\ln x - \int \frac{3}{2}x^2 \cdot \frac{1}{x}\,dx \\ &= \frac{3}{2}x^2\ln x - \int \frac{3}{2}x\,dx \\ &= \frac{3}{2}x^2\ln x - \frac{3}{4}x^2 + C.\end{aligned}$$

b) Use the $u-$substitution $u = x + 3$, $du = dx$:

$$\begin{aligned}\int_0^2 \frac{x}{x+3}\,dx = \int_3^5 \frac{u-3}{u}\,du &= \int_3^5 \left(1 - \frac{3}{u}\right)\,du \\ &= [u - 3\ln u]_3^5 \\ &= (5 - 3\ln 5) - (3 - 3\ln 3) \\ &= 2 - 3\ln 5 + 3\ln 3.\end{aligned}$$

3. Notice that between $x = 0$ and $x = 1$, $x^2 > x^3$ so the graph of x^2 is above that of x^3.

a) The area of R is given by the integral

$$\int_0^1 [x^2 - x^3]\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

b) Use the shell method:

$$V = \int_0^1 2\pi r h\,dx = \int_0^1 2\pi x(x^2 - x^3)\,dx.$$

c) Use the washer method:

$$V = \int_0^1 \pi[R^2 - r^2]\,dx = \int_0^1 \pi\left[(5 - x^3)^2 - (5 - x^2)^2\right]\,dx.$$

4. a) First factor the denominator to obtain

$$\frac{7x-2}{x^3 - 2x^2 + x} = \frac{7x-2}{x(x^2 - 2x + 1)} = \frac{7x-2}{x(x-1)^2}.$$

Based on this factorization the partial fraction decomposition should have the form

$$\frac{7x-2}{x^3 - 2x^2 + x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Find the lowest common denominator on the right-hand side of the above equation and add the terms; we obtain

$$\frac{7x-2}{x^3 - 2x^2 + x} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

and then by cancelling the denominators we see

$$7x - 2 = A(x-1)^2 + Bx(x-1) + Cx.$$

Plug in $x = 1$ to both sides to obtain $5 = C$ and plug in $x = 0$ to both sides to obtain $-2 = A$. Last, plug in $x = 2, A = -2$ and $C = 5$ to both sides to obtain $12 = -2 + 2B + 10$ which gives $B = 2$. Therefore the partial fraction decomposition is

$$\frac{7x-2}{x^3 - 2x^2 + x} = \frac{-2}{x} + \frac{2}{x-1} + \frac{5}{(x-1)^2}.$$

b) First, separate the variables by dividing through by y^2. This yields $\frac{1}{y^2}y' = 2x^2$. Rewriting this as an integral, we obtain

$$\int \frac{1}{y^2}\,dy = \int 2x^2\,dx \Rightarrow \frac{-1}{y} = \frac{2}{3}x^3 + C.$$

To find C, plug in the initial condition $x = 0$, $y = -1$ to obtain $\frac{-1}{-1} = \frac{2}{3}0^3 + C$, i.e. $C = 1$. So the solution is

$$\frac{-1}{y} = \frac{2}{3}x^3 + 1.$$

5. a) Observe that $\sin(3n)$ is always between 0 and 1, hence $n\sin(3n)$ is at most n and therefore $5n - n\sin(3n) \geq 5n - n = 4n$. This means that $\frac{5n-n\sin(3n)}{n^2} \geq \frac{4n}{n^2} = \frac{4}{n}$. But $\sum \frac{4}{n} = 4\sum\frac{1}{n}$ diverges since it is a constant times the harmonic series, so by the Comparison Test, $\sum_{n=1}^{\infty} \frac{5n-n\sin(3n)}{n^2}$ also diverges.

   b) Apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{((n+1)!)^2}{(2(n+1))!}\right|}{\left|\frac{(n!)^2}{(2n)!}\right|} = \lim_{n\to\infty} \frac{(n+1)!(n+1)!(2n)!}{(2n+2)!n!n!} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

   Since $\rho < 1$, the series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$ converges absolutely.

6. a) Factor out $\frac{4}{3}$ from all the terms to obtain

$$\frac{4}{3}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{3^5} + ...\right)$$

   which is a geometric series with $a_0 = \frac{4}{3}$, $r = \frac{1}{3}$. So its sum is $\frac{a_0}{1-r} = \frac{4/3}{1-1/3} = \frac{4/3}{2/3} = 2$.

   b) Factor out 4 from all the terms to obtain

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...\right);$$

   now recall that the Taylor series for $\arctan x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^n$. Therefore the series enclosed by the parentheses above is the same series as the power series for $\arctan x$ with $x = 1$ plugged in, i.e. it sums to $\arctan 1 = \frac{\pi}{4}$. Therefore the original series sums to $4 \cdot \frac{\pi}{4} = \pi$.

7. We know the Taylor series of $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6}! + ...$; therefore the third Taylor polynomial of $\cos x$ is $P_3(x) = 1 - \frac{x^2}{2}$. So $\cos(.25) = \cos(1/4) \approx P_3(1/4) = 1 - \frac{(1/4)^2}{2} = \frac{31}{32}$.

8. a) We know the Taylor series of $\ln(1+x) = \ln(x+1)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$; replacing x with x^4 and then multiplying through all the terms by x, we see that

$$f(x) = x\ln(x^4+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{4n+1} = x^5 - \frac{x^9}{2} + \frac{x^{13}}{3} - \frac{x^{17}}{4} + ...$$

Now to approximate the integral, use the ninth Taylor polynomial which is $x^5 - \frac{x^9}{2}$. We have

$$\int_0^1 f(x)\,dx \approx \int_0^1 \left(x^5 - \frac{x^9}{2}\right) dx = \left[\frac{x^6}{6} - \frac{x^{10}}{20}\right]_0^1 = \frac{1}{6} - \frac{1}{20} = \frac{7}{60}.$$

b) We find power series for the numerator and denominator. First, the numerator is obtained by starting with the power series for $\ln(1+x)$, replacing x with $-x^3$, then adding x^3. We obtain

$$x^3 + \ln(1-x^3) = -\frac{1}{2}x^6 - \frac{1}{3}x^9 - \frac{1}{4}x^{12} - ...$$

Now the denominator: start with the power series for e^x, replace x with $2x^5$, multiply by x and subtract x to get

$$xe^{2x^5} - x = 2x^6 + 2x^{11} + \frac{4}{3}x^{16} + ...$$

Now for the limit:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^3+\ln(1-x^3)}{xe^{2x^5}-x} &= \lim_{x\to 0} \frac{-\frac{1}{2}x^6 - \frac{1}{3}x^9 - \frac{1}{4}x^{12} - ...}{2x^6 + 2x^{11} + \frac{4}{3}x^{16} + ...} \\
&= \lim_{x\to 0} \frac{x^6\left(-\frac{1}{2} - \frac{1}{3}x^3 - \frac{1}{4}x^6 - ...\right)}{x^6\left(2 + 2x^5 + \frac{4}{3}x^{11} + ...\right)} \\
&= \lim_{x\to 0} \frac{-\frac{1}{2} - \frac{1}{3}x^3 - \frac{1}{4}x^6 - ...}{2 + 2x^5 + \frac{4}{3}x^{11} + ...} \\
&= \frac{-\frac{1}{2} - \frac{1}{3}0^3 - \frac{1}{4}0^6 - ...}{2 + 2\cdot 0^5 + \frac{4}{3}0^{11} + ...} \\
&= \frac{-\frac{1}{2}}{2} = \frac{-1}{4}.
\end{aligned}$$

2.5 Spring 2011 Exam 1

1. (2.5, 2.6) For each of these integrals, determine whether it is better to use integration by parts or a $u-$substitution. If integration by parts is better, write "PARTS" and indicate what your choice of u and dv would be. If a substitution is better, write "$u-$SUB" and indicate what you would set u equal to.

   a) $\displaystyle\int x^2 \sin x \, dx$

   b) $\displaystyle\int x^2 \sin x^3 \, dx$

   c) $\displaystyle\int \frac{\ln x}{x} \, dx$

   d) $\displaystyle\int x \ln x \, dx$

   e) $\displaystyle\int \cos^3 x \, dx$

2. (2.8) Calculate one of the following two integrals:

$$\int \frac{-x^2 + 2x + 5}{(x+1)^3} \, dx \qquad\qquad \int \frac{x^2 - x + 3}{x^3 + x} \, dx$$

3. (2.5) Calculate one of the following two integrals:

$$\int x^5 \sqrt{x^2 + 1} \, dx \qquad\qquad \int \frac{x^2}{\sqrt{2x - 1}} \, dx$$

4. Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt[3]{x}$, $x = 27$ and the $x-$axis.

   a) (5.2) Let S_1 be the solid obtained when R is revolved around the $y-$axis. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_1.

   b) (5.2) Let S_1 be as in part (b). Write an expression involving one or more integral(s) with respect to y that gives the volume of S_1.

   c) (5.2) Let S_2 be the solid obtained when R is revolved around the line $y = 8$. Write an expression involving one or more integral(s) with respect to x that gives the volume of S_2.

   d) (5.2) Let S_3 be the solid whose base is R such that cross-sections of S_3 parallel to the $x-$axis are rectangles which are half as tall as they are wide. Write an expression involving one or more integral(s) with respect to whatever variable you like that gives the volume of S_3.

5. (4.2) Determine whether or not the following improper integral converges or diverges. Justify your answer:

$$\int_{1/2}^{\infty} \frac{x+1}{x^4 + 2} \, dx$$

Solutions

1. a) PARTS: Set $u = x^2$, $dv = \sin x\, dx$.
   b) $u-$SUB: Set $u = x^3$.
   c) $u-$SUB: Set $u = \ln x$.
   d) PARTS: Set $u = \ln x$, $dv = x\, dx$.
   e) $u-$SUB: Set $u = \sin x$. This is because the integral can be rewritten as

$$\int \cos^3 x\, dx = \int \cos x \cos^2 x\, dx = \int \cos x(1 - \sin^2 x)\, dx$$

   and so after this $u-$substitution the integral would become $\int (1 - u^2)\, du$.

2. For the first integral, we use partial fractions. The decomposition of the integrand has the form

$$\frac{-x^2 + 2x + 5}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3};$$

writing everything with common denominator we obtain

$$\frac{-x^2 + 2x + 5}{(x+1)^3} = \frac{A(x+1)^2}{(x+1)^3} + \frac{B(x+1)}{(x+1)^3} + \frac{C}{(x+1)^3};$$

cancelling the common denominators we end up with

$$-x^2 + 2x + 5 = A(x+1)^2 + B(x+1) + C.$$

Plug in $x = -1$ to obtain $-1 - 2 + 5 = 0 + 0 + C$, i.e. $C = 2$. Plug in $x = 0$ and $C = 2$ to obtain $5 = A + B + 2$, i.e. $A + B = 3$. Plug in $x = 1$ and $C = 2$ to obtain $6 = 4A + 2B + 2$, i.e. $4A + 2B = 4$, i.e. $2A + B = 2$. The equations $A + B = 3, 2A + B = 2$ can be solved together to obtain $A = -1, B = 4$. All this gives

$$\frac{-x^2 + 2x + 5}{(x+1)^3} = \frac{-1}{x+1} + \frac{4}{(x+1)^2} + \frac{2}{(x+1)^3}.$$

So the integral is

$$\begin{aligned} \int \frac{-x^2 + 2x + 5}{(x+1)^3}\, dx &= \int \left[\frac{-1}{x+1} + \frac{4}{(x+1)^2} + \frac{2}{(x+1)^3}\right] dx \\ &= -\ln|x+1| - \frac{4}{x+1} - \frac{1}{(x+1)^2} + C. \end{aligned}$$

For the second integral, we also use partial fractions. Start by factoring the denominator of the integrand and making a guess as to the form of the decomposition:

$$\frac{x^2 - x + 3}{x^3 + x} = \frac{x^2 - x + 3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Now find a common denominator of this equation:

$$\frac{x^2-x+3}{x(x^2+1)} = \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)}.$$

Forgetting the common denominator, we have

$$x^2-x+3 = A(x^2+1) + (Bx+C)x.$$

Set $x=0$ in this equation to obtain $3 = A+0$, i.e. $A=3$. Now by multiplying out the right-hand side of the above equation and combining like powers of x, we get

$$x^2-x+3 = (A+B)x^2 + Cx + A = (3+B)x^2 + Cx + 3.$$

It must therefore be that $3+B=1$ (i.e. $B=-2$) and $C=-1$, so

$$\frac{x^2-x+3}{x^3+x} = \frac{x^2-x+3}{x(x^2+1)} = \frac{3}{x} + \frac{-2x-1}{x^2+1}.$$

So the integral becomes

$$\int \frac{x^2-x+3}{x^3+x}\,dx = \int \left[\frac{3}{x} + \frac{-2x-1}{x^2+1}\right] dx = \int \left[\frac{3}{x} - \frac{2x}{x^2+1} - \frac{1}{x^2+1}\right] dx.$$

The first and third terms of this integral are "just do it" integrals; the middle term is done by a $u-$substitution $u = x^2+1$. In the end, this integral evaluates to

$$3\ln|x| - \ln|x^2+1| - \arctan x + C.$$

3. For the first integral, set $u = x^2+1$. Then $x = \sqrt{u-1}$ and $du = 2x\,dx$. Dividing the last equation by two, we obtain $\frac{1}{2}\,du = x\,dx$. So

$$\int x^5\sqrt{x^2+1}\,dx = \int xx^4\sqrt{x^2+1}\,dx = \int \frac{1}{2}(\sqrt{u-1})^4\sqrt{u}\,du = \frac{1}{2}\int (u-1)^2\sqrt{u}\,du.$$

Multiplying out the integrand, we see that the integral becomes

$$\frac{1}{2}\int \left((u^2-2u+1)\sqrt{u}\right)\,du = \frac{1}{2}\int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\,du$$

Evaluating this integral with the power rule, and back-substituting for x, we see the answer is

$$\frac{1}{2}\left[\frac{2}{7}u^{7/2} - 2\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right] + C = \frac{1}{7}(x^2+1)^{7/2} - \frac{2}{5}(x^2+1)^{5/2} + \frac{1}{3}(x^2+1)^{3/2} + C.$$

For the second integral, set $u = 2x-1$ so $x = \frac{1}{2}(u+1)$ and $du = 2dx$ so $\frac{1}{2}du = dx$. Plugging all this into the integral, we see

$$\int \frac{x^2}{\sqrt{2x-1}}\,dx = \int \frac{\left[\frac{1}{2}(u+1)\right]^2}{\sqrt{u}}\frac{1}{2}\,du = \frac{1}{8}\int \frac{u^2+2u+1}{\sqrt{u}}\,du.$$

Distributing the terms in the integrand and calculating the integral, we obtain

$$\begin{aligned}\frac{1}{8}\int\left(u^{3/2}+2u^{1/2}+u^{-1/2}\right)du &= \frac{1}{8}\left(\frac{2}{5}u^{5/2}+2\cdot\frac{2}{3}u^{3/2}+2u^{1/2}\right)+C\\ &= \frac{1}{20}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+\frac{1}{4}(2x-1)^{1/2}+C.\end{aligned}$$

(This integral could also be done by using integration by parts twice; the answer you get by this method looks different but is actually the same as the above answer.)

4. The region R is shaped like a triangle with a curved top, having vertices at $(0,0)$, $(27,0)$ and $(27,3)$.

   a) Since the axis of revolution is vertical and the direction of integration is horizontal, we use the shell method. Here a cylindrical cross section has height $\sqrt[3]{x}$ and radius x, so the volume is

   $$V=\int_0^{27}2\pi rh\,dx=\int_0^{27}2\pi x\sqrt[3]{x}\,dx.$$

   b) Since the axis of revolution is vertical and the direction of integration is also vertical, cross-sections will be washers. We see that the outer radius of the washer at height y is $R=27$ and the inner radius is $r=y^3$, so

   $$V=\int_0^3\pi(R^2-r^2)\,dy=\int_0^3\pi\left[27^2-(y^3)^2\right]\,dy.$$

   c) Since the axis of revolution is horizontal and the direction of integration is also horizontal, cross-sections will be washers. We see that the outer radius of the washer at horizontal position x is $R=8$ and the inner radius is $r=8-\sqrt[3]{x}$, so

   $$V=\int_0^{27}\pi(R^2-r^2)\,dx=\int_0^{27}\pi\left[8^2-(8-\sqrt[3]{x})^2\right]\,dx.$$

   d) Since the described cross-sections are parallel to the $x-$axis, we must choose $y-$integration. At height y, the solid has cross section which is a rectangle with width $27-y^3$ and height $\frac{1}{2}(27-y^3)$, so the area of such a cross-section is $A(y)=\frac{1}{2}(27-y^3)^2$. Thus the volume is

   $$V=\int_0^3A(y)\,dy=\int_0^3\frac{1}{2}(27-y^3)^2\,dy.$$

5. We know $\int_1^\infty\frac{x}{x^4}\,dx=\int_1^\infty\frac{1}{x^3}\,dx$ and $\int_1^\infty\frac{1}{x^4}\,dx$ both converge from results in class. Therefore the integrals of the same functions starting at $1/2$ rather than 1 also converge because starting index of an improper integral is irrelevant to convergence. Finally, by result in class the sum of two convergent improper integrals converges, so we now know

   $$\int_{1/2}^\infty\frac{x+1}{x^4}\,dx=\int_{1/2}^\infty\frac{x}{x^4}\,dx+\int_{1/2}^\infty\frac{1}{x^4}\,dx$$

converges. But

$$\frac{x+1}{x^4+2} \leq \frac{x+1}{x^4}$$

since the fractions have the same numerator but the first fraction has larger denominator, so by the Comparison Test for Integrals

$$\int_{1/2}^{\infty} \frac{x+1}{x^4+2}\, dx$$

converges.

2.6 Spring 2011 Exam 2

1. (7.3) Define what it means for a series to *converge*, what it means for a series to *absolutely converge*, and what it means for a series to *conditionally converge*.

2. (7.4) For each of the following infinite series, state the test which will best determine whether the series converges or diverges. You need only state the test; you do not need to carry out an argument or otherwise explain your answer.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{4+\sin 3n - \cos n}{n}$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^n}$

   c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

   d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n (n^2+7)}{2n^2+9}$

3. Find the exact sum of each of these infinite series (simplify your answers). You may assume that these series converge.

   a) (6.2) $\displaystyle\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$

   b) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n-1}}{11^n}$

   c) (8.2) $\displaystyle\frac{1}{3} + \frac{1}{9} + \frac{1}{27 \cdot 2!} + \frac{1}{3^4 \cdot 3!} + \frac{1}{3^5 \cdot 4!} + \frac{1}{3^6 \cdot 5!} + \ldots$

4. (7.4) Determine whether the following infinite series converge absolutely, converge conditionally, or diverge. Justify your reasoning.

   a) $\displaystyle 1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \frac{1}{36} - \frac{1}{49} - \frac{1}{64} + \frac{1}{9^2} + \ldots$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+4^n}{n4^n}$

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{2011}}{2^n}$

5. (8.3) Find the values of x for which the following power series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x-5)^n$$

6. Let $f(x) = x^4 \arctan(2x^3)$.

   a) (8.1) Write the Taylor series (centered at 0) of $f(x)$.

b) (8.2) Estimate $\int_0^1 f(x)\,dx$ by replacing the integrand with its tenth Taylor polynomial.

c) (8.2) Find $f^{(13)}(0)$, the 13^{th} derivative of f at zero.

7. (8.2) Estimate $\sin\dfrac{1}{3}$ by using the Taylor polynomial of order 4 for an appropriately chosen function.

8. a) (7.4) Suppose $\sum a_n$ is some convergent infinite series. Do we know whether or not the series $\sum e^{-a_n}$ converges? Explain your answer.

   b) (7.4) Suppose $\sum a_n$ is some convergent infinite series. Is it possible for $\sum a_n^2$ to diverge? Explain your answer.

Solutions

1. Given an infinite series $\sum a_n$, we say the series *converges* if $\lim_{n\to\infty} S_n$ exists and is finite, where S_n is the n^{th} partial sum of the series. A series $\sum a_n$ *absolutely converges* if $\sum |a_n|$ converges, and *conditionally converges* if it converges but does not absolutely converge.

2. a) Comparison Test (the terms of the given series are at least $2/n$)

   b) Ratio Test (the terms contain n^n)

   c) Alternating Series Test (terms decrease in absolute value and approach zero as $n \to \infty$)

d) $n^{th}-$term Test (limit of the terms does not exist as $n \to \infty$)

3. a)
$$\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n = 4\left(\frac{2}{5}\right)\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{8}{5}\left(\frac{1}{1-2/5}\right) = \frac{8}{5}\cdot\frac{5}{3} = \frac{8}{3}.$$

b)
$$\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n-1}}{11^n} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-9}{11}\right)^n = \frac{1}{3}\left(\frac{1}{1-(-9/11)}\right) = \frac{1}{3}\cdot\frac{11}{20} = \frac{11}{60}.$$

c) First factor out $1/3$ from the series; then what is left is the Taylor series for e^x with $1/3$ plugged in for x:

$$\begin{aligned} & \frac{1}{3}+\frac{1}{9}+\frac{1}{27\cdot 2!}+\frac{1}{3^4\cdot 3!}+\frac{1}{3^5\cdot 4!}+\frac{1}{3^6\cdot 5!}+... \\ = & \frac{1}{3}\left[1+\frac{1}{3}+\frac{1}{9\cdot 2!}+\frac{1}{3^3\cdot 3!}+\frac{1}{3^4\cdot 4!}+\frac{1}{3^5\cdot 5!}+...\right] \\ = & \frac{1}{3}\left[1+\frac{1}{3}+\frac{(1/3)^2}{2!}+\frac{(1/3)^3}{3!}+\frac{(1/3)^4}{4!}+\frac{(1/3)^5}{5!}+...\right] \\ = & \frac{1}{3}e^{1/3}. \end{aligned}$$

4. a) This series is neither positive nor alternating. We forget the series as it is given and consider the series of the absolute values of the given terms, which is $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...$, a convergent $p-$series ($p=2$). The given series then converges absolutely by definition.

b) Rewrite the terms of the series as follows:

$$\sum_{n=1}^{\infty}\frac{2n+4^n}{n4^n} = \sum_{n=1}^{\infty}\left(\frac{2n}{n4^n}+\frac{4^n}{n4^n}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{4^n}+\frac{1}{n}\right).$$

Now $\sum \frac{2}{4^n}$ is a constant times a geometric series with $r=1/4$ so it converges; $\sum\frac{1}{n}$ diverges since it is harmonic. So the original series is a convergent series plus a divergent series which diverges.

c) We use the Ratio Test:

$$\rho = \lim_{n\to\infty}\frac{\left|\frac{(n+1)^{2011}}{2^{n+1}}\right|}{\left|\frac{n^{2011}}{2^n}\right|} = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^{2011} = \frac{1}{2}\cdot 1^{2011} = \frac{1}{2} < 1$$

so the series converges absolutely by the Ratio Test.

5. This power series is centered at $x=5$. We find the radius of convergence by Abel's formula:

$$R = \lim_{n\to\infty}\frac{\left|\frac{(-1)^n}{n3^n}\right|}{\left|\frac{(-1)^{n+1}}{(n+1)3^{n+1}}\right|} = \lim_{n\to\infty}\frac{(n+1)3}{n} = 3$$

so by the Cauchy-Hadamard Theorem, the series converges absolutely when $x \in (5-3, 5+3) = (2, 8)$ and diverges when $x < 2$ or $x > 8$. It is left to check convergence at the endpoints.

When $x = 2$, the series reduces to $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(2-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(-3)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (harmonic).

When $x = 8$, the series reduces to $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(8-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges (alternating harmonic; done in class).

So the given series converges when $x \in (2, 8]$.

6. a) We know $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$; replacing x with $2x^3$ we obtain

$$\begin{aligned}\arctan(2x^3) &= 2x^3 - \frac{(2x^3)^3}{3} + \frac{(2x^3)^5}{5} - \frac{(2x^3)^7}{7} + \ldots \\ &= 2x^3 - \frac{2^3}{3}x^9 + \frac{2^5}{5}x^{15} - \frac{2^7}{7}x^{21} + \ldots;\end{aligned}$$

finally multiplying through by x^4 we obtain

$$f(x) = 2x^7 - \frac{2^3}{3}x^{13} + \frac{2^5}{5}x^{19} - \frac{2^7}{7}x^{25} + \ldots$$

b) From (a), we see that $P_{10}(x) = 2x^7$ as the Taylor series contains no other terms of power less than or equal to 10. Therefore

$$\int_0^1 f(x)\,dx \approx \int_0^1 2x^7\,dx = \left.\frac{x^8}{4}\right|_0^1 = \frac{1}{4}.$$

c) By the uniqueness of power series, $f^{(13)}(0) = 13!\,a_{13}$ where a_{13} is the coefficient on the 13^{th} power term of the Taylor series of $f(x)$ centered at zero. From (a) this coefficient is $-2^3/3 = -8/3$. So $f^{(13)}(0) = 13! \cdot (-8/3)$.

7. We know $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ so $\sin(1/3) \approx P_4(1/3) = (1/3) - \frac{(1/3)^3}{3!}$.

8. a) If $\sum a_n$ converges, then by the n^{th} term test, $\lim_{n\to\infty} a_n = 0$. Then $\lim_{n\to\infty} e^{-a_n} = e^{-0} = 1$ so by the n^{th} term test, the series $\sum e^{-a_n}$ must diverge.

b) Yes; let $a_n = \frac{(-1)^{n+1}}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series test. But for this choice of a_n, $a_n^2 = \left(\frac{(-1)^{n+1}}{\sqrt{n}}\right)^2 = \frac{1}{n}$ so $\sum a_n^2$ is the harmonic series, which diverges.

2.7 Spring 2011 Final Exam

1. (2.2) Evaluate the following three "similar-looking" indefinite integrals:

$$\int \frac{1}{x+1}\,dx \qquad \int \frac{1}{x^2+1}\,dx \qquad \int \frac{1}{(x+1)^2}\,dx$$

2. (2.10) Evaluate two of the following three integrals:

$$\int 24x^2 \cos 2x\,dx \qquad \int_0^3 \frac{x}{x^4+1}\,dx \qquad \int \frac{6}{4-x^2}\,dx$$

3. a) (4.4) Write an expression involving one or more integral(s) (with respect to whatever variable you like) that gives the length of the curve $y = \sin x$ from $x = 0$ to $x = \pi$.

   b) (4.1) Write an expression involving one or more integral(s) (with respect to whatever variable you like) that gives the area of the region R enclosed by the graphs of $y = x^2$ and $y = -4x$.

   c) (4.2) Let Q be the planar region enclosed by the graphs of $y = \ln x$, $y = 3$, and the $x-$ and $y-$ axes. Let S be the solid obtained by revolving Q around the $y-$axis. Write an eqution involving one or more integral(s) (with respect to whatever variable you like) that gives the volume of S.

4. (7.4) Determine whether or not the following infinite series converge absolutely, converge conditionally, or diverge. Justify your reasoning.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n+3^n}$ b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n^{-2/5}$

5. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n)!} x^n$.

   a) (8.3) Find the domain of $f(x)$ (i.e. find the set of x for which the power series converges).

   b) (8.2) Write a power series for the function $f'(x)$. Write your answer in $\Sigma-$notation.

   c) (8.2) Evaluate $\displaystyle\lim_{x\to 0} \frac{2f(x)+x-2}{x^2}$.

   d) (8.1) Write the Taylor series of $f(x)$. Your answer should be in $\Sigma-$notation.

6. Find the exact sum of each of the following (finite or infinite) series:

   a) (6.2) $\dfrac{5}{7} - \dfrac{5}{7^2} + \dfrac{5}{7^3} - \dfrac{5}{7^4} + \dfrac{5}{7^5} + \ldots$

   b) (6.2) $2^{-5} + 2^{-3} + 2^{-1} + \ldots + 2^{17} + 2^{19}$

Solutions

1. a) We have $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$ so $x_0 = 0$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$, $x_4 = 8$. Now by Simpson's Rule,

$$\begin{aligned}\int_0^8 \ln(x+1)\,dx &= \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{8}{12}\left[\ln 1 + 4\ln 3 + 2\ln 5 + 4\ln 7 + \ln 9\right] \\ &= \frac{2}{3}\left(6\ln 3 + 2\ln 5 + 4\ln 7\right).\end{aligned}$$

b) First, let $f(x) = \ln(x+1)$. Then $f^{(4)}(x)$, the fifth derivative of x, is $-6/(x+1)^3$. The absolute value of this expression, $6/(x+1)^3$, is maximized when x is smallest, i.e. when $x = 0$, so $M_4 = \max\{|f^{(4)}(x)| : x \in [0,8]\} = 6/(0+1)^3 = 6$. Now by the error formula for Simpson's Rule,

$$E_S \leq \frac{(b-a)^5}{180n^4} M_4 = \frac{8^5}{180 \cdot 4^4}(6) = \frac{64}{15}.$$

c) In Simpson's Rule, it takes two subintervals to make each parabolic region. Therefore since there are 4 subintervals, there are 2 parabolic regions whose areas are calculated.

2. All these integrals can be done by inspection:

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C; \qquad \int \frac{1}{x^2+1}\,dx = \arctan x + C;$$

$$\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + C.$$

3. The first integral is done via integration by parts: set $u = 12x^2$, $dv = 2\cos 2x\,dx$ so that $du = 24x\,dx$ and $v = \sin 2x$. Then by the integration by parts formula,

$$\int 24x^2 \cos 2x\,dx = \int u\,dv = uv - \int v\,du = 12x^2 \sin 2x - \int 24x \sin 2x\,dx.$$

Now for the remaining integral, use integration by parts again: set $u = 12x$ and $dv = 2\sin 2x\,dx$ so that $du = 12$ and $v = -\cos 2x$. Then by the integration by parts formula,

$$\begin{aligned}\int 24x \sin 2x\,dx. = \int u\,dv &= uv - \int v\,du \\ &= -12x\cos 2x - \int -12\cos 2x \\ &= -12x\cos 2x + 6\sin 2x.\end{aligned}$$

Putting this back into the original integral (and adding the constant C), we obtain

$$\begin{aligned}\int 24x^2\cos 2x\,dx &= 12x^2\sin 2x - \int 24x\sin 2x\,dx\\ &= 12x^2\sin 2x - (-12x\cos 2x + 6\sin 2x)\\ &= 12x^2\sin 2x + 12x\cos 2x - 6\sin 2x + C.\end{aligned}$$

For the second integral, use the $u-$substitution $u = x^2$. Then $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$ and the limits of integration ($x = 0, x = 3$) become $u = 0^2 = 0$ and $u = 3^2 = 9$. Then the integral becomes

$$\int_0^9 \frac{1/2}{u^2+1}\,du = \frac{1}{2}\arctan u|_0^9 = \frac{1}{2}\arctan 9.$$

For the third integral, use partial fractions. Write

$$\frac{6}{4-x^2} = \frac{6}{(2-x)(2+x)} = \frac{A}{2-x} + \frac{B}{2+x},$$

find common denominators and equate the numerators to obtain $6 = A(2+x) + B(2-x)$. Set $x = 2$ to obtain $4A = 6$, i.e. $A = \frac{3}{2}$. Set $x = -2$ to obtain $4B = 6$, i.e. $B = \frac{3}{2}$. So

$$\int \frac{6}{4-x^2}\,dx = \int \left(\frac{3/2}{2-x} + \frac{3/2}{2+x}\right)dx = -\frac{3}{2}\ln|2-x| + \frac{3}{2}\ln|2+x| + C.$$

4. a) In general the length of the curve $y = f(x)$ from $x = a$ to $x = b$ is $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$. In this case since $f(x) = \sin x$, $f'(x) = \cos x$ and $a = 0, b = \pi$ so we have $L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx$.

   b) The region R is entirely in the second quadrant; the intersection points of the two graphs are $(-4, 16)$ and $(0, 0)$. The function $y = -4x$ is the "top" function of R so the area of R is given by $\int_{-4}^0 \left[-4x - x^2\right]\,dx$.

   c) Q is a region in the first quadrant bounded by three straight line segments (namely from $(0,0)$ to $(1,0)$, from $(0,0)$ to $(0,3)$ and from $(0,3)$ to $(e^3,3)$) and one curve ($y = \ln x$ from $(1,0)$ to $(e^3,3)$). Revolving this region around the $y-$axis produces a solid with no hole. It is easier to integrate with respect to y. This way, the direction of integration is parallel to the axis of revolution, so the disk/washer method applies. Here the cross-sections are disks with radius $r = e^y$ (this comes from solving $y = \ln x$ for x) and the solid goes from $y = 0$ (the bottom) to $y = 3$ (the top). So the volume is

$$V = \int_0^3 \pi r^2\,dy = \int_0^3 \pi[e^y]^2\,dy = \int_0^3 \pi e^{2y}\,dy.$$

5. a) Observe $0 \le \frac{6}{n+3^n} \le \frac{6}{3^n}$ because the second fraction has the same numerator but smaller denominator. Now $\sum \frac{6}{3^n} = 6\sum\left(\frac{1}{3}\right)^n$ converges as it is a constant times a geometric series with common ratio less than one in absolute value. So by the Comparison Test, $\sum_{n=1}^{\infty} \frac{6}{n+3^n}$ converges; since this series is positive it converges absolutely.

b) Let $a_n = (-1)^{n+1}n^{-2/5}$; the given series $\sum a_n$ is alternating. Now $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n^{2/5}} = 0$ and for all n, $|a_{n+1}| = \frac{1}{(n+1)^{2/5}} \leq \frac{1}{n^{2/5}} = |a_n|$ so by the Alternating Series Test, $\sum a_n$ converges. However, $\sum |a_n| = \sum \frac{1}{n^{2/5}}$ diverges since it is a $p-$series with $p = \frac{2}{5} < 1$. Therefore $\sum a_n$ converges conditionally.

6. a) Use Abel's Formula:

$$\begin{aligned} R = \lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty}\frac{\frac{1}{n!(2n)!}}{\frac{1}{(n+1)![2(n+1)]!}} &= \lim_{n\to\infty}\frac{(n+1)!(2n+2)!}{n!(2n)!} \\ &= \lim_{n\to\infty}(n+1)(2n+2)(2n+1) = \infty \end{aligned}$$

so since $R = \infty$, the series converges absolutely for all real numbers x.

b) Differentiating term-by-term within the series, we obtain

$$f'(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!(2n)!}nx^{n-1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!(2n)!}x^{n-1} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!(2n+2)!}x^n.$$

c) First, write out some terms of the series defining $f(x)$:

$$f(x) = 1 - \frac{1}{2}x + \frac{1}{2!4!}x^2 - \frac{1}{3!6!}x^3 + ...$$

Now multiply through this series by 2, add x and subtract 2 to obtain

$$2f(x) + x - 2 = +\frac{2}{2!4!}x^2 - \frac{2}{3!6!}x^3 + ...$$

Dividing by x^2 and letting x go to 0, we obtain

$$\lim_{x\to 0}\frac{2f(x) + x - 2}{x^2} = \frac{2}{2!4!} = \frac{1}{24}.$$

d) By uniqueness of power series, the Taylor series of f must be the series originally used to define f, i.e. the Taylor series is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(2n)!}x^n$.

7. a) This is an infinite geometric series:

$$\begin{aligned} \frac{5}{7} - \frac{5}{7^2} + \frac{5}{7^3} - \frac{5}{7^4} + \frac{5}{7^5} + ... &= \frac{5}{7}\left[1 - \frac{1}{7} + \left(\frac{-1}{7}\right)^2 + \left(\frac{-1}{7}\right)^3 + ...\right] \\ &= \frac{5}{7}\left[\frac{1}{1 - \frac{-1}{7}}\right] = \frac{5}{7}\cdot\frac{7}{8} = \frac{5}{8}. \end{aligned}$$

b) This is a finite geometric series:

$$\begin{aligned} 2^{-5} + 2^{-3} + 2^{-1} + ... + 2^{17} + 2^{19} &= 2^{-5}\left[1 + 2^2 + 2^4 + ... + 2^{24}\right] \\ &= 2^{-5}\left[1 + 4 + 4^2 + ... + 4^{12}\right] \\ &= 2^{-5}\left[\frac{4^{13} - 1}{4 - 1}\right] = \frac{4^{13} - 1}{96}. \end{aligned}$$

2.8 Fall 2011 Exam 1

1. (2.6) Evaluate one of the following two integrals:

$$\int_0^1 \arcsin x\,dx \qquad\qquad \int_{1/3}^1 x\ln(3x)\,dx$$

2. (2.5) Evaluate one of the following two integrals:

$$\int \cos^3 x\,dx \qquad\qquad \int \frac{2x}{2x+1}\,dx$$

3. (2.8) Evaluate one of the following two integrals:

$$\int \frac{3x^3+19x^2+21x-9}{x^4+6x^3+9x^2}\,dx \qquad\qquad \int \frac{2x^2}{(x-1)^3}\,dx$$

4. Given the improper integrals below, determine whether or not they converge. Explain your answer.

   a) (3.2) $\displaystyle\int_2^4 \frac{2}{(x-2)^4}\,dx$

   b) (3.3) $\displaystyle\int_5^\infty \frac{2x^2+x^2\sin x}{4x^6}\,dx$

5. Let R be the region under the graph of $y = \arctan x$ (and above the $x-$axis) from $x = 1$ to $x = \sqrt{3}$.

   a) (4.2) Suppose R is revolved around the $x-$axis to produce a solid. Write an integral with respect to x that gives the volume of this solid.

   b) (4.2) Suppose R is revolved around the line $x = -2$ to produce a solid. Write an integral with respect to x that gives the volume of this solid.

   c) (4.2) Suppose R is revolved around the line $y = 3$ to produce a solid. Write an integral with respect to x that gives the volume of this solid.

Solutions

1. For the first integral, use integration by parts with $u = \arcsin x$ and $dv = dx$. Then $du = \frac{1}{\sqrt{1-x^2}}dx$ and $v = x$ so by the integration by parts formula,

$$\int_0^1 \arcsin x\, dx = \int u\, dv = uv - \int v\, du = x \arcsin x|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$$

To handle the integral at the right, perform a $u-$substitution with $u = 1 - x^2$, $du = -2xdx$ so $\frac{-1}{2}du = xdx$ (and change the limits from $x = 0$ to $x = 1$ to $u = 1 - 0^2 = 1$ to $u = 1 - 1^2 = 0$) to obtain

$$\begin{aligned}\int_0^1 \arcsin x\, dx &= x \arcsin x|_0^1 - \int_1^0 \frac{-1}{2}\frac{1}{\sqrt{u}}\, du \\ &= (1 \arcsin 1 - 0 \arcsin 0) - (-\sqrt{u})|_1^0 \\ &= \frac{\pi}{2} - (0+1) \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

For the second integral, use integration by parts with $u = \ln 3x$ and $dv = x\, dx$. Then $du = \frac{1}{3x}3\, dx = \frac{1}{x}\, dx$ and $v = \frac{x^2}{2}$ so by the integration by parts formula,

$$\begin{aligned}\int_{1/3}^1 x \ln(3x)\, dx = \int u\, dv = uv - \int v\, du &= \left.\frac{x^2}{2}\ln(3x)\right|_{1/3}^1 - \int_{1/3}^1 \frac{x^2}{2}\frac{1}{x}\, dx \\ &= \left(\frac{\ln 3}{2} - \frac{2}{9}\cdot 0\right) - \int_{1/3}^1 \frac{x}{2}\, dx \\ &= \frac{\ln 3}{2} - \left.\frac{x^2}{4}\right|_{1/3}^1 = \frac{\ln 3}{2} - \left(\frac{1}{4} - \frac{1}{36}\right) \\ &= \frac{\ln 3}{2} - \frac{2}{9}.\end{aligned}$$

2. Rewrite the first integral as

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx$$

and perform the $u-$substitution $u = \sin x$, $du = \cos x\, dx$ to obtain

$$\int (1-u^2)\, du = u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C.$$

For the second integral, use the substitution $u = 2x + 1$, $du = 2dx$ and $x = \frac{u-1}{2}$ to obtain

$$\begin{aligned}\int \frac{2x}{2x+1}\, dx = \int \frac{u-1}{2}\frac{1}{u}\, du = \int \left(\frac{1}{2} - \frac{1}{2u}\right)\, du &= \frac{1}{2}u - \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}(2x+1) - \frac{1}{2}\ln|2x+1| + C.\end{aligned}$$

3. Both these integrals use partial fractions. For the first integral, start by factoring the denominator:
$$x^4 + 6x^3 + 9x^2 = x^2(x^2 + 6x + 9) = x^2(x + 3)^2$$
Now the "guessed" form of the decomposition is
$$\frac{3x^3 + 19x^2 + 21x - 9}{x^4 + 6x^3 + 9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 3} + \frac{D}{(x + 3)^2};$$
add the fractions on the right-hand side and clear denominators to obtain
$$3x^3 + 19x^2 + 21x - 9 = Ax(x + 3)^2 + B(x + 3)^2 + Cx^2(x + 3) + Dx^2.$$
Plug in $x = 0$ to both sides to obtain $-9 = 9B$, i.e. $B = -1$. Plug in $x = -3$ to both sides to obtain $-81 + 171 - 63 - 9 = 9D$, i.e. $18 = 9D$, i.e. $D = 2$. Now our equation is
$$3x^3 + 19x^2 + 21x - 9 = Ax(x + 3)^2 - (x + 3)^2 + Cx^2(x + 3) + 2x^2;$$
plugging in $x = 1$ to both sides yields $3 + 19 + 21 - 9 = 16A - 16 + 4C + 2$, i.e. $48 = 16A + 4C$, i.e. $12 = 4A + C$. Plugging in $x = -1$ to both sides yields $-3 + 19 - 21 - 9 = -4A - 4 + 2C + 2$, i.e. $-12 = -4A + 2C$. Now solve the two equations
$$12 = 4A + C, \qquad -12 = -4A + 2C$$
for A and C to get $A = 3, C = 0$. Thus the partial fraction decomposition is
$$\frac{3x^3 + 19x^2 + 21x - 9}{x^4 + 6x^3 + 9x^2} = \frac{3}{x} - \frac{1}{x^2} + \frac{2}{(x + 3)^2}$$
so the integral is
$$\begin{aligned}\int \frac{3x^3 + 19x^2 + 21x - 9}{x^4 + 6x^3 + 9x^2}\, dx &= \int \left(\frac{3}{x} - \frac{1}{x^2} + \frac{2}{(x + 3)^2}\right) dx \\ &= 3\ln|x| + \frac{1}{x} - \frac{2}{x + 3} + C.\end{aligned}$$
For the second integral, the denominator is already factored. The "guessed" form of the decomposition is
$$\frac{2x^2}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3};$$
add the fractions on the right-hand side and clear denominators to obtain
$$2x^2 = A(x - 1)^2 + B(x - 1) + C;$$
substitute in $x = 1$ to both sides to get $2 = C$. Now the equation reduces to
$$2x^2 = A(x - 1)^2 + B(x - 1) + 2;$$

plug in $x = 0$ to get $0 = A - B + 2$, i.e. $A = B - 2$, and plug in $x = 2$ to get $8 = A + B + 2$. Therefore $A = 2$ and $B = 4$ so

$$\frac{2x^2}{(x-1)^3} = \frac{2}{x-1} + \frac{4}{(x-1)^2} + \frac{2}{(x-1)^3}$$

and the integral is therefore

$$\begin{aligned}\int \frac{2x^2}{(x-1)^3}\,dx &= \int \left(\frac{2}{x-1} + \frac{4}{(x-1)^2} + \frac{2}{(x-1)^3}\right) dx \\ &= 2\ln|x-1| - \frac{4}{x-1} - \frac{1}{(x-1)^2} + C.\end{aligned}$$

4. a) Notice that the integrand has a vertical asymptote at $x = 2$, the left-hand limit of this improper integral. Rewriting this as a limit of a definite integral and evaluating, we see that

$$\begin{aligned}\int_2^4 \frac{2}{(x-2)^4}\,dx &= \lim_{b\to 2^+} \int_b^4 \frac{2}{(x-2)^4}\,dx \\ &= \lim_{b\to 2^+} \left.\frac{-2}{3(x-2)^3}\right|_b^4 \\ &= \lim_{b\to 2^+} \left(\frac{-2}{3(4-2)^3} - \frac{-2}{3(b-2)^3}\right) \\ &= \infty.\end{aligned}$$

   Therefore the integral diverges.

   b) Notice that $2x^2 + x^2\sin x = x^2(2+\sin x) \le x^2(2+1) = 3x^2$. Dividing through by $4x^6$, we see that

$$\frac{2x^2 + x^2\sin x}{4x^6} \le \frac{3x^2}{4x^6} = \frac{3}{4x^4}.$$

   We know that

$$\int_5^\infty \frac{3}{4x^4}\,dx = \frac{3}{4}\int_5^\infty \frac{1}{x^4}\,dx$$

   converges (it is a constant times the convergent integral $\int_5^\infty \frac{1}{x^p}\,dx$ where $p = 4 > 1$), so by the Comparison Test for Integrals,

$$\int_5^\infty \frac{2x^2 + x^2\sin x}{4x^6}\,dx \text{ converges.}$$

5. a) Since the direction of integration is parallel to the axis of revolution, we use washers (in this case, disks as the solid has no “hole”): $A(x) = \pi r^2 = \pi\arctan^2 x$ so

$$V = \int_1^{\sqrt{3}} \pi\arctan^2 x\,dx.$$

   Note: the notation $\arctan^2 x$ means $(\arctan x)^2$.

b) Now the direction of integration is perpendicular to the axis of revolution, so we use the shell method. $A(x) = 2\pi rh$ where $r = x - (-2) = x + 2$ and $h = \arctan x$; we have

$$V = \int_1^{\sqrt{3}} 2\pi(x+2)\arctan x\, dx.$$

c) As in part (a), the direction of integration is parallel to the axis of revolution, so we use washers as our cross-sections. Notice that $\arctan\sqrt{3} = \pi/3 < 3$, so the axis of revolution is above the region R. Therefore, we have $R = 3$ and $r = 3 - \arctan x$, so $A(x) = \pi R^2 - \pi r^2 = 9\pi - \pi(3 - \arctan x)^2$. Therefore

$$V = \int_1^{\sqrt{3}} \left[9\pi - \pi(3 - \arctan x)^2\right] dx.$$

2.9 Fall 2011 Exam 2

1. (8.3) Two concepts we have encountered in our study of series are *power series* and *Taylor series*. Explain how these two concepts relate to one another and explain the distinction between the two concepts.

2. Find the sum of each of the following series (you may assume without justification that each series converges). Your answers should be completely simplified:

   a) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{1+3^n}{7^n}$

   b) (6.2) $72 - 24 + 8 - \dfrac{8}{3} + \dfrac{8}{9} - \dfrac{8}{27} + \dfrac{8}{81} - \dfrac{8}{3^5} + \ldots$

   c) (8.2) $\pi - \dfrac{\pi^3}{2^2 \cdot 3!} + \dfrac{\pi^5}{2^4 \cdot 5!} - \dfrac{\pi^7}{2^6 \cdot 7!} + \ldots$

3. (10.1) Determine whether or not each of the following series converges absolutely, converges conditionally, or diverges. Justify your reasoning.

   a) (7.4) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{e^n + e^{-n}}$

   b) (7.4) $\displaystyle\sum_{n=6}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2-4}}$

4. (8.3) Find the interval of convergence of the following power series:
$$\sum_{n=1}^{\infty} \frac{(-2)^n (x+1)^n}{n}$$

5. (8.2) Use the third Taylor polynomial for an appropriately chosen function to estimate $\sqrt{e}$. Simplify your answer, writing it as a fraction in lowest terms.

6. Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{(2n)!}$.

   a) (8.3) Show that the domain of $f(x)$ is the set of all real numbers.

   b) (8.2) Find $f^{(2011)}(0)$, the 2011^{th} derivative of f at zero.

   c) (8.2) Find a power series representation of $f''(x)$. Write your answer in $\Sigma-$notation.

7. a) (8.1) Suppose you were to write the Taylor series centered at $x = 4$ of the function $f(x) = \dfrac{6}{\sqrt[3]{1-x}}$. Without actually writing this series, what would you expect the radius of convergence of this Taylor series to be? Why?

   b) (8.3) Determine whether or not the series $\sum_{n=0}^{\infty} \left(e^{1/n} - 1\right)$ converges.

Solutions

1. A power series is an arbitrary function of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$. A Taylor series is a specific power series associated to a function f; in particular the Taylor series of f centered at a is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ and is the only possible power series centered at a that can represent the function f on an open interval containing a.

2. a) This series can be split into the sum of two convergent geometric series:

$$\sum_{n=0}^{\infty} \frac{1+3^n}{7^n} = \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^n + \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n = \frac{1}{1-\frac{1}{7}} + \frac{1}{1-\frac{3}{7}} = \frac{7}{6} + \frac{7}{4} = \frac{35}{12}.$$

   b) This is a geometric series; begin by factoring out the first term:

$$\begin{aligned}
&72 - 24 + 8 - \frac{8}{3} + \frac{8}{9} - \frac{8}{27} + \frac{8}{81} - \frac{8}{3^5} + \dots \\
=&72\left[1 - \frac{1}{3} + \frac{1}{9} - \right] \\
=&72\left[1 + \left(\frac{-1}{3}\right) + \left(\frac{-1}{3}\right)^2 + \left(\frac{-1}{3}\right)^3 + \dots\right] \\
=&72\left[\frac{1}{1-\left(\frac{-1}{3}\right)}\right] = 72\left[\frac{3}{4}\right] = 54.
\end{aligned}$$

   c) This is (almost) the power series for $\sin x$ with $\frac{\pi}{2}$ plugged in for x:

$$\begin{aligned}
&\pi - \frac{\pi^3}{2^2 \cdot 3!} + \frac{\pi^5}{2^4 \cdot 5!} - \frac{\pi^7}{2^6 \cdot 7!} + \dots \\
=&2\left[\frac{\pi}{2} - \frac{\pi^3}{2^3 \cdot 3!} + \frac{\pi^5}{2^5 \cdot 5!} - \frac{\pi^7}{2^7 \cdot 7!} + \dots\right] \\
=&2\left[\frac{\left(\frac{\pi}{2}\right)}{1!} - \frac{\left(\frac{\pi}{2}\right)^3}{3!} + \frac{\left(\frac{\pi}{2}\right)^5}{5!} - \frac{\left(\frac{\pi}{2}\right)^7}{7!} + \dots\right] \\
=&2\sin\left(\frac{\pi}{2}\right) = 2 \cdot 1 = 2.
\end{aligned}$$

3. a) Observe that $\frac{3}{e^n+e^{-n}} \leq \frac{3}{e^n}$ and $\sum \frac{3}{e^n} = 3\sum\left(\frac{1}{e}\right)^n$ converges (since it is a geometric series with $r = 1/e$). Therefore by the Comparison Test, $\sum \frac{3}{e^n+e^{-n}}$ converges as well; it must **converge absolutely** since it is a positive series.

   b) Observe that $\sum \frac{(-1)^{n+1}}{\sqrt{n^2-4}}$ is an alternating series; we see

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\sqrt{n^2-4}}\right| = \lim_{n\to\infty}\frac{1}{\sqrt{n^2-4}} = 0$$

and

$$\begin{aligned}
&|a_n| \geq |a_{n+1}| \\
\Leftrightarrow\ &\frac{1}{\sqrt{n^2-4}} \geq \frac{1}{\sqrt{(n+1)^2-4}} \\
\Leftrightarrow\ &\sqrt{n^2-4} \leq \sqrt{(n+1)^2-4} \\
\Leftrightarrow\ &n^2-4 \leq (n+1)^2-4 \\
\Leftrightarrow\ &n^2 \leq (n+1)^2.
\end{aligned}$$

Since the last inequality is clearly true, so is the first one. Therefore by the Alternating Series Test, $\sum \frac{(-1)^{n+1}}{\sqrt{n^2-4}}$ converges.

Now we look at the series $\sum \left|\frac{(-1)^{n+1}}{\sqrt{n^2-4}}\right| = \sum \frac{1}{\sqrt{n^2-4}}$. Notice that for positive n, $\sqrt{n^2-4} \leq \sqrt{n^2} = n$, so taking reciprocals of both sides of this inequality yields $\frac{1}{\sqrt{n^2-4}} \geq \frac{1}{n}$. Now $\sum \frac{1}{n}$ diverges (it is the harmonic series, so by the Comparison Test, $\sum \frac{1}{\sqrt{n^2-4}}$ diverges as well. Putting this together, we can conclude that the original series $\sum \frac{(-1)^{n+1}}{\sqrt{n^2-4}}$ **converges conditionally**.

4. Notice that this power series is centered at -1 since it contains powers of $x+1$. Now, find the radius of convergence using Abel's Formula:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\left|\frac{(-2)^n}{n}\right|}{\left|\frac{(-2)^{n+1}}{n+1}\right|} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

This tells us that the series converges absolutely for $|x+1| < \frac{1}{2}$, i.e. when $x \in (-3/2, -1/2)$ and diverges when $x > -1/2$ or $x < -3/2$. Last, we check the endpoints:

When $x = -1/2$, the series becomes $\sum \frac{(-2)^n(1/2)^n}{n} = \sum \frac{(-1)^n}{n}$ which converges (it is the alternating harmonic series).

When $x = -3/2$, the series becomes $\sum \frac{(-2)^n(-1/2)^n}{n} = \sum \frac{1}{n}$ which diverges (this is the harmonic series).

Thus, the interval of convergence of the power series is $\left(\frac{-3}{2}, \frac{-1}{2}\right]$.

5. We have $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots$ so its third Taylor polynomial is $P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. Therefore

$$\sqrt{e} = e^{1/2} \approx P_3(1/2) = 1 + \frac{1}{2} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} = \frac{79}{48}.$$

6. a) We find the radius of convergence by Abel's Formula:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\left|\frac{n^2}{(2n)!}\right|}{\left|\frac{(n+1)^2}{[2(n+1)]!}\right|}$$
$$= \lim_{n\to\infty} \frac{n^2(2n+2)!}{(n+1)^2(2n)!}$$
$$= \lim_{n\to\infty} \frac{n^2(2n+2)(2n+1)}{(n+1)^2} = \infty$$

since the numerator has degree 4 and the denominator has degree 2. Since $R = \infty$, the series converges absolutely for all x by the Cauchy-Hadamard Theorem, and therefore the domain of this function is all real numbers.

b) Thinking of the power series as $\sum_{n=0}^{\infty} a_n x^n$, we see that $a_{2011} = \frac{2011^2}{2\cdot 2011)!} = \frac{2011^2}{4022!}$. At the same time, by uniqueness of coefficients of power series, we know that $a_{2011} = \frac{f^{(2011)}(0)}{2011!}$. Equating the two known expressions for a_{2011}, we obtain

$$\frac{2011^2}{4022!} = \frac{f^{(2011)}(0)}{2011!}$$

so we can solve for $f^{(2011)}(0)$ to obtain

$$= f^{(2011)}(0) = \frac{2011^2 2011!}{4022!}.$$

c) Differentiating the series twice term-by-term, we obtain

$$f''(x) = \sum_{n=2}^{\infty} \frac{n^3(n-1)x^{n-2}}{(2n)!}.$$

Notice that the series starts at $n = 2$ since the $n = 0$ and $n = 1$ terms of the series of $f(x)$ become zero when differentiated twice.

7. a) Notice that this function is continuous for all values of x except $x = 1$. This means that the Taylor series of this function, being a power series, cannot converge at $x = 1$ (since power series are continuous everywhere they converge). It stands to reason, therefore, that the radius of convergence of the Taylor series is the distance from the place the power series is centered (namely $x = 4$) to the closest value of x where the series cannot converge (namely $x = 1$). This distance is $4 - 1 = 3$. Therefore we should expect the radius of convergence to be 3.

b) We know that $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ for all x, and therefore $e^x - 1 = x+\frac{x^2}{2!}+\frac{x^3}{3!}$. Note that $e^x - 1 \geq x$ since you have to add positive terms to x to get the power series of $e^x - 1$. Therefore, by substituting in $\frac{1}{n}$ for x in the above inequality, we see

$$e^{1/n} - 1 \geq \frac{1}{n}.$$

(Notice that the inequality does not reverse, i.e. we are not taking reciprocals of both sides, we are just doing a substitution.) Since $\sum \frac{1}{n}$ diverges (it is harmonic), so does $\sum \left(e^{1/n} - 1\right)$ by the Comparison Test.

2.10 Fall 2011 Final Exam

1. a) (7.3) This semester we saw many different versions of the Triangle Inequality. Precisely state any one version of the Triangle Inequality.

   b) (5.1) Explain why adding up an infinite list of numbers is a "hard" problem. In particular, why are the ideas of calculus necessary to add up an infinite list of numbers?

2. (8.2) Consider the integral $\int_0^1 \sqrt{x+1}\,dx$. Approximate this integral by replacing the integrand with its second Taylor polynomial.

3. (2.5) Evaluate one of the following two integrals:

$$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx \qquad\qquad \int \frac{x}{(x-1)^{10}}\,dx$$

4. (2.3, 2.5) Evaluate one of the following two integrals:

$$\int \frac{\cos^2 x}{\sin^2 x}\,dx \qquad\qquad \int \sin^2 x \cos^2 x\,dx$$

5. a) (4.1) Find the area enclosed by the graphs of $y = 5x - x^2$ and $y = 9x + 3$. Simplify your answer.

   b) (4.3) Find the length of the graph of $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 0$ to $x = 2$. Simplify your answer.

6. Find the sum of each of the following series; simplify your answers.

   a) (6.2) $\frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{3^5} + \ldots$

   b) (8.2) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$

7. Consider the infinite series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$.

   a) (5.2) Find the third partial sum of the series. Simplify your answer completely.

   b) (5.2) Find a formula (in terms of N) for the N^{th} partial sum of the series. Your formula should be in "closed form", i.e. should not have any $\Sigma-$notation or "..." in it.

   c) (5.2) Use your answer to part (b) to find the sum of the infinite series.

8. (7.4) Determine whether or not each of the following series converges absolutely, converges conditionally, or diverges. Justify your reasoning.

$$\sum_{n=1}^{\infty}\left(\frac{3}{n}+\frac{2^n}{3^n}\right) \qquad\qquad \sum_{n=0}^{\infty}\frac{n!}{2012^n}$$

9. (8.1) Write the Taylor series (centered at $x = 0$) for each of the following functions.

   a) $f(x) = \ln(1 + 3x)$
   b) $g(x) = 4x^2 e^{x^2}$
   c) $h(x) = \frac{x}{(1-x)^3}$

10. (8.2) Evaluate the following limit:

$$\lim_{x\to 0}\frac{12\cos x^6 + 6x^{12} - 12}{x^{24}}$$

Solutions

1. a) **Triangle Inequality for real numbers:** For any real numbers a and b, $|a + b| \leq |a| + |b|$.

   Generalized Triangle Inequality for real numbers: Let $a_1, ..., a_n$ be real numbers. Then $|a_1 + ... + a_n| \leq |a_1| + ... + |a_n|$.

   Triangle Inequality for integrals: Let f be integrable on $[a, b]$. Then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

   Triangle Inequality for infinite series: If an infinite series converges absolutely, then it converges. Equivalently, $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$.

b) Addition of numbers is defined as a binary operation (there are two inputs and one output). To add a finite list of numbers, one adds them two at a time using the associative property. Since the list is finite, this process terminates since there are less numbers left to add after each addition. But for infinite series, numbers cannot be added two at a time because the process would never terminate. Additionally, the associative property fails for infinite series (they cannot be legally rearranged or regrouped unless they are known to absolutely converge).

2. We calculate the second Taylor polynomial by the definition. Notice that if we let $f(x) = \sqrt{x+1}$, then $f^{(0)}(0) = f(0) = 1$, $f'(0) = \frac{1}{2}(0+1)^{-1/2} = \frac{1}{2}$ and $f''(0) = \frac{-1}{4}(0+1)^{-3/2}$. Therefore the second Taylor polynomial of f centered at zero is

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

so the integral is approximated as follows:

$$\begin{aligned}\int_0^1 \sqrt{x+1}\,dx \approx \int_0^1 P_2(x)\,dx &= \int_0^1 \left[1 + \frac{1}{2}x - \frac{1}{8}x^2\right]\,dx \\ &= \left[x + \frac{1}{4}x^2 - \frac{1}{24}x^3\right]_0^1 \\ &= 1 + \frac{1}{4} - \frac{1}{24} = \frac{29}{24}.\end{aligned}$$

3. For the first integral, use a $u-$substitution $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$ so $2du = \frac{1}{\sqrt{x}}\,dx$. Then the integral becomes

$$\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx = \int 2\sqrt{u}\,du = 2 \cdot \frac{2}{3}u^{3/2} + C = \frac{4}{3}(1+\sqrt{x})^{3/2} + C.$$

For the second integral, use the $u-$substitution $u = x - 1$, $du = dx$. Then since $x = u + 1$, the integral becomes

$$\begin{aligned}\int \frac{x}{(x-1)^{10}}\,dx = \int \frac{u+1}{u^{10}}\,dx &= \int \left(u^{-9} + u^{-10}\right)\,dx \\ &= \frac{u^{-8}}{-8} + \frac{u^{-9}}{-9} + C \\ &= \frac{-1}{8}(x-1)^{-8} - \frac{1}{9}(x-1)^{-9} + C.\end{aligned}$$

4. For the first integral, use trig identities to rewrite the integrand:

$$\int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \cot^2 x\,dx = \int (\csc^2 x - 1)\,dx = -cotx - x + C.$$

For the second integral, also use trig identities to rewrite the integrand:

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\
&= \frac{1}{4}\int (1-\cos^2(2x))\, dx \\
&= \frac{1}{4}\int \sin^2(2x)\, dx \\
&= \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx \\
&= \frac{1}{8}\int (1-\cos 4x)\, dx \\
&= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + C = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.
\end{aligned}$$

5. a) First, find the $x-$coordinates of the points of intersection of the graphs; setting $5x - x^2 = 9x + 3$ yields $x^2 + 4x + 3 = 0$ which has solutions $x = -3, x = -1$. So the area between the graphs is

$$\begin{aligned}
\int_{-3}^{-1} [(5x - x^2) - (9x+3)]\, dx &= \int_{-3}^{-1} (-x^2 - 4x - 3)\, dx \\
&= \left[\frac{-1}{3}x^3 - 2x^2 - 3x\right]_{-3}^{-1} \\
&= (\frac{1}{3} - 2 + 3) - (9 - 18 + 9) = \frac{4}{3}.
\end{aligned}$$

b) First, by direct calculation we have $y' = \sqrt{x-1}$. Then, by the formula for arc length, we have

$$\begin{aligned}
L = \int_0^2 \sqrt{1 + [y']^2}\, dx &= \int_0^2 \sqrt{1 + (x-1)}\, dx \\
&= \int_0^2 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_0^2 = \frac{2}{3}\cdot 2^{3/2} = \frac{4}{3}\sqrt{2}.
\end{aligned}$$

6. a) This series is geometric:

$$\begin{aligned}
\frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{3^5} + \ldots &= \frac{2}{9}\left[1 + \frac{1}{3} + \frac{1}{9} + \ldots\right] \\
&= \frac{2}{9}\left[\frac{1}{1-\frac{1}{3}}\right] = \frac{2}{9}\cdot\frac{3}{2} = \frac{1}{3}.
\end{aligned}$$

b) This is the Taylor series for $\arctan x$ with $x = 1$ plugged in:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots = \arctan 1 = \frac{\pi}{4}.$$

7. a) $S_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = 1 + \frac{1}{4} - \frac{1}{5} = \frac{21}{20}.$

b) Notice that most of the numbers in this partial sum cancel, because they are both added and subtracted:

$$\begin{aligned}
S_N &= a_1 + a_2 + ... + a_N \\
&= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + ... + \left(\frac{1}{N-1} - \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{1}{N+2}\right) \\
&= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}
\end{aligned}$$

c) By defintition,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right). = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left[1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right] = \frac{3}{2}.$$

8. a) This series splits into the sum of two series, one which converges and one which diverges:

$$\sum_{n=1}^{\infty} \left(\frac{3}{n} + \frac{2^n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.$$

The first series diverges since it is a constant times the harmonic series, and the second series converges since it is geometric with $r = \frac{2}{3}$. Therefore the original series, being the sum of a convergent and divergent series, **diverges**.

b) Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{2012^{n+1}}}{\frac{n!}{2012^n}} = \lim_{n\to\infty} \frac{(n+1)!}{2012^{n+1}} \cdot \frac{2012^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2012} = \infty$$

and since $\rho > 1$, the series **diverges**.

9. a) Start with the Taylor series for $\ln(1+x)$ and replace all the x's with $3x$:

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \\
\ln(1+3x) &= 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + ... \\
&= 3x - \frac{3^2x^2}{2} + \frac{3^3x^3}{3} - \frac{3^4x^4}{4} + ... \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n} x^n.
\end{aligned}$$

b) Start with the Taylor series for e^x, replace the x's with x^2 and then multiply all the terms by $4x^2$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \\ e^{x^2} &= 1 + x^2 + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + ... \\ &= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + ... \\ 4x^2 e^{x^2} &= (4x^2)1 + (4x^2)x^2 + (4x^2)\frac{x^4}{2} + (4x^2)\frac{x^6}{3!} + (4x^2)\frac{x^8}{4!} + ... \\ &= 4x^2 + 4x^4 + \frac{4x^6}{2} + \frac{4x^8}{3!} + \frac{4x^{10}}{4!} + ... = \sum_{n=0}^{\infty} \frac{4}{n!} x^{2n+2}. \end{aligned}$$

c) Start with the Taylor series for $\frac{1}{1-x}$, differentiate twice and then multiply all the terms by $\frac{x}{2}$:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + ... \\ \frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) &= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + ... \\ \frac{2}{(1-x)^3} = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) &= 2 + 6x + 12x^2 + 20x^3 + 30x^4 + ... \\ \frac{x}{(1-x)^3} &= (x/2)2 + (x/2)6x + (x/2)12x^2 + (x/2)20x^3 + ... \\ &= x + 3x^2 + 6x^3 + 10x^4 + 15x^5 + ... = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-1} \end{aligned}$$

10. Start by writing the Taylor series of the function whose limit is being taken:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... \\ \cos x^6 &= 1 - \frac{x^{12}}{2} + \frac{x^{24}}{4!} - \frac{x^{36}}{6!} + \frac{x^{48}}{8!} - ... \\ 12\cos x^6 &= 12 - 12\frac{x^{12}}{2} + 12\frac{x^{24}}{4!} - 12\frac{x^{36}}{6!} + 12\frac{x^{48}}{8!} - ... \\ &= 12 - 6x^{12} + \frac{12}{4!}x^{24} - \frac{12}{6!}x^{36} + \frac{12}{8!}x^{48} - ... \\ 12\cos x^6 + 6x^{12} - 12 &= \frac{12}{4!}x^{24} - \frac{12}{6!}x^{36} + \frac{12}{8!}x^{48} - ... \\ \frac{12\cos x^6 + 6x^{12} - 12}{x^{24}} &= \frac{12}{4!} - \frac{12}{6!}x^{12} + \frac{12}{8!}x^{24} - ... \end{aligned}$$

As $x \to 0$, this quantity therefore approaches $\frac{12}{4!} = \frac{12}{24} = \frac{1}{2}$.

Chapter 3

Exams from 2012 to 2014

3.1 Spring 2013 Exam 1

1. (2.10) Each of the following integrals is evaluated either by a $u-$substitution or by integration by parts. If the integral is to be evaluated with a $u-$substitution, write "$U-$SUB" and say what u should be (for example, "$u = x^3$"). If the integral should be evaluated by parts, write "PARTS" and write what you would assign to be u and dv. You do not need to completely evaluate the integrals.

   a) $\int 3x^2 e^{5x}\, dx$

   b) $\int \frac{x}{4x^2+3}\, dx$

   c) $\int \arctan x\, dx$

   d) $\int 4x^2 \ln x\, dx$

2. Evaluate each of the following three integrals:

   a) (2.1) $\int \left(\frac{5}{3x} + 2x^4 - \sqrt{x}\right) dx$

   b) (2.2) $\int 2\sin 5x\, dx$

   c) (2.4) $\int 2x\cos(x^2+1)\, dx$

3. (2.6) Evaluate one of the following two integrals:

$$\int x\sec^2 x\, dx \qquad \int \frac{x}{e^x}\, dx$$

4. (2.5) Evaluate one of the following two integrals:

$$\int \sin^3 x\, dx \qquad \int \frac{8x^2}{2x+1}\, dx$$

5. (2.8) Evaluate one of the following two integrals:

$$\int \frac{4x^2 - 3x + 2}{x^3 - x^2}\,dx \qquad\qquad \int \frac{2x + 3}{(x-1)^3}\,dx$$

6. Given the improper integrals below, determine whether or not they converge. You must give sufficient reasoning or calculations to justify your answer.

   a) (3.2) $\displaystyle\int_2^4 \frac{2}{(x-2)^4}\,dx$ b) (3.4) $\displaystyle\int_1^\infty \frac{x^2 + e^{-x}}{4x^3}\,dx$

7. (**Bonus**) (2.9) Evaluate the following integral:

$$\int e^{\sqrt{x}}\,dx$$

Solutions

1. a) PARTS: $u = 3x^2$ and $dv = e^{5x}\,dx$.

   b) U-SUB: $u = 4x^2 + 3$.

   c) PARTS: $u = \arctan x$ and $dv = dx$.

   d) PARTS: $u = \ln x$ and $dv = 4x^2\,dx$.

2. a) $\displaystyle\int \left(\frac{5}{3x} + 2x^4 - \sqrt{x}\right)dx = \frac{5}{3}\int \frac{1}{x}\,dx + 2\int x^4\,dx - \int x^{1/2}\,dx = \frac{5}{3}\ln|x| + \frac{2}{5}x^5 - \frac{2}{3}x^{3/2} + C.$

   b) $\displaystyle\int 2\sin 5x\,dx = 2\int \sin 5x\,dx = \frac{-2}{5}\cos 5x + C.$

c) Let $u = x^2 + 1$ so that $du = 2x\,dx$. Then $\displaystyle\int 2x\cos(x^2+1)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2+1) + C$.

3. For the first integral, use parts: set $u = x$ and $dv = \sec^2 x\,dx$ so that $du = dx$ and $v = \tan x$. Then, by the parts formula,

$$\int x\sec^2 x\,dx = \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x\,dx.$$

The remaining integral is done with a $u-$substitution: set $u = \cos x\,dx$ so that $du = -\sin x\,dx$ and $-du = \sin x\,dx$. Then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$

Putting this together, we see that

$$\int x\sec^2 x\,dx = x\tan x + \ln|\cos x| + C.$$

For the second integral, rewrite the integral as $\int xe^{-x}\,dx$. Then use parts, setting $u = x$ and $du = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}$. By the parts formula,

$$\int xe^{-x}\,dx = \int u\,dv = uv - \int v\,du = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

4. For the first integral, start by rewriting it:

$$\int \sin^3 x\,dx = \int \sin^2 x\sin x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

Now perform a $u-$substitution on this integral; set $u = \cos x$ so that $du = -\sin x\,dx$ and $-du = \sin x\,dx$. The integral then becomes

$$\int -(1-u^2)\,du = \int -(1-u^2)\,du = -u + \frac{u^3}{3} + C = -\cos x + \frac{1}{3}\cos^3 x + C.$$

For the second integral, first think of it as

$$\int \frac{8x^2}{2x+1}\,dx = \int 4x^2 \cdot \frac{1}{2x+1} \cdot 2\,dx$$

This suggests the $u-$substitution $u = 2x+1$; $du = 2\,dx$. Now we solve for $4x^2$ to get $x = \frac{1}{2}(u-1)$ so $x^2 = \frac{1}{4}(u-1)^2$ so $4x^2 = (u-1)^2$. Now the integral becomes

$$\begin{aligned}
\int \frac{8x^2}{2x+1}\,dx = \int \frac{(u-1)^2}{u}\,du &= \int \frac{u^2-2u+1}{u}\,du \\
&= \int (u - 2 + \frac{1}{u})\,du \\
&= \frac{u^2}{2} - 2u + \ln|u| + C \\
&= \frac{(2x+1)^2}{2} - 2(2x+1) + \ln|2x+1| + C.
\end{aligned}$$

5. The first integral uses partial fractions:

$$\frac{4x^2-3x+2}{x^3-x^2} = \frac{4x^2-3x+2}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$$

Multiplying out, this produces the equation

$$4x^2-3x+2 = Ax(x-1)+B(x-1)+Cx^2.$$

Substituting in $x=0$ to this equation, we get $2=0-B+0$ so $B=-2$. Substituting in $x=1$ to the same equation, we get $4-3+2=0+0+C$ so $C=3$. Substituting in $x=2, B=-2$ and $C=3$ to this equation, we get $16-6+2=2A-2+12$, i.e. $A=1$. So the integral becomes

$$\int \frac{4x^2-3x+2}{x^3-x^2}\,dx = \int\left(\frac{1}{x}-\frac{2}{x^2}+\frac{3}{x-1}\right)\,dx = \ln|x|+\frac{2}{x}+3\ln|x-1|+C.$$

For the second integral, there are two methods of solution. The first solution is to use partial fractions:

$$\frac{2x+3}{(x-1)^3} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

Multiplying out, this produces the equation

$$2x+3 = A(x-1)^2+B(x-1)+C.$$

Substituting in $x=1$ to this equation gives $5=C$ and substituting in $x=0, C=5$ gives $3=A-B+5$, i.e. $A-B=-2$. Substituting in $x=2, C=5$ to the same equation yields $7=A+B+5$, i.e. $A+B=2$. Solving the two equations together for A and B yields $A=0, B=2$. Thus

$$\int \frac{2x+3}{(x-1)^3}\,dx = \int\left(\frac{2}{(x-1)^2}+\frac{5}{(x-1)^3}\right)\,dx = \frac{-2}{x-1}+\frac{-5/2}{(x-1)^3}+C.$$

An alternate solution to the second integral is to perform the $u-$substitution $u=x-1$ so $du=dx$. Solving for x we see $x=u+1$ so altogether this substitution yields

$$\begin{aligned}
\int \frac{2x+3}{(x-1)^3}\,dx &= \int 2(u+1)+3u^3\,du \\
&= \int \frac{2u+5}{u^3}\,du \\
&= \int\left(2u^{-2}+5u^{-3}\right)\,du \\
&= -2u^{-1}+\frac{-5}{2}u^{-2}+C \\
&= \frac{-2}{x-1}+\frac{-5/2}{(x-1)^2}+C.
\end{aligned}$$

6. a) This integral is improper because the integrand has a vertical asymptote when $x = 2$:

$$\begin{aligned}\int_2^4 \frac{2}{(x-2)^4}\,dx &= \lim_{b\to 2^+}\int_b^4 \frac{2}{(x-2)^4}\,dx\\ &= \lim_{b\to 2^+} \frac{-2}{3}\cdot\frac{1}{(x-2)^3}\bigg|_b^4\,dx\\ &= \lim_{b\to 2^+}\left[\frac{-2}{3}\cdot\frac{1}{2^3}-\frac{-2}{3}\cdot\frac{1}{(b-2)^3}\right]\\ &= \lim_{b\to 2^+}\left[\frac{-1}{12}+\frac{2}{3(b-2)^3}\right]\\ &= \frac{-1}{12}+\frac{2}{3\cdot 0}=\infty.\end{aligned}$$

Thus the integral diverges.

b) Apply the Comparison Test for Improper Integrals. Notice that

$$\frac{x^2+e^{-x}}{4x^3} \geq \frac{x^2}{4x^3} = \frac{1}{4}\frac{1}{x}.$$

Now we proved in class that $\int_1^\infty \frac{1}{x}\,dx$ diverges; therefore so does $\int_1^\infty \frac{1}{4}\frac{1}{x}\,dx$. By the Comparison Test,

$$\int_1^\infty \frac{x^2+e^{-x}}{4x^3}\,dx$$

also diverges.

7. Perform the $t-$substitution (like a $u-$sub but I will use the letter t) $t = \sqrt{x}$. Then $dt = \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2t}dx$ so $dx = 2t\,dt$. Now

$$\int e^{\sqrt{x}}\,dx = \int e^t 2t\,dt.$$

Now perform integration by parts on the right-hand integral: set $u = 2t$ and $dv = e^t\,dt$ so that $du = 2\,dt$ and $v = e^t$. By the parts formula,

$$\int e^t 2t\,dt = \int u\,dv = uv - \int v\,du = 2te^t - \int 2e^t\,dt = 2te^t - 2e^t + C.$$

Finally, back-substitute for x to get

$$\int e^{\sqrt{x}}\,dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

3.2 Spring 2013 Exam 2

1. Let R be the region enclosed by the graphs of $y = x^3$, $y = 0$, and $y = -2x+12$.

   a) (4.1) Write an expression involving one or more integral(s) with respect to x that gives the area of R.

   b) (4.1) Write an expression involving one or more integral(s) with respect to y that gives the area of R.

   c) (4.2) Let S_1 be the solid formed by revolving R around the $y-$axis. Write an expression involving one or more integrals with respect to x that gives the volume of S_1.

   d) (4.2) Write an expression involving one or more integrals with respect to y that gives the volume of S_1, where S_1 is the solid in the previous problem.

   e) (4.2) Let S_2 be the solid formed by revolving R around the line $y = -2$. Write an expression involving one or more integrals with respect to whatever variable you like that gives the volume of S_2.

2. a) (4.5) Consider a wire of length 4 m whose density x inches from the left endpoint of the wire is given by $\delta(x) = \cos x + 1$ mg/m. Write an expression involving one or more integral(s) which gives how far from the left endpoint of the wire its center of mass is.

   b) (4.6) Write expressions involving one or more integral(s) which find the center of mass of the triangle whose vertices are $(0, 0)$, $(0, 1)$ and $(1, 0)$, assuming the triangle has constant density.

3. Suppose that X is a random variable whose range is $[0, 2]$ and whose density function is $f(x) = cx^2 + x$ for some constant c.

   a) (4.8) Find c.

   b) (4.8) Find the probability that X is less than 1.

   c) (4.8) Find the expected value of X.

4. (4.4) Write an expression which gives the length of the curve $y = e^{3x}$ from $x = 1$ to $x = 8$.

5. (4.3) In physics, if the net current going through a wire at time t is a constant I, then the electric charge Q transferred over Δt units of time is given by $Q = I \cdot \Delta t$. Suppose the net current going through a wire at time t is **not constant**, and is equal to $I(t) = 2t - 3$. What is the electric charge Q transferred from time $t = 0$ to time $t = 2$? (I actually want you to find the answer.)

Solutions

1. Solve for the corner points of R:

   - the curves $y = x^3$ and $y = 0$ meet at $(0, 0)$;
   - the curves $y = x^3$ and $y = -2x + 12$ meet at $(2, 8)$;
   - the curves $y = 0$ and $y = -2x + 12$ meet at $(6, 0)$.

   Then:

   a) $A = \int_0^2 \left[x^3 - 0\right] \, dx + \int_2^6 \left[(-2x+12) - 0\right] \, dx = \int_0^2 x^3 \, dx + \int_2^6 (-2x+12) \, dx.$

   b) Solve the equations for x in terms of y: $y = x^3$ is $x = \sqrt[3]{y}$ and $y = -2x + 12$ is $x = \frac{y-12}{-2} = \frac{12-y}{2}$. Then:

   $$A = \int_0^8 \left[\frac{12-y}{2} - \sqrt[3]{y}\right] \, dy.$$

   c) Since the direction of integration is horizontal but the axis of revolution is vertical, this is the shell method (the cross-sectional area is $2\pi r h$):

   $$V = \int_0^2 2\pi x(x^3) \, dx + \int_2^6 2\pi x(-2x+12) \, dx.$$

   d) Since the direction of integration and axis of revolution are vertical, this is the washer method (the cross-sectional area is $\pi R^2 - \pi r^2$):

   $$V = \int_0^8 \left[\pi \left(\frac{12-y}{2}\right)^2 - \pi \left(\sqrt[3]{y}\right)^2\right] \, dy.$$

   e) If one integrates with respect to x, this is the washer method since both the direction of integration and axis of revolution are horizontal:

   $$V = \int_0^2 \left[\pi(x^3+2)^2 - \pi 2^2\right] \, dx + \int_2^6 2 \left[\pi(-2x+12+2)^2 - \pi 2^2\right] \, dx.$$

   If one integrates with respect to y, this is the shell method since the direction of integration is vertical but the axis of revolution is horizontal:

   $$V = \int_0^8 2\pi(y+2) \left(\frac{12-y}{2} - \sqrt[3]{y}\right) \, dy.$$

2. a) We have

   $$\overline{x} = \frac{M_0}{M} = \frac{\int_0^4 x\delta(x) \, dx}{\int_0^4 \delta(x) \, dx} = \frac{\int_0^4 x(\cos x + 1) \, dx}{\int_0^4 (\cos x + 1) \, dx}.$$

b) The top of this triangle is the line $y = -x+1$. So this is a region bounded above by $f(x) = -x + 1 = 1 - x$ and below by $y = 0$ from $x = 0$ to $x = 1$. So by the formulas derived in class, the center of mass is $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{M_y}{M} = \frac{\int_0^1 x\delta(x)[f(x) - g(x)]\,dx}{\int_0^1 \delta(x)[f(x) - g(x)]\,dx} = \frac{\int_0^1 x(1-x)\,dx}{\int_0^1 (1-x)\,dx};$$

$$\overline{y} = \frac{M_x}{M} = \frac{\int_0^1 x\frac{1}{2}\delta(x)[(f(x))^2 - (g(x))^2]\,dx}{\int_0^1 \delta(x)[f(x) - g(x)]\,dx} = \frac{\int_0^1 \frac{1}{2}x(1-x)^2\,dx}{\int_0^1 (1-x)\,dx}.$$

(By symmetry, it is clear that $\overline{x} = \overline{y}$.)

3. Suppose that X is a random variable whose range is $[0, 2]$ and whose density function is $f(x) = cx^2 + x$ for some constant c.

a) We know the density function must integrate to 1, so we have

$$1 = \int_0^2 (cx^2 + x)\,dx = \left[\frac{c}{3}x^3 + \frac{1}{2}x^2\right]_0^2 = \frac{8c}{3} + 2.$$

Solving the equation $\frac{8c}{3} + 2 = 1$ for c, we see $c = \frac{-3}{8}$.

b) $P(X < 1) = \int_0^1 f(x)\,dx = \int_0^1 \left(\frac{-3}{8}x^2 + x\right)\,dx = \left[\frac{-1}{8}x^3 + \frac{1}{2}x^2\right]_0^1 = \frac{-1}{8} + \frac{1}{2} = \frac{3}{8}$.

c) $EX = \int_0^2 x\,f(x)\,dx = \int_0^2 x(cx^2+x)\,dx = \int_0^2 (cx^3+x^2)\,dx = \left[\frac{c}{4}x^4 + \frac{1}{3}x^3\right]_0^2 = \frac{7}{6}$.

4. If $f(x) = e^{3x}$, then $f'(x) = 3e^{3x}$ so the length is

$$s = \int_1^8 \sqrt{1 + [f'(x)]^2}\,dx = \int_1^8 \sqrt{1 + (3e^{3x})^2}\,dx.$$

5. By the general principal of applications of integration, the formula $Q = I \cdot \Delta t$ translates into the formula $Q = \int_a^b I(t)\,dt$. Here the answer is given by

$$Q = \int_0^2 (2t - 3)\,dt = \Big[t^2 - 3t\Big]_0^2 = -2.$$

3.3 Spring 2013 Exam 3

1. a) (7.3) Define **precisely** what it means for a series $\sum a_n$ to *converge absolutely*.

   b) (7.3) Give a specific example of a series that converges absolutely (no justification required).

   c) (7.3) Define **precisely** what it means for a series $\sum a_n$ to *converge conditionally*.

   d) (7.3) Give a specific example of a series that converges conditionally (no justification required).

   e) (7.3) Why do we care whether or not a series converges absolutely (as opposed to just knowing whether or not a series converges)?

2. Find the sum of each of these series. Please simplify your answer:

   a) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{3 \cdot 2^{3n-1}}{5 \cdot 3^{2n+1}}$

   b) (6.2) $18 - 6 + 2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81} - \ldots$

   c) (8.2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n}$

3. (7.4) Determine, with justification, whether each of the following series converges absolutely, converges conditionally, or diverges:

   a) $\displaystyle\sum_{n=3}^{\infty} \frac{5}{\sqrt{n}-1}$

   b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^8} + \frac{1}{8^n}\right)$

4. (8.3) Determine, with justification, the set of x for which the following power series converges:
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n\sqrt{n}}(x-5)^n$$

5. a) (8.2) Approximate $\cos\frac{1}{4}$ by computing the third Taylor polynomial for an appropriately chosen function.

   b) (8.2) Approximate the integral $\int_0^1 e^{-x^2}\,dx$ by replacing the integrand with its fourth Taylor polynomial.

   c) (8.2) Evaluate the following limit without using L'Hôpital's Rule:
$$\lim_{x\to 0} \frac{\sin x^2 - x^2}{\cos x^3 - 1}$$

Solutions

1. a) $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.

   b) $\sum \frac{1}{n^2}$

   c) $\sum a_n$ *converges conditionally* if $\sum a_n$ converges but $\sum |a_n|$ diverges.

   d) $\sum \frac{(-1)^{n+1}}{n}$

   e) The terms of a series can be rearranged/regrouped without changing the sum if and only if the series converges absolutely.

2. a) $\sum_{n=0}^{\infty} \frac{3 \cdot 2^{3n-1}}{5 \cdot 3^{2n+1}} = \frac{3 \cdot 2^{-1}}{3} \sum_{n=0}^{\infty} \frac{8^n}{9^n} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{8}{9}} = \frac{9}{2}$

   b) $18-6+2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\frac{2}{81}-... = 18\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = 18\left[\frac{1}{1-\frac{-1}{3}}\right] = 18\left(\frac{3}{4}\right) = \frac{27}{2}$

   c) $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n} = 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} 3^n = 3\ln(3+1) = 3\ln 4.$

3. a) Notice that $0 \leq \frac{5}{\sqrt{n}} \leq \frac{5}{\sqrt{n}-1}$. Since $\sum \frac{5}{\sqrt{n}}$ diverges (it is a $p-$series with $p = \frac{1}{2} \leq 1$), by the Comparison Test $\sum \frac{5}{\sqrt{n}-1}$ **diverges** as well.

   b) Notice $\sum \frac{1}{n^8}$ converges (it is a $p-$series with $p = 8 > 1$) and $\sum \frac{1}{8^n}$ converges (it is geometric with $r = \frac{1}{8}$). Therefore the series $\sum_{n=1}^{\infty} \left(\frac{1}{n^8} + \frac{1}{8^n}\right)$ is the sum of two convergent series, hence converges. Since this series is positive, it **converges absolutely**.

4. Apply Abel's Formula to find the radius of convergence:

$$\begin{aligned} R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{2^n\sqrt{n}}\right|}{\left|\frac{(-1)^{n+1}}{2^{n+1}\sqrt{n+1}}\right|} \\ &= \lim_{n\to\infty} \frac{1}{2^n\sqrt{n}} \cdot \frac{2^{n+1}\sqrt{n+1}}{1} \\ &= \lim_{n\to\infty} \frac{2\sqrt{n+1}}{\sqrt{n}} \\ &= 2\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} = 2\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 2 \end{aligned}$$

Thus the interval of convergence goes from $a - R = 5 - 2 = 3$ to $a + R = 5 + 2 = 7$. Now test the endpoints:

$$x = 3 \Rightarrow \sum \frac{(-1)^n}{2^n\sqrt{n}}(3-5)^n = \sum \frac{(-1)^n}{2^n\sqrt{n}}(-2)^n = \sum \frac{1}{\sqrt{n}} \text{diverges } (p\text{−series}, p = \frac{1}{2} \leq 1)$$

$$x = 7 \Rightarrow \sum \frac{(-1)^n}{2^n\sqrt{n}}(7-5)^n = \sum \frac{(-1)^n}{2^n\sqrt{n}}(2)^n = \sum \frac{(-1)^n}{\sqrt{n}} \text{converges (Alt. Series Test))}$$

Summarizing, this series converges when $x \in (3, 7]$.

5. a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ so $P_3(x) = 1 - \frac{x^2}{2!} = 1 - \frac{1}{2}x^2$. Therefore

$$\cos\frac{1}{4} \approx P_3(\frac{1}{4}) = 1 - \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{31}{32}.$$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ so $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...$ and therefore $P_4(x) = 1 - x^2 + \frac{1}{2}x^4$. Therefore

$$\begin{aligned}\int_0^1 e^{-x^2}\,dx \approx \int_0^1 P_4(x)\,dx &= \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4\right)\,dx \\ &= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30}.\end{aligned}$$

c) Write Taylor series for the functions $\sin x^2$ and $\cos x^3$ and substitute:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x^2 - x^2}{\cos x^3 - 1} &= \lim_{x\to 0}\frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}\cdots - x^2}{1 - \frac{x^6}{2!} + \frac{x^{12}}{4!}\cdots - 1} \\ &= \lim_{x\to 0}\frac{-\frac{x^6}{3!} + \frac{x^{10}}{5!}\cdots}{-\frac{x^6}{2!} + \frac{x^{12}}{4!}\cdots} \\ &= \lim_{x\to 0}\frac{x^6\left[\frac{-1}{3!} + \frac{x^4}{5!}\cdots\right]}{x^6\left[\frac{-1}{2!} + \frac{x^6}{4!}\cdots\right]} \\ &= \lim_{x\to 0}\frac{\frac{-1}{3!} + \frac{x^4}{5!}\cdots}{\left[\frac{-1}{2!} + \frac{x^6}{4!}\cdots\right]} \\ &= \frac{\frac{-1}{3!}}{\frac{-1}{2!}} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}.\end{aligned}$$

3.4 Spring 2013 Final Exam

1. For each part of this problem, you are given two integrals. For each part, choose one of the two integrals and evaluate it :

   a) (2.3)
   $$\int \tan x\,dx \qquad \int \tan^2 x\,dx$$

   b) (2.8)
   $$\int \frac{x-1}{x^2+3x+2}\,dx \qquad \int \frac{2x-3}{(x+1)^2}\,dx$$

   c) (2.6)
   $$\int 3x\cos 2x\,dx \qquad \int x^2(x-1)^{2/3}\,dx$$

2. a) (3.3) Determine whether the following improper integral converges or diverges:
   $$\int_4^\infty \frac{3x}{x^4+7}\,dx$$

   b) (3.2) Determine whether the following improper integral converges or diverges:
   $$\int_1^\infty \frac{\ln x}{x}\,dx$$

3. Let R be the region in the $xy-$plane lying below the graph of $y=\sqrt{x}$, above the $x-$axis and to the left of the line $x=4$.

   a) (4.1) Find the area of R.

   b) (4.2) Write an integral (or integrals) in whatever variable you like that gives the volume of the solid obtained by revolving R around the line $y=-1$.

   c) (4.2) Write an integral (or integrals) in whatever variable you like that gives the volume of the solid obtained by revolving R around the $y-$axis.

4. a) (4.4) Write an integral which gives the arc length of the curve $y=\sin x$ from $x=0$ to $x=2\pi$.

   b) (4.8) Suppose X is a random variable taking values between 0 and 4 whose density function is $f(x)=\frac{1}{8}x$. Find the expected value of X.

5. Find the sum of each of these series. Please simplify your answer:

   a) (6.2) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)^n$ b) (6.2) $\displaystyle\sum_{n=0}^{\infty}\frac{3^{2n+1}}{7\cdot 2^{4n-1}}$ c) (8.2) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)!}$

6. (7.4) Choose two of the following three series. For the two series you choose, determine (with justification) whether each of the following series converges absolutely, converges conditionally, or diverges:

$$\sum_{n=3}^{\infty} \frac{2+\sin n}{3n^2} \qquad\qquad \sum_{n=0}^{\infty} \frac{n!(n+1)!}{(2n)!} \qquad\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4\sqrt[3]{n}}$$

7. Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{e^n}{n!} x^n$.

   a) (8.3) Find the set of x for which the power series converges.

   b) (8.2) Let $g(x) = x^3 f(2x^2)$. Write the power series for g in $\Sigma-$notation.

   c) (8.2) Find $g^{(2013)}(0)$, the 2013^{th} derivative of g at zero.

8. a) (8.2) Approximate $\sqrt[4]{e}$ (the fourth root of e) by computing the second Taylor polynomial for an appropriately chosen function.

   b) (8.2) Approximate the integral $\int_0^1 \ln(x^3+1)\, dx$ by replacing the integrand with its sixth Taylor polynomial.

   c) (8.2) Evaluate the following limit without using L'Hôpital's Rule:

   $$\lim_{x \to 0} \frac{\arctan x - x}{2x^3}$$

   d) (6.3) Write the repeating decimal $.02525252525...$ as a fraction in lowest terms. *Hint:* write the fraction as a geometric series.

Solutions

1. a) i. Start by rewriting the integral as $\int \frac{\sin x}{\cos x}\,dx$. Then use the u-sub $u = \cos x$, $du = -\sin x\,dx$ so $-du = \sin x\,dx$ to obtain $\int -\frac{1}{u}\,du = -\ln u + C = -\ln\cos x + C$.

   ii. Start by rewriting the integral as $\int(\sec^2 x - 1)\,dx$. This can be done directly to obtain $\tan x - x + C$.

   b) i. Rewrite the integrand using partial fractions. The denominator factors as $(x+1)(x+2)$ so we have $\frac{A}{x+1} + \frac{B}{x+2} = \frac{x-1}{x^2+3x+2}$. Finding a common denominator and then clearing denominators, we obtain $A(x+2) + B(x+1) = x - 1$. Plug in $x = -2$ to get $-B = -3$, i.e. $B = 3$. Plug in $x = -1$ to get $A = -2$. Thus the integral is

$$\int \frac{x-1}{x^2+3x+2}\,dx = \int \left(\frac{-2}{x+1} + \frac{3}{x+2}\right)\,dx = -2\ln(x+1) + 3\ln(x+2) + C.$$

   ii. This integral can be done with either partial fractions or a u-substitution. For the partial fractions method, the appropriate guessed form of the decomposition is $\frac{A}{x+1} + \frac{B}{(x+1)^2}$; after finding common denominators and then clearing denominators, we obtain $A(x+1) + B = 2x - 3$. Thus $A = 2$ and $B = -5$. Therefore the integral is

$$\int \frac{2x-3}{(x+1)^2}\,dx = \int \left(\frac{2}{x+1} + \frac{-5}{(x+1)^2}\right)\,dx = 2\ln(x+1) + 5(x+1)^{-1} + C.$$

   For the u-substitution method, let $u = x+1$ so $du = dx$. Therefore $x = u-1$ so the integral becomes

$$\begin{aligned}\int \frac{2(u-1)-3}{u^2}\,du = \int \frac{2u-5}{u^2}\,du &= \int \left(\frac{2}{u} - 5u^{-2}\right)\,du \\ &= 2\ln u + 5u^{-1} + C \\ &= 2\ln(x+1) + 5(x+1)^{-1} + C.\end{aligned}$$

   c) i. Use parts with $u = 3x$ and $dv = \cos 2x\,dx$. Therefore $du = 3\,dx$ and $v = \int dv = \int \cos 2x\,dx = \frac{1}{2}\sin 2x$. Therefore by the parts formula, we have

$$\begin{aligned}\int 3x\cos 2x\,dx = \int u\,dv = uv - \int v\,du \\ &= 3x\cdot\frac{1}{2}\sin 2x - \int 3\cdot\frac{1}{2}\sin 2x\,dx \\ &= \frac{3}{2}x\sin 2x - \frac{3}{2}\int \sin 2x\,dx \\ &= \frac{3}{2}x\sin 2x - \frac{3}{2}\cdot\frac{-1}{2}\cos 2x + C \\ &= \frac{3}{2}x\sin 2x + \frac{3}{4}\cos 2x + C.\end{aligned}$$

ii. This integral is best handled with the u-substitution $u = x - 1$. Therefore $x = u + 1$ and $du = dx$ so the integral becomes

$$\begin{aligned}\int (u+1)^2 u^{2/3}\,du &= \int (u^2 + 2u + 1)u^{2/3}\,du \\ &= \int \left(u^{8/3} + 2u^{5/3} + u^{2/3}\right)\,du \\ &= \frac{3}{11}u^{11/3} + \frac{3}{4}u^{8/3} + \frac{3}{5}u^{5/3} + C.\end{aligned}$$

(This can also be done by using integration by parts twice (starting with $u = x^2$ and $dv = (x-1)^{2/3}$) but I won't give that solution here.)

2. a) Notice that $0 \leq \frac{3x}{x^4+7} \leq \frac{3x}{x^4} = \frac{3}{x^3}$. Since $\int_4^\infty \frac{3}{x^3}\,dx$ converges (it is a p-integral with $p = 3 > 1$), so does $\int_4^\infty \frac{3x}{x^4+7}\,dx$ by the Comparison Test.

b) This integral is computed directly:

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x}\,dx \\ &\qquad u\text{-sub } u = \ln x, du = \frac{1}{x}\,dx \\ &= \lim_{b\to\infty}\int_0^{\ln b} u\,du \\ &= \lim_{b\to\infty} \left.\frac{u^2}{2}\right|_0^{\ln b} \\ &= \lim_{b\to\infty}\frac{1}{2}\ln^2 b = \frac{1}{2}(\ln\infty)^2 = \infty.\end{aligned}$$

Therefore the integral diverges.

3. Let R be the region in the $xy-$plane lying below the graph of $y = \sqrt{x}$, above the $x-$axis and to the left of the line $x = 4$.

a) Integrate the function from 0 to 4:

$$A = \int_0^4 \sqrt{x}\,dx = \left.\frac{2}{3}x^{3/2}\right|_0^4 = \frac{2}{3}4^{3/2} = \frac{16}{3}.$$

b) Integrating with respect to x, the direction of integration is parallel to the axis of revolution, so you get washers:

$$V = \int_0^4 \left[\pi R^2 - \pi r^2\right]\,dx = \int_0^4 \left[\pi(\sqrt{x}+1)^2 - \pi(1)^2\right]\,dx.$$

Integrating with respect to y (solve the equation $y = \sqrt{x}$ for y to get $x = y^2$), the direction of integration is perpendicular to the axis of revolution, so you get shells. Notice also that the top corner of the region is the point $(4, 2)$ so y goes from 0 to 2:

$$V = \int_0^2 2\pi r h\,dy = 2\pi(y+1)(2-y^2)\,dy.$$

c) Integrating with respect to x, the direction of integration is perpendicular to the axis of revolution, so you get shells:

$$V = \int_0^4 2\pi r h\, dx = 2\pi(x)(\sqrt{x})\, dx.$$

Integrating with respect to y, the direction of integration is parallel to the axis of revolution, so you get washers:

$$V = \int_0^2 \left[\pi R^2 - \pi r^2\right]\, dx = \int_0^2 \left[\pi(4^2)^2 - \pi(y^2)^2\right]\, dx.$$

4. a) Note $\frac{dy}{dx} = \cos x$ so by the arc length formula,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{2\pi} \sqrt{1 + \cos^2 x}\, dx.$$

b) Applying the formula for expected value:

$$EX = \int_a^b x f(x)\, dx = \int_0^4 x \frac{1}{8} x\, dx = \int_0^4 \frac{1}{8} x^2\, dx = \left.\frac{x^3}{24}\right|_0^4 = \frac{64}{24} = \frac{8}{3}.$$

5. a) This is a geometric series. First, factor out $\frac{-1}{5}$ and rewrite the series so that it starts at $n = 0$:

$$\sum_{n=1}^{\infty} \left(\frac{-1}{5}\right)^n = \frac{-1}{5} \sum_{n=0}^{\infty} \left(\frac{-1}{5}\right)^n = \frac{-1}{5} \cdot \frac{1}{1 - (-1/5)} = \frac{-1}{5} \cdot \frac{5}{6} = \frac{-1}{6}.$$

b) This is a geometric series:

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{7 \cdot 2^{4n-1}} = \sum_{n=0}^{\infty} \frac{3 \cdot 9^n}{7 \cdot 12 \cdot 16^n} = \frac{6}{7} \sum_{n=0}^{\infty} \left(\frac{9}{16}\right)^n = \frac{6}{7} \cdot \frac{1}{1 - \frac{9}{16}} = \frac{6}{7} \cdot \frac{16}{7} = \frac{96}{49}.$$

c) Note that $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. The given series is the same with π plugged in for x, so

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} = \sin \pi = 0.$$

6. a) Note that $0 \leq \frac{2+\sin n}{3n^2} \leq \frac{2+1}{3n^2} = \frac{1}{n^2}$. The series $\sum \frac{1}{n^2}$ converges (p-series, $p = 2 > 1$) so by the Comparison Test, the given series converges as well.

b) Use the Ratio Test:

$$\begin{aligned}
\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\left|\frac{(n+1)!(n+2)!}{(2(n+1))!}\right|}{\left|\frac{n!(n+1)!}{(2n)!}\right|} \\
&= \lim_{n\to\infty} \frac{(n+1)!(n+2)!}{(2(n+1))!} \cdot \frac{(2n)!}{n!(n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+2)(n+1)}{(2n+2)!} \cdot \frac{(2n)!}{1} \\
&= \lim_{n\to\infty} \frac{(n+2)(n+1)}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{n^2+3n+2}{4n^2+6n+2} = \frac{1}{4}.
\end{aligned}$$

(If necessary, use L'Hôpital's Rule twice to get the last step.) Since $\rho < 1$, the series converges absolutely by the Ratio Test.

c)
- This series is alternating;
- notice that $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}}{4\sqrt[3]{n}}\right| = 0$;
- note that the terms are decreasing in absolute value: $|a_{n+1}| = \frac{1}{4\sqrt[3]{n+1}} \leq \frac{1}{4\sqrt[3]{n}} = |a_n|$.

Therefore the series converges by the Alternating Series Test. To determine whether the convergence is absolute or conditional, consider $\sum |a_n| = \sum \frac{1}{4\sqrt[3]{n}} = \frac{1}{4}\sum \frac{1}{n^{1/3}}$. This is a p-series with $p = \frac{1}{3} \leq 1$ which diverges. Therefore the original series only converges conditionally.

7. Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{e^n}{n!} x^n$.

a) The series is centered at $a = 0$. By Abel's Formula, the radius of convergence is

$$\begin{aligned}
R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} &= \lim_{n\to\infty} \frac{\left|\frac{e^n}{n!}\right|}{\left|\frac{e^{n+1}}{(n+1)!}\right|} \\
&= \lim_{n\to\infty} \frac{e^n}{n!} \cdot \frac{(n+1)!}{e^{n+1}} \\
&= \lim_{n\to\infty} \frac{n+1}{e} = \infty.
\end{aligned}$$

Since the radius of convergence is infinite, the series converges for all $x \in (-\infty, \infty)$.

b) Start with the given series, substitute $2x^2$ for x and then multiply through by

x^3:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{e^n}{n!} x^n \\ f(2x^2) &= \sum_{n=0}^{\infty} \frac{e^n}{n!} (2x^2)^n \\ g(x) = x^3 f(2x^2) &= \sum_{n=0}^{\infty} \frac{e^n}{n!} (2x^2)^n x^3 \\ &= \sum_{n=0}^{\infty} \frac{e^n 2^n}{n!} x^{2n+3}. \end{aligned}$$

c) By the uniqueness of power series, the coefficient on the x^{2013} term of the Taylor series of $g(x)$ is $\frac{g^{(2013)}(0)}{2013!}$. This occurs when $2n+3 = 2013$, i.e. when $2n = 2010$, i.e. when $n = 1005$. So the coefficient on the x^{2013} term is $\frac{e^{1005}2^{1005}}{1005!}$. Therefore

$$\frac{g^{(2013)}(0)}{2013!} = \frac{e^{1005}2^{1005}}{1005!} \quad \Rightarrow \quad g^{(2013)}(0) = \frac{2013! e^{1005}2^{1005}}{1005!}.$$

8. a) Let $f(x) = e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$ We are trying to find $\sqrt[4]{e} = e^{1/4} = f(1/4)$. This is roughly $P_2(\frac{1}{4}) = 1 + \frac{1}{4} + \frac{(1/4)^2}{2} = \frac{41}{32}$.

b) We know $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - ...$ so by replacing x with x^3 we get $\ln(x^3 + 1) = x^3 - \frac{x^6}{2} + \frac{x^9}{3}....$ Thus the sixth Taylor polynomial of the integrand is $P_6(x) = x^3 - \frac{x^6}{2}$. Therefore

$$\int_0^1 \ln(x^3+1)\,dx \approx \int_0^1 \left[x^3 - \frac{x^6}{2}\right] dx = \left[\frac{x^4}{4} - \frac{x^7}{14}\right]_0^1 = \frac{1}{4} - \frac{1}{14} = \frac{5}{28}.$$

c) Write the Taylor series of the function you are taking the limit of:

$$\begin{aligned} \lim_{x\to 0} \frac{\arctan x - x}{2x^3} &= \lim_{x\to 0} \frac{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... - x}{2x^3} \\ &= \lim_{x\to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...}{2x^3} \\ &= \lim_{x\to 0} \left[-\frac{1}{6} + \frac{x^2}{10} - \frac{x^4}{14} + ...\right] \\ &= \frac{-1}{6} + 0 - 0 + ... = \frac{-1}{6}. \end{aligned}$$

d) Following the hint,

$$
\begin{aligned}
.02525252525\ldots &= .025 + .00025 + .0000025 + \ldots \\
&= \frac{25}{10^3} + \frac{25}{10^5} + \frac{25}{10^7} + \ldots \\
&= \frac{25}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right] \\
&= \frac{25}{10^3}\left[1 + \frac{1}{100} + \left(\frac{1}{100}\right)^2 + \ldots\right] \\
&= \frac{25}{1000}\frac{1}{1-\frac{1}{100}} \\
&= \frac{25}{1000}\cdot\frac{100}{99} = \frac{25}{990} = \frac{5}{198}.
\end{aligned}
$$

3.5 Fall 2013 Exam 1

1. a) Suppose that a student is trying to differentiate $f(x) = e^{x^2-3}$ using *Mathematica*, and types in the commands you see below:

```
f[x_] = e^(x^2 - 3)
f'[x]
```

   This won't give the right answer. What is wrong with what the student typed in?

   b) Write the *Mathematica* code that will produce a plot of the function $f(x) = \ln x$, for $x-$values between 0 and 7.

   c) Suppose you were to type in the following command into *Mathematica*:

```
Integrate[x^2/3 Sin[x], {x, 0, 3}]
```

   This will compute some definite integral of some function. Write the integral that is being computed in handwritten notation (i.e. with an integral sign, upper and lower limits, integrand, etc. written as you would write these things by hand).

2. (2.10) For each of the given integrals, determine what the **first step** is in the **best method** to compute the integral by hand. For each integral, your choices are:

   A. Just write the answer.
   B. Rewrite the integrand with a trig identity.
   C. Use a $u-$substitution.
   D. Use integration by parts.
   E. Find a partial fraction decomposition.

   a) $\displaystyle\int \tan x\,dx$

   b) $\displaystyle\int \csc^2 x\,dx$

   c) $\displaystyle\int \sin^2 x\,dx$

   d) $\displaystyle\int x\sin^2 x\,dx$

   e) $\displaystyle\int \sin^3 x\cos x\,dx$

3. (2.2) Compute $\displaystyle\int_0^{\pi} \cos\left(\frac{x}{4}\right)\,dx$.

4. (3.3) Determine, with justification, whether the following integral converges or diverges:

$$\int_4^{\infty} \left(x^{-2/3} - \frac{2}{x^7}\right)\,dx$$

5. (2.5) Evaluate one of the following two integrals.

$$\int 2x^2\sqrt{2-x}\,dx \qquad\qquad \int \frac{x^2}{2x+3}\,dx$$

6. (2.6) Evaluate one of the following two integrals.

$$\int 3x^2 \sin x \, dx \qquad\qquad \int x^{2/3} \ln x \, dx$$

7. (2.8) Find the partial fraction decomposition of one of the following two expressions.

$$\frac{x^2 - 10x - 3}{x(x-3)(x+1)} \qquad\qquad \frac{10 + 3x}{(x+2)^2}$$

8. a) (3.1) Given the following improper integral:

$$\int_1^\infty 2e^{-x} \, dx$$

rewrite the integral as a limit, and then compute the value to which the integral converges. (Trust me, this integral does converge.)

b) (3.3) Using part (a), determine (with appropriate justification) whether the following improper integral converges or diverges:

$$\int_1^\infty e^{-x} \sin^2 x \, dx$$

Solutions

1. a) In *Mathematica* the number e is `E`, not `e`.

   b) `Plot[Log[x], {x, 0, 7}]`

   c) $\displaystyle\int_0^3 \frac{x^2 \sin x}{3}\, dx$

2. a) B (write $\tan x$ as $\frac{\sin x}{\cos x}$)

   b) A (answer is $-\cot x + C$)

   c) B (write $\sin^2 x$ as $\frac{1}{2}(1 - \cos 2x)$)

   d) D ($u = x, dv = \sin^2 x$ will eventually lead to the answer)

   e) C ($u = \sin x$)

3. The integral is done either with the $u-$sub $u = \frac{x}{4}$, or by the linear replacement principle:

$$\int_0^\pi \cos\left(\frac{x}{4}\right)\, dx = 4\sin\frac{x}{4}\bigg|_0^\pi = 4\sin\frac{\pi}{4} - 4\sin 0 = 4\left(\frac{\sqrt{2}}{2}\right) - 4\cdot 0 = 2\sqrt{2}.$$

4. $$\int_4^\infty \left(x^{-2/3} - \frac{2}{x^7}\right)\, dx = \int_4^\infty \frac{1}{x^{2/3}}\, dx - 2\int_4^\infty \frac{1}{x^7}\, dx$$

   Both of these integrals are $p-$integrals. The first integral diverges ($p = 2/3 \leq 1$); the second integral converges ($p = 7 > 1$), so the whole thing, being a sum of a divergent integral and a convergent integral, diverges.

5. For the first integral, set $u = 2 - x$ so that $du = -dx$ and therefore $-du = dx$. Solving for x, we also see that $x = 2 - u$ so the integral becomes

$$\begin{aligned}
\int 2(2-u)^2\sqrt{u}\,(-du) &= \int -2(4 - 4u + u^2)\sqrt{u}\, du \\
&= \int \left(-8 + 8u - 2u^2\right) u^{1/2}\, du \\
&= \int \left(-8u^{1/2} + 8u^{3/2} - 2u^{5/2}\right)\, du \\
&= -8\cdot\frac{2}{3}u^{3/2} + 8\cdot\frac{2}{5}u^{5/2} - 2\cdot\frac{2}{7}u^{7/2} + C \\
&= \frac{-16}{3}(2-x)^{3/2} + \frac{16}{5}(2-x)^{5/2} - \frac{4}{7}(2-x)^{7/2} + C.
\end{aligned}$$

For the second integral, set $u = 2x+3$ so $du = 2dx$ and $dx = \frac{1}{2}du$. Also, solving for x we get $x = \frac{1}{2}(u-3)$ so the integral becomes

$$\begin{aligned}
\int \frac{\left[\frac{1}{2}(u-3)\right]^2}{u} \cdot \frac{1}{2}\,du &= \frac{1}{8}\int \frac{u^2-6u+9}{u}\,du \\
&= \frac{1}{8}\int \left(u - 6 + \frac{9}{u}\right)du \\
&= \frac{1}{8}\left(\frac{u^2}{2} - 6u + 9\ln|u|\right) + C \\
&= \frac{1}{16}(2x+3)^2 - \frac{3}{4}(2x+3) + \frac{9}{8}\ln|2x+3| + C.
\end{aligned}$$

6. For the first integral, use integration by parts twice. First, set $u = 3x^2$, $dv = \sin x\,dx$ and solve for $du = 6x\,dx$ and $v = -\cos x$. Thus

$$\int 3x^2 \sin x\,dx = \int u\,dv = uv - \int v\,du = -3x^2\cos x + \int 6x\cos x\,dx.$$

Now use integration by parts again on the remaining integral. Set $u = 6x$ and $dv = \cos x$ and solve for $du = 6\,dx$ and $v = \sin x$ to obtain

$$\int 6x\cos x\,dx = \int u\,dv = uv - \int v\,du = 6x\sin x - \int 6\sin x\,dx = 6x\sin x + 6\cos x + C$$

and substitute back into the original problem to get

$$\int 3x^2\sin x\,dx = -3x^2\cos x + 6x\sin x + 6\cos x + C.$$

For the second integral, use integration by parts with $u = \ln x$, $dv = x^{2/3}\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = \frac{3}{5}x^{5/3}$ so

$$\begin{aligned}
\int x^{2/3}\ln x\,dx = \int u\,dv = uv - \int v\,du &= \frac{3}{5}x^{5/3}\ln x - \int \frac{3}{5}x^{5/3}\frac{1}{x}\,dx \\
&= \frac{3}{5}x^{5/3}\ln x - \frac{3}{5}\int x^{2/3}\,dx \\
&= \frac{3}{5}x^{5/3}\ln x - \frac{9}{25}x^{5/3} + C.
\end{aligned}$$

7. For the first expression, all the factors in the denominator are distinct, so the appropriate guess is

$$\frac{x^2-10x-3}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}.$$

Find a common denominator and clear denominators to obtain

$$x^2 - 10x - 3 = A(x-3)(x+1) + B(x)(x+1) + C(x)(x-3)$$

Now substitute to solve for A, B, C:

$$x = 0 \Rightarrow -3 = A(-3)1 + 0 + 0 \Rightarrow -3 = -3A \Rightarrow A = 1$$
$$x = 3 \Rightarrow 9 - 30 - 3 = 0 + B(3)4 + 0 \Rightarrow -24 = 12B \Rightarrow B = -2$$
$$x = -1 \Rightarrow 1 + 10 - 3 = 0 + 0 + C(-1)(-4) \Rightarrow 8 = 4C \Rightarrow C = 2$$

Thus the partial fraction decomposition is

$$\frac{x^2 - 10x - 3}{x(x-3)(x+1)} = \frac{1}{x} - \frac{2}{x-3} + \frac{2}{x+1}.$$

For the second expression, the appropriate guess is

$$\frac{10 + 3x}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}.$$

Find a common denominator and clear denominators to obtain

$$10 + 3x = A(x+2) + B$$

Plug in $x = -2$ to obtain $10 - 6 = 0 + B$, i.e. $B = 4$. Now plug in $x = 0$ and $B = 4$ to get $10 = 2A + 4$, i.e. $A = 3$. Thus the partial fraction decomposition is

$$\frac{10 + 3x}{(x+2)^2} = \frac{3}{x+2} + \frac{4}{(x+2)^2}.$$

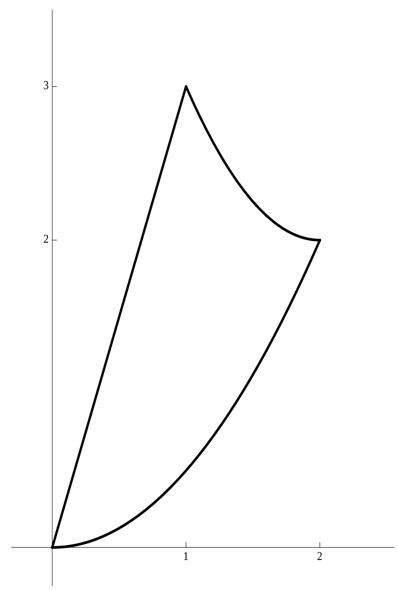
8. a) This region is horizontally unbounded since the upper limit is ∞:

$$\begin{aligned}\int_1^\infty 2e^{-x}\,dx &= \lim_{b\to\infty} \int_1^b 2e^{-x}\,dx \\ &= \lim_{b\to\infty} -2e^{-x}\Big|_1^b \\ &= \lim_{b\to\infty} \left(-2e^{-b} - -2e^{-1}\right) \\ &= \lim_{b\to\infty} \left(\frac{-2}{e^b} + \frac{2}{e}\right) \\ &= 0 + \frac{2}{e} = \frac{2}{e}.\end{aligned}$$

b) Notice that $0 \le \sin^2 x \le 1$, so multiplying through by e^{-x} gives $0 \le e^{-x}\sin^2 x \le e^{-x}$. From part (a) (and linearity), $\int_1^\infty e^{-x}\,dx$ converges, so $\int_1^\infty e^{-x}\sin^2 x\,dx$ also converges by the Comparison Test.

3.6 Fall 2013 Exam 2

1. Let R be the region of points in the $xy-$plane which are located to the right of $y = 3x$, above the equation $y = \frac{1}{2}x^2$, and below the curve $y = (x-2)^2 + 2$. A picture of this region is shown below:



   a) (4.1) Write down an expression involving one or more integrals with respect to x which gives the area of R.

   b) (4.1) Write down an expression involving one or more integrals with respect to y which gives the area of R.

2. Let Q be the region of points in the $xy-$plane located to the right of the $y-$axis, below the line $y = 4$, and above the curve $y = \sqrt{x}$.

   a) (4.2) Suppose Q is revolved around the $x-$axis to produce a solid. Write down an expression involving one or more integrals with respect to x which will compute the volume of this solid.

   b) (4.2) Suppose Q is revolved around the $y-$axis to produce a solid. Write down an expression involving one or more integrals with respect to x which will compute the volume of this solid.

   c) (4.2) Suppose Q is revolved around the line $y = -3$ to produce a solid. Write down an expression involving one or more integrals (with respect to whatever variable you like) which will compute the volume of this solid.

d) (4.2) Suppose Q is revolved around the line $x = 20$ to produce a solid. Write down an expression involving one or more integrals (with respect to whatever variable you like) which will compute the volume of this solid.

3. (4.6) Write down formulas (in terms of integrals) that could be used to find the centroid of the triangle whose vertices are $(0, 0)$, $(0, 9)$ and $(1, 3)$. (You do not need to evaluate any of the integrals.)

   Hint: Write the equation of the lines which comprise two of the sides of the triangle.

4. Suppose X is a random variable whose density function is

$$f(x) = \begin{cases} \dfrac{1}{4\sqrt{x}} & \text{if } 1 \leq x \leq 9 \\ 0 & \text{else} \end{cases}$$

a) (4.8) Find the probability that $X \leq 4$.

b) (4.8) Find the expected value of X.

Solutions

1. a) You need two integrals; each integral is the "top function" minus the "bottom function":

$$A = \int_0^1 \left[3x - \frac{1}{2}x^2\right]\, dx + \int_1^2 \left[(x-2)^2 + 2 - \frac{1}{2}x^2\right]\, dx$$

   b) Again, you need two integrals; each integral is the "right-most function" minus the "left-most" function. To integrate with respect to y, you need to solve all the equations for x:

$$A = \int_0^2 \left[\sqrt{2y} - \frac{1}{3}y\right]\, dy + \int_2^3 \left[(2 + \sqrt{y-2}) - \frac{1}{3}y\right]\, dy$$

2. The corner points of the region are $(0,0)$, $(0,4)$ and $(16,4)$. The last point is found by solving $y = \sqrt{x}$ with $y = 4$.

   a) Since the direction of integration and the axis of revolution are both horizontal, use the washer formula:

$$V = \int_0^{16} \left[\pi R^2 - \pi r^2\right]\, dx = \int_0^{16} \left[\pi 4^2 - \pi(\sqrt{x})^2\right]\, dx.$$

   b) Since the direction of integration is horizontal, but the axis of revolution is vertical, use shells:

$$V = \int_0^{16} 2\pi r h\, dx = \int_0^{16} 2\pi x(4 - \sqrt{x})\, dx.$$

   c) Since the axis of revolution is horizontal, to write an integral with respect to x one must use the washer formula:

$$V = \int_0^{16} \left[\pi R^2 - \pi r^2\right]\, dx = \int_0^{16} \left[\pi(4+3)^2 - \pi(\sqrt{x}+3)^2\right]\, dx.$$

   To write an integral with respect to y, use shells and solve the equation $y = \sqrt{x}$ for x:

$$V = \int_0^4 2\pi r h\, dy = \int_0^4 2\pi(y+3)y^2\, dy.$$

   d) Since the axis of revolution is vertical, to write an integral with respect to x one must use the shell formula:

$$V = \int_0^{16} 2\pi r h\, dx = \int_0^{16} 2\pi(20 - x)(4 - \sqrt{x})\, dx.$$

   To write an integral with respect to y, use washers:

$$V = \int_0^4 \left[\pi R^2 - \pi r^2\right]\, dy = \int_0^4 \left[\pi(20)^2 - \pi(20 - y^2)^2\right]\, dy.$$

3. The left side of the triangle is the $y-$axis; the bottom-right side of the triangle passing through $(0,0)$ and $(1,3)$ has equation $y = 3x$; the top-right side of the triangle passing through $(0,9)$ and $(1,3)$ has equation $y = 9-6x$. Now use the formulas from class with $a=0$, $b=1$, the top function $9-6x$ and the bottom function $3x$:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_0^1 x(9-6x-3x)\,dx}{\int_0^1 (9-6x-3x)\,dx}; \quad \overline{y} = \frac{M_x}{M} = \frac{\int_0^1 \frac{1}{2}\left[(9-6x)^2-(3x)^2\right]\,dx}{\int_0^1 (9-6x-3x)\,dx}$$

4. a)

$$P(X \leq 4) = \int_1^4 f(x)\,dx = \int_1^4 \frac{1}{4\sqrt{x}}\,dx = \frac{1}{2}\sqrt{x}\Big|_1^4 = 1 - \frac{1}{2} = \frac{1}{2}.$$

b)

$$EX = \int_1^9 xf(x)\,dx = \int_1^9 \frac{1}{4}\sqrt{x}\,dx = \frac{1}{6}x^{3/2}\Big|_1^9 = \frac{27}{6} - \frac{1}{6} = \frac{13}{3}.$$

3.7 Fall 2013 Exam 3

1. (7.4) Tell me whether each of the following series converges or diverges (no justification is required in this question, and you do not need to distinguish between absolute and conditional convergence):

   a) $\sum_{n=1}^{\infty} \frac{1}{n}$

   b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

   c) $\sum_{n=1}^{\infty} \frac{3}{7^n}$

   d) $1-1+1-1+1-1+...$

   e) $\sum_{n=1}^{\infty} \frac{3}{n^5}$

   f) $\sum_{n=1}^{\infty} \frac{-3}{\sqrt{n}}$

2. a) (7.2) Find the third partial sum of the series

$$2-\frac{1}{2}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-...$$

   Simplify your answer.

   b) (7.3) Why do we care whether or not a series converges absolutely (as opposed to just knowing whether or not it converges)?

3. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

$$\sum \frac{\sin n + 3}{5^n} \qquad \sum \frac{(-1)^n n!}{10^n} \qquad \sum \frac{4n^2}{3n^2+5n^4} \qquad \sum \frac{4(-1)^n}{n^{2/5}}$$

4. a) (8.2) Estimate the following integral by replacing the integrand with its sixth Taylor polynomial (simplify your answer)

$$\int_0^{1/2} 2e^{x^4}\,dx$$

   b) (8.2) Evaluate the following limit without using L'Hôpital's Rule (simplify your answer):

$$\lim_{x\to 0} \frac{e^x - x - 1}{\cos 8x - 1}$$

   c) (8.2) Approximate $\cos\left(\frac{1}{6}\right)$ by evaluating the third Taylor polynomial for an appropriately chosen function. Simplify your answer.

5. Find and simplify the sum of each of the following series (you may assume that all these series converge):

a) (6.2) $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \frac{3}{5^4} - \frac{3}{5^5} + \ldots$

b) (8.2) $1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \ldots$

c) (6.2) $\sum_{n=2}^{\infty} \frac{4 \cdot 3^{n-1}}{2^{2n} 5^n}$

6. (8.3) Find the interval of convergence of the following power series:

$$\sum_{n=0}^{\infty} \frac{(-2)^n (x+1)^n}{n^2 6^n}$$

Solutions

1. a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic)

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (alternating harmonic)

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{7^n}$ converges (geometric $r = 1/7$)

   d) $1 - 1 + 1 - 1 + 1 - 1 + ...$ diverges (geometric $r = -1$)

   e) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^5}$ converges ($p-$series $p = 5$)

   f) $\displaystyle\sum_{n=1}^{\infty} \frac{-3}{\sqrt{n}}$ diverges ($p-$series $p = 1/2$)

2. a) $S_3 = 2 - \dfrac{1}{2} + \dfrac{1}{8} = \dfrac{13}{8}$.

   b) Because the terms of an absolutely convergent series can be rearranged and/or regrouped arbitrarily without affecting the sum of the series, but the terms of a conditionally convergent series cannot be regrouped legally.

3. a) Observe $0 \leq \dfrac{\sin n + 3}{5^n} \leq \dfrac{4}{5^n}$. Now $\sum \frac{4}{5^n}$ converges since it is geometric with $r = \frac{1}{5}$, so the original series $\sum \frac{\sin x+3}{5^n}$ converges as well by the Comparison Test. Since the series is positive, it converges absolutely.

   b) Apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(n+1)!}{10^{n+1}}\right|}{\left|\frac{(-1)^n n!}{10^n}\right|} = \lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} = \infty$$

   Since $\rho > 1$, the series diverges by the Ratio Test.

   c) Observe $0 \leq \frac{4n^2}{5n^4+3n^2} \leq \frac{4n^2}{5n^4} = \frac{4}{5n^2}$. Since $\sum \dfrac{4}{5n^2}$ converges (it is a $p-$series with $p = 2 > 1$), so does $\sum \dfrac{4n^2}{5n^4 + 3n^2}$ by the Comparison Test. Since the series is positive, it converges absolutely.

   d) This is an alternating series;

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|\frac{4(-1)^n}{n^{2/5}}\right| = \lim_{n\to\infty} \frac{4}{n^{2/5}} = 0;$$

   and as n increases, $|a_n| = \frac{4}{n^{2/5}}$ decreases. Thus by the Alternating Series Test the series converges. However, if one considers the series

$$\sum |a_n| = \sum \frac{4}{n^{2/5}},$$

this is a $p-$series with $p = 2/5 \le 1$, so $\sum |a_n|$ diverges. Thus the series $\sum \dfrac{4(-1)^n}{n^{2/5}}$ converges conditionally.

4. a) Start with $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$ and replace x with x^4 to get $e^{x^4} = 1 + x^4 + \frac{x^8}{2!} + ...$ Multiply everything by 2 to get $2e^{x^4} = 2 + 2x^4 + x^8 + ...$ Now

$$\int_0^{1/2} 2e^{x^4}\,dx \approx \int_0^{1/2} P_6(x)\,dx = \int_0^{1/2} (2 + 2x^4)\,dx = \left[2x + \frac{2}{5}x^5\right]_0^{1/2}$$
$$= 2 + \frac{2}{5}\left(\frac{1}{2}\right)^5 = 2 + \frac{5}{16} = \frac{37}{16}.$$

b) Write the Taylor series for both the numerator and denominator:

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - x - 1}{\cos 8x - 1} &= \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ... - x - 1}{1 - \frac{(8x)^2}{2!} + \frac{(8x)^4}{4!} - ... - 1} \\
&= \lim_{x\to 0} \frac{\frac{x^2}{2} + \frac{x^3}{3!} + ...}{-\frac{(8x)^2}{2!} + \frac{(8x)^4}{4!} - ...} \\
&= \lim_{x\to 0} \frac{\frac{1}{2} + \frac{x}{3!} + ...}{-\frac{64}{2!} + \frac{8^4x^2}{4!} - ...} \\
&= \lim_{x\to 0} \frac{\frac{1}{2} + 0 + 0 + ...}{-\frac{64}{2!} + 0 - 0 + ...} \\
&= \lim_{x\to 0} \frac{1}{2} \cdot \frac{-2}{64} = \frac{-1}{64}.
\end{aligned}$$

c) We know $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$ so the third Taylor polynomial for $\cos x$ is $P_3(x) = 1 - \frac{1}{2}x^2$. Now

$$\cos\left(\frac{1}{6}\right) \approx P_3\left(\frac{1}{6}\right) = 1 - \frac{1}{2}\left(\frac{1}{6}\right)^2 = 1 - \frac{1}{72} = \frac{71}{72}.$$

5. The first series is geometric:

$$3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \frac{3}{5^4} - \frac{3}{5^5} + ... = 3\sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n = \frac{3}{1-(-1/5)} = \frac{3}{6/5} = \frac{15}{6} = \frac{5}{2}.$$

The second series is the Taylor series for e^x with 4 plugged in for x. Therefore

$$1 + 4 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + ... = e^4.$$

The last series is geometric:

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{4 \cdot 3^{n-1}}{2^{2n}5^n} = \sum_{n=2}^{\infty} \frac{4 \cdot 3^n}{3 \cdot 4^n 5^n} &= \frac{4}{3} \sum_{n=2}^{\infty} \frac{3^n}{20^n} \\ &= \frac{4}{3}\left(\frac{3}{20}\right)^2 \sum_{n=0}^{\infty}\left(\frac{3}{20}\right)^n \\ &= \frac{4}{3}\left(\frac{3}{20}\right)^2 \frac{1}{1-3/20} \\ &= \frac{4}{3} \cdot \left(\frac{3}{20}\right)^2 \cdot \frac{20}{17} \\ &= 4 \cdot \frac{3}{20}\frac{1}{17} = \frac{3}{85}.\end{aligned}$$

6. By Abel's Formula, the radius of convergence is

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\left|\frac{(-2)^n}{n^2 6^n}\right|}{\left|\frac{(-2)^{n+1}}{(n+1)^2 6^{n+1}}\right|} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{6}{2} = 3.$$

Since the power series is centered at -1, by the Cauchy-Hadamard Theorem the series converges absolutely on $(a-R, a+R) = (-1-3, -1+3) = (-4, 2)$ and diverges on $(-\infty, -4)$ and $(2, \infty)$. Last, we test the endpoints $x = -4$ and $x = 2$:

- $x = -4$: Here the series becomes $\sum \frac{(-2)^n(-3)^n}{n^2 6^n} = \sum \frac{1}{n^2}$, a convergent $p-$series.
- $x = 2$: Here the series becomes $\sum \frac{(-2)^n(3)^n}{n^2 6^n} = \sum \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test.

Thus the interval of convergence includes both endpoints; this interval is $[-4, 2]$.

3.8 Fall 2013 Final Exam

1. a) (2.6) Evaluate one of the following two integrals.

$$\int 2x^2 \cos x \, dx \qquad\qquad \int xe^{-4x} \, dx$$

   b) (2.8) Evaluate one of the following two integrals.

$$\int \frac{x^2 - 3x + 1}{x^3 + x} \, dx \qquad\qquad \int \frac{5x - 9}{x^2 - 1} \, dx$$

   c) (2.5) Evaluate one of the following two integrals.

$$\int x(2 + x)^{2/5} \, dx \qquad\qquad \int \frac{6x^2}{\sqrt{1 - x}} \, dx$$

2. Let R be the region in the $xy-$plane which lies above the parabola $y = x^2$, to the right of the line $x = 1$, and below the line $y = 16$.

   a) (4.1) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the area of R.
   b) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the $x-$axis.
   c) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the $y-$axis.

3. Let X be a random variable taking values in $[0, 4]$ whose density function is

$$f(x) = \frac{3}{32}x(4 - x).$$

   a) (4.8) Find the probability that $X \leq 2$.
   b) (4.8) Find the probability that $X = 3$.
   c) (4.8) Find the expected value of X.

4. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{n=2}^{\infty} \frac{(n + 5)}{(2n)!} \qquad\qquad \sum_{n=0}^{\infty} \frac{n}{n + 2}$$

$$\sum_{n=1}^{\infty} \left(\frac{7^n}{3^{2n}} - \frac{2}{\sqrt{n}} \right) \qquad\qquad \sum_{n=3}^{\infty} \frac{\ln n}{n}$$

5. a) (8.2) Estimate the following integral by replacing the integrand with its eighth Taylor polynomial:

$$\int_0^1 x^2 \sin(x^2)\, dx$$

b) (8.2) Estimate $\ln\left(\frac{7}{6}\right)$ by computing the third Taylor polynomial for an appropriately chosen function.

c) (8.2) Let f be an infinitely differentiable function such that $f(0) = 0$; $f'(0) = 1$; $f''(0) = 2$; $f'''(0) = 3$; $f^{(4)}(0) = 4$; etc. (i.e. in general, $f^{(n)}(0) = n$). Find the exact value of $f(2)$.

d) (6.3) A ball is dropped from a height of 8 meters. Each time the ball bounces, it rebounds to $1/4$ of its previous height. Find the total distance travelled by the ball before it comes to rest.

6. Find the sum of each of these series:

a) (6.2) $12 - 2 + \dfrac{1}{3} - \dfrac{1}{18} + \dfrac{1}{108} - ...$

b) (8.2) $1 - 2 + \dfrac{4}{2!} - \dfrac{8}{3!} + \dfrac{16}{4!} - \dfrac{32}{5!} + ...$

c) (6.2) $\displaystyle\sum_{n=0}^{104} \left(\frac{4}{7}\right)^n$

Hints and some solutions

Solutions to all problems on this exam have not been typed yet. Here are some solutions, and hints to the other problems:

1. a) For the first integral, use integration by parts twice. For the second integral, use parts once.
   b) Both integrals use partial fractions.
   c) For the first integral, use the u-sub $u = 2 + x$. For the second integral, use the u-sub $u = 1 - x$.

2. a) If using x as the variable of integration, integrate from the left to the right, where the integrand is the top function minus the bottom function:

$$A = \int_1^4 (16 - x^2)\, dx$$

   If using y, integrate from the bottom to the top, where the integrand is the right-most function minus the left-most function:

$$A = \int_1^{16} (\sqrt{y} - 1)\, dy$$

   b)

$$V = \int_1^4 \left[\pi(16^2) - \pi(x^2)^2\right] dx = \int_1^{16} 2\pi y(\sqrt{y} - 1)\, dy$$

   c)

$$V = \int_1^4 2\pi x(16 - x^2)\, dx = \int_1^{16} \left[\pi(\sqrt{y})^2 - \pi(1)^2\right] dy$$

3. a) $P(X \leq 2) = \int_0^2 f(x)\, dx = \int_0^2 \frac{3}{32}x(4-x)\, dx = \int_0^2 \left(\frac{3}{8}x - \frac{3}{32}x^2\right) dx$.
   b) $P(X = 3) = \int_3^3 f(x)\, dx = 0$.
   c) $EX = \int_a^b xf(x)\, dx = \int_0^4 x\frac{3}{32}x(4-x)\, dx = \int_0^4 \left(\frac{3}{8}x^2 - \frac{3}{32}x^3\right) dx$.

4. a) converges absolutely (use Ratio Test)
   b) diverges (converges minus diverges = diverges)
   c) diverges (use N^{th} term test)
   d) converges absolutely (use Integral Test)

5. a) The Taylor series of $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!}$.... Substitute x^2 for x to get $\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$..., then multiply by x^2 to get

$$x^2 \sin x^2 = x^4 - \frac{x^8}{3!} + \frac{x^{12}}{5!} - \ldots$$

   So the eighth Taylor polynomial of the integrand is $P_8(x) = x^4 - \frac{1}{6}x^8$. Therefore

$$\int_0^1 x^2 \sin(x^2)\, dx \approx \int_0^1 (x^4 - \frac{1}{6}x^8)\, dx = \left[\frac{x^5}{5} - \frac{1}{54}x^9\right]_0^1 = \frac{1}{5} - \frac{1}{54}.$$

b) Use the function $f(x) = \ln(x+1)$. The Taylor series of f is $x - \frac{x^2}{2} + \frac{x^3}{3} - ...$ so the third Taylor polynomial is $P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. So

$$\ln\frac{7}{6} = f(\frac{1}{6}) \approx P_3(\frac{1}{6}) = \frac{1}{6} - \frac{(1/6)^2}{2} + \frac{(1/6)^3}{3}.$$

c) Write the Taylor series of f, using the given information that $f^{(n)}(0) = n$ (drop the $n = 0$ term because it doesn't contribute to the sum since it is zero)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n.$$

Now rewrite the series so that its initial term is $n = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = xe^x.$$

Therefore $f(2) = 2e^2$.

d) The total distance travelled by the ball is

$$\begin{aligned}
8 + 2 + 2 + \frac{1}{2} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + ... &= 8 + 4 + 1 + \frac{1}{4} + ... \\
&= 8 + 4\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + ...\right] \\
&= 8 + 4\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \\
&= 8 + 4 \cdot \frac{1}{1 - \frac{1}{4}} \\
&= 8 + 4 \cdot \frac{4}{3} = \frac{40}{3}.
\end{aligned}$$

Note that every term other than the 8 appears twice because the ball bounces up, then down each time it bounces.

6. a) This series is geometric:

$$12 - 2 + \frac{1}{3} - \frac{1}{18} + \frac{1}{108} - ... = 12\sum_{n=0}^{\infty}\left(\frac{-1}{6}\right)^n = 12 \cdot \frac{1}{1 - (-1/6)} = 12 \cdot \frac{6}{7} = \frac{72}{7}.$$

b) Use the Taylor series for the exponential function:

$$1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - \frac{32}{5!} + ... = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2}.$$

c) Use the finite sum formula for geometric series:

$$\sum_{n=0}^{104}\left(\frac{4}{7}\right)^n = \frac{1 - (4/7)^{105}}{1 - 4/7}.$$

3.9 Spring 2014 Exam 1

1. a) Write down the *Mathematica* code that would be used to define the function $f(x) = x\cos x$.

   b) Suppose you type the following code into *Mathematica*:

   ```
   Plot[x^2 + 1/x, {x, -3, 5}, PlotRange -> {-4,10}]
   ```

   i. What function is being graphed? Write the answer in "hand-written" notation.

   ii. What is the right-most $x-$value that will appear on the graph?

   c) Write down the *Mathematica* code that evaluates the integral

$$\int_4^7 \ln x \, dx.$$

2. (2.10) For each of the given integrals, determine what the **best method** to compute the integral by hand (there might be more than one "best method").

   - If the best method is to just write the answer, write "JUST DO IT" and write the answer.
   - If the best method is to use a $u-$substitution, write "U-SUB" and write what the u would be in your $u-$sub.
   - If the best method is integration by parts, write "PARTS" and write what you would assign to be u and dv.
   - If the best method is partial fractions, write "PARTIAL FRACTIONS" and write your 'guessed' form of the decomposition.

   You do not need to actually compute these integrals unless you can "just do it".

   a) $\displaystyle\int \frac{1}{(x+1)^2}\, dx$

   b) $\displaystyle\int \frac{1}{x^2+1}\, dx$

   c) $\displaystyle\int \frac{1}{x^2-1}\, dx$

   d) $\displaystyle\int \frac{x}{x^2+1}\, dx$

   e) $\displaystyle\int \frac{x^3}{x^2-1}\, dx$

3. (2.1) Compute each of the following integrals:

   a) $\displaystyle\int \left(6x^2 - 5\sqrt{x} + \frac{4}{7x}\right) dx$

   b) $\displaystyle\int \left(2\sin x + \frac{e^x}{3}\right) dx$

c) $\displaystyle\int \left(\csc x \cot x - 4\csc^2 x\right)\, dx$

4. (3.2) Determine, with justification, whether the following integral converges or diverges:
$$\int_2^4 \frac{2}{\sqrt{x-2}}\, dx$$

5. (2.8) Compute the partial fraction decomposition of the following expression:
$$\frac{7x^2 - x + 4}{x(x-2)(x+1)}$$

6. (2.5) Evaluate one of the following two integrals.
$$\int \tan^7 x \sec^4 x\, dx \qquad\qquad \int \frac{(x-1)^2}{x+2}\, dx$$

7. (2.6) Evaluate one of the following two integrals.
$$\int \arctan x\, dx \qquad\qquad \int 4x \sec^2 x\, dx$$

Solutions

1. a) `f[x_] = x Cos[x]`
   b) i. $f(x) = x^2 + \frac{1}{x}$
      ii. 5
   c) `Integrate[Log[x], {x, 4, 7}]`

2. a) Either "JUST DO IT" (using the linear replacement principle, the answer is $-(x+1)^{-1} + C$) or "U-SUB" $u = x + 1$.
   b) JUST DO IT: $\arctan x + C$
   c) PARTIAL FRACTIONS: $\frac{A}{x-1} + \frac{B}{x+1}$.
   d) U-SUB: $u = x^2 + 1$
   e) U-SUB: $u = x^2 - 1$ (this will be a more complicated u-sub)

3. a) $\displaystyle\int \left(6x^2 - 5\sqrt{x} + \frac{4}{7x}\right) dx = 2x^3 - \frac{10}{3}x^{3/2} + \frac{4}{7}\ln x + C$
   b) $\displaystyle\int \left(2\sin x + \frac{e^x}{3}\right) dx = -2\cos x + \frac{e^x}{3} + C$
   c) $\displaystyle\int \left(\csc x \cot x - 4\csc^2 x\right) dx = -\csc x + 4\cot x + C$

4. Compute the improper integral directly:

$$\begin{aligned}\int_2^4 \frac{2}{\sqrt{x-2}}\,dx &= \lim_{b\to 2^+} \int_b^4 \frac{2}{\sqrt{x-2}}\,dx \\ &= \lim_{b\to 2^+} \int_b^4 2(x-2)^{-1/2}\,dx \\ &= \lim_{b\to 2^+} 4(x-2)^{1/2}\Big|_b^4 \\ &= \lim_{b\to 2^+} \left[4\sqrt{2} - 4\sqrt{b-2}\right] \\ &= 4\sqrt{2} - 4\sqrt{0} = 4\sqrt{2}.\end{aligned}$$

   Therefore the integral converges to $4\sqrt{2}$.

5. The guessed form of the decomposition is $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}$; finding a common denominator and clearing denominators, we obtain

$$A(x-2)(x+1) + Bx(x+1) + Cx(x-2) = 7x^2 - x + 4.$$

   Substitute $x = 2$ to get $6B = 28 - 2 + 4$, i.e. $B = 5$. Substitute $x = -1$ to get $3C = 7 + 1 + 4$, i.e. $C = 4$. Substitute $x = 0$ to get $-2A = 4$, i.e. $A = -2$. Therefore

$$\frac{7x^2 - x + 4}{x(x-2)(x+1)} = \frac{-2}{x} + \frac{5}{x-2} + \frac{4}{x+1}.$$

6. a) For the first integral, rewrite with a trig identity and then use a u-sub:

$$\begin{aligned}
\int \tan^7 x \sec^4 x \, dx &= \int \tan^7 x \sec^2 x \sec^2 x \, dx \\
&= \int \tan^7 x (\tan^2 x + 1) \sec^2 x \, dx \\
&= \int (\tan^9 x + \tan^7 x) \sec^2 x \, dx \\
&\quad (u\text{-sub } u = \tan x; \, du = \sec^2 x \, dx) \\
&= \int (u^9 + u^7) \, du \\
&= \frac{u^{10}}{10} + \frac{u^8}{8} + C \\
&= \frac{\tan^{10} x}{10} + \frac{\tan^8 x}{8} + C.
\end{aligned}$$

b) For the second integral, use a u-substitution $u = x+2$ so $du = dx$ and $x = u-2$. Therefore

$$\begin{aligned}
\int \frac{(x-1)^2}{x+2} \, dx &= \int \frac{(u-2-1)^2}{u} \, du \\
&= \int \frac{(u-3)^2}{u} \, du \\
&= \int \frac{u^2 - 6u + 9}{u} \, du \\
&= \int \left(u - 6 + \frac{9}{u} \right) \, du \\
&= \frac{u^2}{2} - 6u + 9 \ln u + C \\
&= \frac{(x+2)^2}{2} - 6(x+2) + 9 \ln(x+2) + C.
\end{aligned}$$

7. a) Use parts with $u = \arctan x$ and $dv = dx$. Then $du = \frac{1}{x^2+1}$ and $v = \int dv = x$. Therefore, by the parts formula

$$\int \arctan x \, dx = \int u \, dv = uv - \int v \, du = x \arctan x - \int \frac{x}{x^2+1} \, dx.$$

Now, use a u-sub $u = x^2 + 1; du = 2x dx$ on the second integral to obtain

$$\begin{aligned}
\int \arctan x \, dx &= x \arctan x - \int \frac{x}{x^2+1} \, dx \\
&= x \arctan x - \int \frac{1}{2} \frac{1}{u} \, du \\
&= x \arctan x - \frac{1}{2} \ln u + C \\
&= x \arctan x - \frac{1}{2} \ln(x^2+1) + C.
\end{aligned}$$

b) Use parts with $u = 4x$ and $dv = \sec^2 x\,dx$, so that $du = 4\,dx$ and $v = \int dv = \tan x$. Therefore, by the parts formula,

$$\int 4x \sec^2 x\,dx = \int u\,dv = uv - \int v\,du = 4x\tan x - \int 4\tan x\,dx.$$

To do the last integral, rewrite and use a u-substitution $u = \cos x$; $du = -\sin x\,dx$:

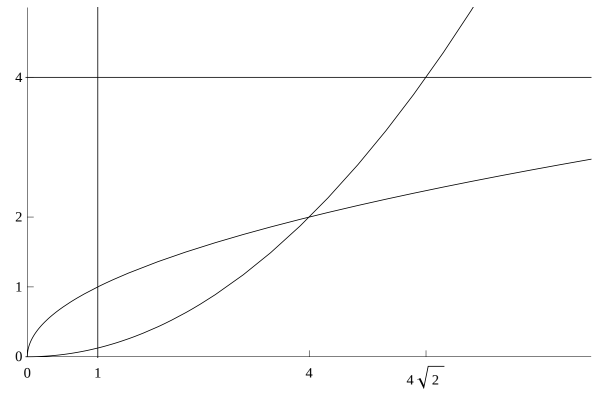
$$\int 4\tan x\,dx = \int \frac{4\sin x}{\cos x}\,dx = \int \frac{-4}{u}\,du = -4\ln u + C = -4\ln(\cos x) + C.$$

Putting all this together, we have

$$\begin{aligned}\int 4x\sec^2 x\,dx &= 4x\tan x - \int 4\tan x\,dx \\ &= 4x\tan x + 4\ln(\cos x) + C.\end{aligned}$$

3.10 Spring 2014 Exam 2

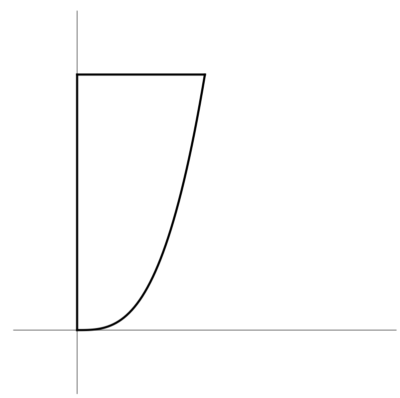
1. Let Q be the region of points in the $xy-$plane located to the right of the line $x = 1$, below the graph of $y = 4$ and above both of the curves $y = \sqrt{x}$ and $y = \frac{1}{8}x^2$. The graphs of all these equations are given at right, and the $x-$ and $y-$ coordinates of the intersection points of these curves are given on the axes.



   a) (4.1) Write an expression involving one or more integrals with respect to the variable x which gives the area of Q.

   b) (4.1) Write an expression involving one or more integrals with respect to the variable y which gives the area of region Q.

2. Let R be the region of points in the $xy-$plane which are located to the right of the $y-$axis, above the equation $y = \frac{1}{2}x^3$, and below the line $y = 4$. A picture of this region is shown to the right:



   a) (4.2) Suppose R is revolved around the $x-$axis to produce a solid. Write an integral with respect to the variable x which gives the volume of this solid.

   b) (4.2) Write an integral with respect to y which gives the volume of the same solid as in part (a).

   c) (4.2) Suppose R is revolved around the $y-$axis to produce a solid. Write an integral with respect to whatever variable you like that gives the volume of this solid.

   d) (4.2) Suppose R is revolved around the line $y = -3$ to produce a solid. Write an integral with respect to whatever variable you like that gives the volume of this solid.

3. Suppose X is a continuous random variable whose density function is

$$f(x) = \begin{cases} cx^3 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$$

a) (4.8) Find the value of c.

b) (4.8) Find the probability that $X = 1$.

c) (4.8) Find the probability that $X > 1$.

d) (4.8) Find the expected value of X.

4. (4.4) Write an integral which gives the length of the curve $y = 5\sin 2x$ from $x = 0$ to $x = 2\pi$.

Solutions

1. a) Integrate the (top function minus the bottom function) from the left to the right to get

$$A = \int_1^4 (4 - \sqrt{x})\, dx + \int_4^{4\sqrt{2}} (4 - \frac{1}{8}x^2)\, dx.$$

b) Solve the equations for x to get $x = y^2$ and $x = \sqrt{8y}$. Thus the area is the integral of (the right-most function minus the left-most function) from the bottom to the top, i.e.

$$A = \int_1^2 (y^2 - 1)\, dy + \int_2^4 (\sqrt{8y} - 1)\, dy.$$

2. Notice first that the corner point of the region is $(2, 4)$ (to get the x-coordinate, solve $\frac{1}{2}x^3 = 4$ for x).

a) Since the direction of integration (left to right) is parallel to the axis of revolution, you get washers:

$$V = \int_a^b \left[\pi R^2 - \pi r^2\right] \, dx = \int_0^2 \left[\pi(4)^2 - \pi(\frac{1}{2}x^3)^2\right] \, dx.$$

b) Solve the equation $y = \frac{1}{2}x^3$ for x to obtain $x = \sqrt[3]{2y}$. Since the direction of integration (bottom to top) is perpendicula to the axis of revolution, you get shells:

$$V = \int_c^d 2\pi r h \, dy = \int_0^4 2\pi y(\sqrt[3]{2y}) \, dy.$$

c) With respect to x, use shells; with respect to y, use washers (actually disks because there is no hole):

$$\begin{aligned} V &= \int_a^b 2\pi r h \, dx = \int_0^2 2\pi x(\frac{1}{2}x^3) \, dx \\ &= \int_c^d \left[\pi R^2 - \pi r^2\right] \, dy = \int_0^4 \left[\pi(\sqrt[3]{2y})^2\right] \, dy \end{aligned}$$

d) With respect to x, use washers; with respect to y, use shells:

$$\begin{aligned} V &= \int_a^b \left[\pi R^2 - \pi r^2\right] \, dx = \int_0^2 \left[\pi(4+3)^2 - \pi(\frac{1}{2}x^3 + 3)^2\right] \, dx \\ &= \int_c^d 2\pi r h \, dy = \int_0^4 2\pi(y+3)(\sqrt[3]{2y}) \, dy \end{aligned}$$

3. a) We know $\int_a^b f(x)\, dx = 1$ for any density function f, so

$$\begin{aligned} 1 &= \int_0^2 cx^3 \, dx \\ 1 &= \left[\frac{cx^4}{4}\right]_0^2 \\ 1 &= \frac{16c}{4} \Rightarrow 1 = 4c \Rightarrow c = \frac{1}{4}. \end{aligned}$$

b) $P(X = 1) = 0$ since X is continuous.

c) $P(X > 1) = \displaystyle\int_1^2 f(x)\, dx = \int_1^2 \frac{1}{4}x^3 \, dx = \left[\frac{x^4}{16}\right]_1^2 = 1 - \frac{1}{16} = \frac{15}{16}.$

d) $EX = \displaystyle\int_a^b xf(x)\, dx = \int_0^2 x\frac{x^3}{16}\, dx = \int_0^2 \frac{x^4}{16}\, dx = \left[\frac{x^5}{20}\right]_0^2 = \frac{32}{20} = \frac{8}{5}.$

4. First, $\frac{dy}{dx} = 10\cos 2x$ by the Chain Rule. Now by the formula for arc length,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^{2\pi} \sqrt{1 + (10\cos 2x)^2} \, dx.$$

3.11 Spring 2014 Exam 3

1. a) (5.2) Find the third partial sum of the series $\sum_{n=0}^{\infty}(n+1)2^n$ (simplify your answer).

   b) (5.3) Write the following series in Σ notation:

   i. $\frac{3}{8}+\frac{3}{13}+\frac{3}{18}+\frac{3}{23}+...$

   ii. $\frac{1}{3}-\frac{1}{3^4}+\frac{1}{3^7}-\frac{1}{3^{10}}+...$

   c) (5.5) Rewrite the series $\sum_{n=3}^{\infty}\frac{2^n}{(n-1)!}$ so that its initial index is $n=0$.

2. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{n=2}^{\infty}\frac{2^n}{5^n+3n+2} \qquad \sum_{n=2}^{\infty}\frac{(-1)^n}{3\sqrt[5]{n}}$$

$$\sum_{n=1}^{\infty}\frac{n!8^n}{(2n)!} \qquad \sum_{n=1}^{\infty}\left(\frac{3}{2^n}-\frac{4}{n}\right)$$

3. a) (8.2) Estimate the following integral by replacing the integrand with its seventh Taylor polynomial (simplify your answer)

$$\int_0^1 \sin(x^2)\,dx$$

   b) (8.2) Evaluate the following limit without using L'Hôpital's Rule (simplify your answer):

$$\lim_{x\to 0}\frac{\arctan x^8 - x^8}{\cos x^{12}-1}$$

   c) (8.2) Approximate $e^{1/5}$ by evaluating the second Taylor polynomial for an appropriately chosen function. Simplify your answer.

4. Find and simplify the sum of each of the following series (you may assume that all these series converge):

   a) (6.2) $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+...$

   b) (8.2) $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-...$

   c) (6.2) $\sum_{n=0}^{\infty}\frac{3\cdot 2^{2n+1}}{7^{n-1}}$

5. (8.2) Suppose $f(x)$ is a function such that for each n, the n^{th} derivative of f at $x = 0$ is $f^{(n)}(0) = \frac{1}{n+1}$. If $g(x) = 2x^2 f(x^4)$, find $g^{(70)}(0)$, the 70th derivative of g at $x = 0$.

6. (7.4) Classify each of the following statements as true or false (circle your answer; no justification is required):

   (a) TRUE FALSE If $\sum a_n$ and $\sum b_n$ both converge, then $\sum(a_n b_n)$ must also converge.

   (b) TRUE FALSE If $\sum a_n$ and $\sum b_n$ both diverge, then $\sum(a_n b_n)$ must also diverge.

   (c) TRUE FALSE If $\sum a_n$ converges but $\sum b_n$ diverges, then $\sum(a_n b_n)$ must diverge.

(Bonus) Justify any one of your answers to Question 6, either by explaining why the statement is true or giving a counterexample showing that it is false.

Solutions

1. a) $S_3 = (0+1)2^0 + (1+1)2^1 + (2+1)2^2 + (3+1)2^3 = 1 + 4 + 12 + 32 = 49.$

   b) Answers may vary:

      i. $\dfrac{3}{8} + \dfrac{3}{13} + \dfrac{3}{18} + \dfrac{3}{23} + ... = \displaystyle\sum_{n=1}^{\infty} \frac{3}{5n+3}$

      ii. $\dfrac{1}{3} - \dfrac{1}{3^4} + \dfrac{1}{3^7} - \dfrac{1}{3^{10}} + ... = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{3n+1}}$

   c) Replace each n with $n+3$ to get $\displaystyle\sum_{n=3}^{\infty} \frac{2^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{2^{n+3}}{(n+2)!}$.

2. a) Notice $0 \le \frac{2^n}{5^n+3n+2} \le \frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n$. Now $\sum \left(\frac{2}{5}\right)^n$ converges (geometric with $r = \frac{2}{5}$), so the original series converges by the Comparison Test (since it is positive, it converges absolutely).

   b) Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)!8^{n+1}}{(2n+2)!}}{\frac{n!8^n}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)8}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{4}{2n+1} = 0$$

   so the series converges absolutely since $\rho < 1$.

   c) This series alternates; $\lim_{n\to\infty} \left|\frac{(-1)^n}{3\sqrt[5]{n}}\right| = 0$ and as n increases, the absolute values of the terms decrease. Therefore the series converges by the Alternating Series Test. However, $\sum \left|\frac{(-1)^n}{3\sqrt[5]{n}}\right| = \sum \frac{1}{n^{1/5}}$ diverges (p-series with $p \le 1$) so the original series converges conditionally.

   d) Split this series; the first part converges (it is geometric with $r = \frac{1}{2}$) and the second part diverges (it is harmonic) so the whole thing diverges.

3. a) Since $\sin x = x - \frac{x^3}{3!} + ...$, $\sin x^2 = x^2 - \frac{x^6}{3!} + ...$ so $P_6(x) = x^2 - \frac{x^6}{6}$. Thus

$$\int_0^1 \sin(x^2)\,dx \approx \int_0^{1/3} (x^2 - \frac{x^6}{6})\,dx = \left[\frac{x^3}{3} - \frac{x^7}{42}\right]_0^1 = \frac{1}{3} - \frac{1}{42} = \frac{13}{42}.$$

   b) Replace the numerator and denominator with their Taylor series:

$$\lim_{x\to 0} \frac{\arctan x^8 - x^8}{\cos x^{12} - 1} = \lim_{x\to 0} \frac{x^8 - \frac{x^{24}}{3} + ... - x^8}{1 - \frac{x^{24}}{2!} + ... - 1} = \lim_{x\to 0} \frac{-\frac{1}{3}x^{24} + ...}{-\frac{1}{2}x^{24} + ...} = \frac{2}{3}.$$

   c) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$ so $P_2(x) = 1 + x + \frac{1}{2}x^2$. Thus $e^{1/5} \approx 1 + \frac{1}{5} + \frac{1}{2}\left(\frac{1}{5}\right)^2 = 1 + \frac{1}{5} + \frac{1}{50} = \frac{61}{50}$.

4. a) $\dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} + \dfrac{1}{32} + ... = \dfrac{1}{4}\left(1 + \dfrac{1}{2} + \dfrac{1}{4} + ...\right) = \dfrac{1}{4}\left(\dfrac{1}{1-(1/2)}\right) = \dfrac{1}{2}.$

b) $\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\frac{1}{5\cdot 2^5}-... = \left[x-\frac{x^2}{2}+\frac{x^3}{3}-...\right]_{x=1/2} = \ln(1+1/2) = \ln\left(\frac{3}{2}\right)$.

c) $\sum_{n=0}^{\infty}\frac{3\cdot 2^{2n+1}}{7^{n-1}} = \sum_{n=0}^{\infty}\frac{42\cdot 4^n}{7^n} = 42\sum_{n=0}^{\infty}\left(\frac{4}{7}\right)^n = 42\cdot\frac{1}{1-(4/7)} = 42\cdot\frac{7}{3} = 98.$

5. We have $f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{(n+1)n!}x^n = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}x^n$. Therefore $g(x) = 2x^2 f(x^4) = \sum_{n=0}^{\infty}\frac{2}{(n+1)!}x^{4n+2}$. Thus the 70^{th} derivative of g at zero is related to the coefficient on the x^{70} term, which is when $4n+2=70$, i.e. $n=17$. So by uniqueness of power series,

$$\frac{2}{18!} = \frac{g^{(70)}(0)}{70!}$$

so $g^{(70)}(0) = \frac{2\cdot 70!}{18!}$.

6. a) is FALSE: set $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. Both $\sum a_n$ and $\sum b_n$ converge by the Alternating Series Test, but $\sum(a_n b_n) = \sum \frac{1}{n}$ is harmonic.

b) is FALSE: set $a_n = b_n = \frac{1}{n}$. Both $\sum a_n$ and $\sum b_n$ are harmonic, so they diverge. But $\sum(a_n b_n) = \sum\frac{1}{n^2}$ is a convergent p-series.

c) is FALSE: set $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$. $\sum a_n$ converges; $\sum b_n$ diverges; $\sum a_n b_n = \sum\frac{1}{n^3}$ converges.

3.12 Spring 2014 Final Exam

1. (2.6) Evaluate one of the following two integrals.

$$\int x\sec^2 x\,dx \qquad\qquad \int 12x^2e^{3x}\,dx$$

2. (2.8) Evaluate one of the following two integrals.

$$\int \frac{24}{x^2-16}\,dx \qquad\qquad \int \frac{-30}{x(x+2)(x-3)}\,dx$$

3. Determine, with justification, whether or not the following two improper integrals converge or diverge:

   a) (3.2) $\displaystyle\int_2^4 \frac{3}{(2-x)^2}\,dx$

   b) (3.4) $\displaystyle\int_0^\infty xe^{-x}\,dx$

4. Let R be the region in the $xy-$plane which lies above the graph of $y=\frac{1}{2}x$, to the right of the $y-$axis, and below the graph of $y=\sqrt{x}+4$.

   Note: The $x-$coordinate of one corner point of this region is 16.

   a) (4.1) Write an expression involving one or more integrals (with respect to the variable x) which gives the area of R.
   b) (4.1) Write an expression involving one or more integrals (with respect to the variable y) which gives the area of R.
   c) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the $x-$axis.
   d) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the $y-$axis.

5. Let X be a random variable taking values in $[0,1]$ whose density function is

$$f(x)=cx^2(1-x).$$

   a) (4.8) Find c.
   b) (4.8) Find the probability that $X<\frac{1}{4}$.
   c) (4.8) Find the expected value of X.

6. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n!)^2}{4(2n)!} \qquad \sum_{n=1}^{\infty} \frac{n}{n+3\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \left(\frac{11^n}{4^{2n}} - \frac{5}{n^3\sqrt{n}} \right) \qquad \sum_{n=2}^{\infty} \frac{n^2}{n^5+7n^2+2}$$

7. (7.4) For each series, determine whether or not the series converges or diverges, along with a "quick" reason:

a) $\sum \frac{1}{n}$

b) $\sum \frac{3}{4+9n}$

c) $\sum \frac{3}{n^4}$

d) $\sum \frac{5}{\sqrt{n}}$

e) $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$

f) $\sum \frac{n-1}{n+1}$

g) $\sum \frac{1}{3^n}$

h) $\sum (-1)^n$

8. a) (8.2) Estimate the following integral by replacing the integrand with its eighth Taylor polynomial:

$$\int_0^{1/2} x^2 \arctan(x^3)\, dx$$

b) (8.2) Estimate $\sin(\frac{1}{3})$ by computing the fourth Taylor polynomial for an appropriately chosen function.

9. (6.2) Find the sum of each of these series:

a) $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$ b) $\sum_{n=0}^{\infty} \frac{2\cdot 5^{n-3}}{11\cdot 3^{2n} 7^{n+1}}$ c) $\sum_{n=0}^{104} \left(\frac{4}{7}\right)^n$

Solutions

1. a) i. Use parts with $u = x$ and $dv = \sec^2 x\,dx$ so that $du = dx$ and $v = \int \sec^2 x\,dx = \tan x$. Therefore by the parts formula,

$$\int x\sec^2 x\,dx = \int u\,dv = uv - \int v\,du = x\tan x - \int \tan x\,dx.$$

To integrate $\tan x$, rewrite it using a trig identity and then use the u-substitution $u = \cos x$, $du = -\sin x\,dx$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{1}{u}\,du = -\ln u = -\ln(\cos x) + C$$

Putting all this together,

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln(\cos x) + C.$$

ii. Use integration by parts twice. For the first step, set $u = x^2$ and $dv = 12e^{3x}\,dx$ so that $du = 2x\,dx$ and $v = \int dv = \int 12e^{3x}\,dx = 4e^{3x}$. Now by the parts formula,

$$\int 12x^2e^{3x}\,dx = \int u\,dv = uv - \int v\,du = 4x^2e^{3x} - \int 8xe^{3x}\,dx.$$

Now in the remaining integral, use parts again with $u = x$ and $dv = 8e^{3x}$ so that $du = dx$ and $v = \frac{8}{3}e^{3x}$. By the parts formula,

$$\int 8xe^{3x}\,dx = \int u\,dv = \int v\,du = \frac{8}{3}xe^{3x} - \int \frac{8}{3}e^{3x}\,dx = \frac{8}{3}xe^{3x} - \frac{8}{9}e^{3x}$$

so putting all this together we have

$$\begin{aligned}\int 12x^2e^{3x}\,dx &= 4x^2e^{3x} - \int 8xe^{3x}\,dx \\ &= 4x^2e^{3x} - \left[\frac{8}{3}xe^{3x} - \frac{8}{9}e^{3x}\right] + C \\ &= 4x^2e^{3x} - \frac{8}{3}xe^{3x} + \frac{8}{9}e^{3x} + C.\end{aligned}$$

b) i. Use partial fractions: first, factor the denominator as $(x-4)(x+4)$ so that the guessed form of the decomposition is

$$\frac{24}{x^2-16} = \frac{A}{x-4} + \frac{B}{x+4};$$

finding common denominator and clearing denominators, we obtain

$$24 = A(x+4) + B(x-4).$$

Now plug in values of x:

$$\begin{aligned} x = -4 &\Rightarrow -24 = -8B \Rightarrow B = 3 \\ x = 4 &\Rightarrow -24 = 8A \Rightarrow A = -3 \end{aligned}$$

Therefore

$$\int \frac{24}{x^2-16}\,dx = \int \left(\frac{-3}{x-4} + \frac{3}{x+4}\right) dx = -3\ln(x-4) + 3\ln(x+4) + C.$$

ii. Use partial fractions: the guessed form of the decomposition is

$$\frac{-30}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3};$$

finding common denominator and clearing denominators, we obtain

$$-30 = A(x+2)(x-3) + Bx(x-3) + Cx(x+2).$$

Now plug in values of x:

$$\begin{aligned} x = 0 &\Rightarrow -30 = -6A \Rightarrow A = 5 \\ x = -2 &\Rightarrow -30 = 10B \Rightarrow B = -3 \\ x = 3 &\Rightarrow -30 = 15C \Rightarrow C = -2 \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{-30}{x(x+2)(x-3)}\,dx &= \int \left(\frac{5}{x} + \frac{-3}{x+2} + \frac{-2}{x-3}\right) dx \\ &= 5\ln x - 3\ln(x+2) - 2\ln(x-3) + C. \end{aligned}$$

c) i. This integral is vertically unbounded at $x = 2$. So it is rewritten as

$$\begin{aligned} \int_2^4 \frac{3}{(2-x)^2}\,dx = \lim_{b\to 2^+} \int_b^4 \frac{3}{(2-x)^2}\,dx \\ &= \lim_{b\to 2^+} \left[\frac{3}{2-x}\right]_b^4 \\ &= \lim_{b\to 2^+} \left[\frac{3}{2-4} - \frac{3}{2-b}\right] \\ &= \frac{-3}{2} - \frac{3}{0} = \infty. \end{aligned}$$

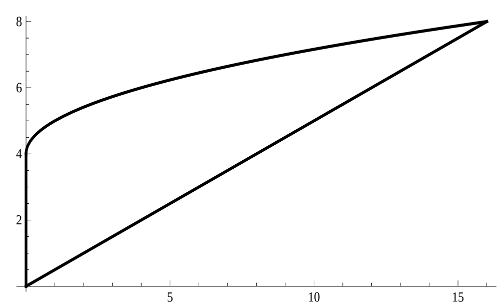
Therefore this integral diverges.

ii. For this integral, use parts with $u = x$ and $dv = e^{-x}\,dx$. Therefore $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$ so by the parts formula (together with rewriting the improper integral as a limit),

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\,dx \\
&= \lim_{b\to\infty}\int_0^b u\,dv \\
&= \lim_{b\to\infty}\left([uv]_0^b - \int_0^b v\,du\right) \\
&= \lim_{b\to\infty}\left(\left[-xe^{-x}\right]_0^b - \int_0^b -e^{-x}\,du\right) \\
&= \lim_{b\to\infty}\left(-be^{-b} - e^{-x}|_0^b\,du\right) \\
&= \lim_{b\to\infty}\left(-be^{-b} - e^{-x}|_0^b\,du\right) \\
&= \lim_{b\to\infty}\left(-be^{-b} - e^{-b} + 1\,du\right) \\
&= \lim_{b\to\infty}\left(-be^{-b}\right) - 0 + 1 \\
&= \lim_{b\to\infty}\frac{-b}{e^b} - 0 + 1 \\
&\overset{L}{=} \lim_{b\to\infty}\frac{-1}{e^b} - 0 + 1 = 0 - 0 + 1 = 1.
\end{aligned}$$

Therefore the integral converges to 1.

2. First, quickly sketching the given graphs gives the following picture of R:



Plug in $x = 16$ to either $y = \frac{1}{2}x$ or $y = \sqrt{x} + 4$ to find the corresponding y-coordinate of the upper-right corner, which is $y = 8$.

a) Integrate from the left to the right, where the integrand is the top function minus the bottom function:

$$A = \int_0^{16} (\sqrt{x} + 4 - \frac{1}{2}x)\,dx$$

b) First, solve the equations for x in terms of y to get $x = 2y$ and $x = (y-4)^2$. Now integrate from the bottom to the top, where the integrand is the right-most function minus the left-most function (you need two integrals because the left-most function changes at $y = 4$):

$$A = \int_0^4 (2y - 0)\,dy + \int_4^8 (2y - (y-4)^2)\,dy$$

c) Using x as the variable, we see that the direction of integration (left to right) is parallel to the axis of revolution, so use washers:

$$V = \int_0^{16} [\pi R^2 - \pi r^2]\,dx = \int_0^{16} [\pi(\sqrt{x}+4)^2 - \pi(\frac{1}{2}x)^2]\,dx$$

(Using y as the variable, you use shells (and need two integrals).)

$$V = \int_0^8 2\pi r h\,dy = \int_0^4 2\pi y(2y-0)\,dy + \int_4^8 2\pi y(2y - (y-4)^2)\,dy$$

d) Using x as the variable, we see that the direction of integration (left to right) is perpendicular to the axis of revolution, so use shells:

$$V = \int_0^{16} 2\pi r h\,dx = \int_0^{16} 2\pi x(\sqrt{x} + 4 - \frac{1}{2}x)\,dx$$

(Using y as the variable, you use washers (and need two integrals).)

$$V = \int_0^8 [\pi R^2 - \pi r^2]\,dy = \int_0^4 \left[\pi(2y)^2 - \pi 0^2\right]\,dy + \int_4^8 \left[\pi(2y)^2 - \pi((y-4)^2)^2\right]\,dy$$

3. a) The density function must integrate to 1:

$$1 = \int_0^1 cx^2(1-x)\,dx = \int_0^1 (cx^2 - cx^3)\,dx = \left[\frac{cx^3}{3} - \frac{cx^4}{4}\right]_0^1$$
$$= \frac{c}{3} - \frac{c}{4} = \frac{c}{12}.$$

Therefore $c = 12$.

b) $P(X < \frac{1}{4}) = \int_0^{1/4} 12x^2(1-x)\,dx = \int_0^{1/4}(12x^2 - 12x^3)\,dx = \left[4x^3 - 3x^4\right]_0^{1/4} = 4(1/4)^3 - 3(1/4)^4 = \frac{4}{64} - \frac{3}{256} = \frac{9}{256}.$

c) $EX = \int_a^b xf(x)\,dx = \int_0^1 x12x^2(1-x)\,dx = \int_0^1 (12x^3 - 12x^4)\,dx = \left[3x^4 - \frac{12}{5}x^5\right]_0^1 = 3 - \frac{12}{5} = \frac{3}{5}.$

4. a) Use the Ratio Test:

$$\begin{aligned}
\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}((n+1)!)^2}{4(2(n+1))!}\right|}{\left|\frac{(-1)^n (n!)^2}{4(2n)!}\right|} \\
&= \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} \\
&= \lim_{n\to\infty} \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2(n+1)(2n+1)} \\
&= \lim_{n\to\infty} \frac{n+1}{(2(2n+1)} = \frac{\infty}{\infty} \\
&\stackrel{L}{=} \lim_{n\to\infty} \frac{1}{4} = \frac{1}{4}.
\end{aligned}$$

Since $\rho < 1$, the series **converges absolutely** by the Ratio Test.

b) The first part of the series can be rewritten using exponent rules as $\sum \left(\frac{11}{16}\right)^n$; this is a geometric series with $r = \frac{11}{16} \in (-1, 1)$ so it converges absolutely by the Geometric Series Test.

The second part of the series can be rewritten as $\sum \frac{5}{n^{7/2}}$; this is a p-series with $p = \frac{7}{2} > 1$ so it converges absolutely.

Since the difference of two absolutely convergent series is absolutely convergent, the entire series **absolutely converges**.

c) Use the N^{th}-term test:

$$\lim_{N\to\infty} |a_N| = \lim_{N\to\infty} \frac{N}{N + 3\sqrt{N}} = \frac{\infty}{\infty} \stackrel{L}{=} \lim_{N\to\infty} \frac{1}{1 + \frac{3}{2\sqrt{N}}} = \frac{1}{1+0} = 1.$$

Since this limit is not zero, the series **diverges** by the N^{th} term test.

d) Notice $0 \leq \frac{n^2}{n^5+7n^2+2} \leq \frac{n^2}{n^5} = \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ converges (it is a p-series with $p = 3 > 1$), so does the given series by the Comparison Test. Since the given series is positive, it must **converge absolutely**.

5. a) $\sum \frac{1}{n}$ diverges (harmonic)

b) $\sum \frac{3}{4+9n}$ diverges (harmonic)

c) $\sum \frac{3}{n^4}$ converges (p-series, $p = 4 > 1$)

d) $\sum \frac{5}{\sqrt{n}}$ diverges (p-series, $p = \frac{1}{2} \leq 1$)

e) $\sum \frac{(-1)^{n+1}}{\sqrt{n}}$ converges (Alternating Series Test)

f) $\sum \frac{n-1}{n+1}$ diverges (N^{th} term Test)

g) $\sum \frac{1}{3^n}$ converges (geometric, $r = \frac{1}{3}$)

h) $\sum (-1)^n$ diverges (geometric, $r = -1$)

6. a) First, the Taylor series of $\arctan x$ is $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$. Substitute x^3 for x to obtain

$$\arctan x^3 = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \ldots$$

and multiply through to get the Taylor series of the integrand:

$$x^2 \arctan x^3 = x^5 - \frac{x^{11}}{3} + \frac{x^{17}}{5} - \ldots$$

So the eighth Taylor polynomial of the integrand is $P_8(x) = x^5$. Therefore

$$\int_0^{1/2} x^2 \arctan(x^3)\,dx \approx \int_0^{1/2} x^5\,dx = \left[\frac{x^6}{6}\right]_0^{1/2} = \frac{(1/2)^6}{6} = \frac{1}{384}.$$

b) The Taylor series of $\sin x$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ so the fourth Taylor polynomial of $\sin x$ is $P_4(x) = x - \frac{x^3}{3!} = x - \frac{1}{6}x^3$. Therefore

$$\sin\frac{1}{3} \approx P_4(\frac{1}{3}) = \frac{1}{3} - \frac{1}{6}\left(\frac{1}{3}\right)^3 = \frac{1}{3} - \frac{1}{162} = \frac{53}{162}.$$

7. a) This series is geometric:

$$\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n = 4\left(\frac{2}{5}\right)\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{8}{5}\cdot\frac{1}{1-2/5} = \frac{8}{5}\cdot\frac{5}{3} = \frac{8}{3}.$$

b) This series is geometric:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2\cdot 5^{n-3}}{11\cdot 3^{2n}7^{n+1}} &= \frac{2\cdot 5^{-3}}{11\cdot 7}\sum_{n=0}^{\infty}\frac{5^n}{9^n 7^n} \\ &= \frac{2}{11\cdot 7\cdot 5^3}\sum_{n=0}^{\infty}\left(\frac{5}{63}\right)^n \\ &= \frac{2}{11\cdot 7\cdot 5^3}\cdot\frac{1}{1-5/63} \\ &= \frac{2}{11\cdot 7\cdot 5^3}\cdot\frac{63}{58} \\ &= \frac{9}{11\cdot 5^3\cdot 29}.\end{aligned}$$

c) This series is a finite geometric series, so by the finite sum formula for geometric series we have

$$\sum_{n=0}^{104}\left(\frac{4}{7}\right)^n = \frac{1-(4/7)^{105}}{1-4/7}.$$

3.13 Fall 2014 Exam 1

1. a) Suppose you want to define the function $f(x) = \ln(x^4+1)$ in *Mathematica*. If you type `f[x_] = Ln[x^4 + 1]`, you will get an error. Why?

   b) Suppose you want to compute the integral $\int_0^3 \sin 2x\, dx$ in *Mathematica*. If you type `Integrate[sin[2x], {x, 0, 3}]`, the command won't work. Why?

   c) Suppose you want to define the function $f(x) = e^{3x}$ in *Mathematica*. If you type `f[x_] = E^3x`, your function will not be defined correctly. Why not?

2. (2.1, 2.2) Evaluate each of the following integrals:

$$\int \frac{4}{x^2+1}\, dx \qquad \int 4e^{x/2}\, dx \qquad \int x(x-2)^2\, dx \qquad \int \left(\frac{\sec^2 x}{5} + 3\sin x\right) dx$$

3. (2.3) Evaluate one of the following two integrals.

$$\int -3\cot^2 x\, dx \qquad \int \frac{x^2-1}{x^2+1}\, dx$$

4. (2.5) Evaluate one of the following two integrals.

$$\int 12x^5\sqrt{x^3+2}\, dx \qquad \int x^2\left(\frac{x}{2}-1\right)^{12} dx$$

5. (2.8) Evaluate one of the following two integrals.

$$\int \frac{3x^3-3x^2+2x+2}{x(x+1)(x-1)(x-2)}\, dx \qquad \int \frac{x+1}{x^2(x-1)}\, dx$$

6. (2.6) Evaluate one of the following two integrals.

$$\int 4x\cos 2x\, dx \qquad \int \ln^2 x\, dx$$

7. (3.3) Determine, with justification, whether the following improper integral converges or diverges:

$$\int_3^\infty \frac{10}{x^2+3x+1}\, dx$$

Solutions

1. a) In *Mathematica*, you need to type `Log` for natural log, not `Ln`.
   b) `Sin` needs a capital S.
   c) The `3x` in the exponent needs parenthesis around it; as typed, the function is $f(x) = e^3 x$.

2. a) $4 \arctan x + C$.
   b) By the Linear Replacement Principle, this is $8e^{x/2} + C$.
   c) First, multiply out to get $\int (x^3 - 4x^2 + 4x)\, dx$. Then integrate to get $\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 + C$.
   d) $\frac{1}{5} \tan x - 3 \cos x + C$.

3. a) Rewrite the integrand:
$$\int -3 \cot^2 x \, dx = \int -3(\csc^2 x - 1)\, dx = 3 \cot x + 3x + C.$$
   b) Rewrite the integrand:
$$\int \frac{x^2 - 1}{x^2 + 1}\, dx = \int \frac{(x^2 + 1) - 2}{x^2 + 1}\, dx = \int \left(1 - \frac{2}{x^2 + 1}\right) dx = x - 2 \arctan x + C.$$

4. a) Let $u = x^3 + 2$ so that $du = 3x^2\, dx$ and $x^3 = u - 2$. Now write the integral as
$$\begin{aligned}\int 12x^5 \sqrt{x^3 + 2}\, dx = \int 4(3x^2)x^3\sqrt{x^3 + 2}\, dx &= \int 4(u-2)\sqrt{u}\, du \\ &= \int 4(u-2)\sqrt{u}\, du \\ &= \int (4u^{3/2} - 2u^{1/2})\, du \\ &= 4\left(\frac{2}{5}\right) x^{5/2} - 2\left(\frac{3}{2}\right) u^{3/2} + C \\ &= \frac{8}{5}(x^3 + 2)^{5/2} - 3(x^3 + 2)^{3/2} + C.\end{aligned}$$
   b) Let $u = \frac{x}{2} - 1$ so that $du = \frac{1}{2}dx$, i.e. $dx = 2du$. Also, $x = 2u + 2$ so the integral becomes
$$\begin{aligned}\int x^2 \left(\frac{x}{2} - 1\right)^{12} dx &= \int (2u+2)^2 u^{12} 2du \\ &= \int (8u^2 + 16u + 8)u^{12}\, du \\ &= \int (8u^{14} + 16u^{13} + 8u^{12})\, du \\ &= \frac{8}{15}u^{15} + \frac{16}{14}u^{14} + \frac{8}{13}u^{13} + C \\ &= \frac{8}{15}\left(\frac{x}{2} - 1\right)^{15} + \frac{16}{14}\left(\frac{x}{2} - 1\right)^{14} + \frac{8}{13}\left(\frac{x}{2} - 1\right)^{13} + C.\end{aligned}$$

5. a) The partial fraction decomposition is

$$\frac{3x^3 - 3x^2 + 2x + 2}{x(x+1)(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} + \frac{D}{x-2}.$$

Finding a common denominator and equating the numerators, we obtain

$$\begin{aligned} 3x^3 - 3x^2 + 2x + 2 = A(x+1)(x-1)(x-2) + Bx(x-1)(x-2) \\ + Cx(x+1)(x-2) + Dx(x+1)(x-1). \end{aligned}$$

Set $x = 0$ to obtain $2 = 2A$, i.e. $A = 1$. Next, set $x = 1$ to obtain $4 = -2C$, i.e. $C = -2$. Next, set $x = 2$ to obtain $18 = 6D$, i.e. $D = 3$. Last, set $x = -1$ to obtain $-6 = -6B$, i.e. $B = 1$. So the integral is

$$\begin{aligned} \int \frac{3x^3 - 3x^2 + 2x + 2}{x(x+1)(x-1)(x-2)}\, dx &= \int \left[\frac{1}{x} + \frac{1}{x+1} + \frac{-2}{x-1} + \frac{3}{x-2}\right] dx \\ &= \ln|x| + \ln|x+1| - 2\ln|x-1| + 3\ln|x-2| + C. \end{aligned}$$

b) The partial fraction decomposition is

$$\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

Finding a common denominator and equating the numerators, we obtain

$$x + 1 = Ax(x-1) + B(x-1) + Cx^2.$$

Now, set $x = 0$ to get $1 = -B$, i.e. $B = -1$. Next, set $x = 1$ to get $2 = C$. Finally, plug in the known values of B and C, and set $x = 2$ to get $3 = A(2)(1) + (-1)(1) + (2)(4)$, i.e. $3 = 2A + 7$, i.e. $A = -2$. Thus the integral is

$$\int \frac{x+1}{x^2(x-1)}\, dx = \int \left[\frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-1}\right] dx = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C.$$

6. a) Use integration by parts; set $u = 4x$ and $dv = \cos 2x$ so that $du = 4dx$ and $v = \frac{1}{2}\sin 2x$. Now by the parts formula the integral is

$$\begin{aligned} \int 4x\cos 2x\, dx = uv - \int v\, du &= 4x\left(\frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\sin 2x(4)dx \\ &= 2x\sin 2x - \int 2\sin 2x\, dx \\ &= 2x\sin 2x + \cos 2x + C. \end{aligned}$$

b) Use integration by parts; set $u = \ln^2 x$ and $dv = dx$ so that $du = 2\ln x\left(\frac{1}{x}\right) dx$ and $v = x$. So by the parts formula the integral is

$$\int \ln^2 x\, dx = uv - \int v\, du = x\ln^2 x - \int x 2\ln x\left(\frac{1}{x}\right) dx = x\ln^2 x - \int 2\ln x\, dx.$$

To do the last integral, use integration by parts again (if you don't remember what it was from class); set $u = 2\ln x$ and $dv = dx$ so that $du = \frac{2}{x}\,dx$ and $v = x$. So by the parts formula

$$\int 2\ln x\,dx = uv - \int v\,du = 2x\ln x - \int x\frac{2}{x}\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C.$$

So the entire integral is

$$x\ln^2 x - \int 2\ln x\,dx = x\ln^2 x - 2x\ln x + 2x + C.$$

7. Notice that $0 \le \dfrac{10}{x^2+3x+1} \le \dfrac{10}{x^2}$. We know from class that $\displaystyle\int_3^\infty \frac{10}{x^2}\,dx$ converges (it is a p-integral with $p = 2 > 1$ and the starting index is irrelevant), so by the Comparison Test $\displaystyle\int_3^\infty \frac{10}{x^2+3x+1}\,dx$ converges as well.

3.14 Fall 2014 Exam 2

1. Let R be the region of points in the $xy-$plane which are located to the right of the graph of $y = 8x$, below the graph of $y = 8$ and above the graph of $y = x^3$.

   a) (4.1) Write down an expression involving one or more integrals with respect to x which gives the area of R.

   b) (4.1) Write down an expression involving one or more integrals with respect to y which gives the area of R.

2. Let Q be the region of points in the $xy-$plane located below the curve $y = 2\sqrt{x}$ and above the curve $y = 2x^4$ (these curves meet at the origin and at the point $(1, 2)$).

   a) (4.2) Suppose Q is revolved around the $x-$axis to produce a solid. Write down an expression involving one or more integrals with respect to x which will compute the volume of this solid.

   b) (4.2) Write down an expression involving one or more integrals with respect to y which will compute the volume of this solid described in part (a).

   c) (4.2) Suppose Q is revolved around the line $y = 10$ to produce a solid. Write down an expression involving one or more integrals (with respect to whatever variable you like) which will compute the volume of this solid.

   d) (4.2) Suppose a solid is constructing whose base is Q and whose cross-sections parallel to the y-axis are squares. Write down an expression involving one or more integrals (with respect to whatever variable you like) which will compute the volume of this solid.

3. Suppose X is a random variable whose density function is

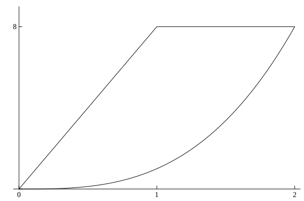
$$f(x) = \begin{cases} 2e^{-2x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

   a) (4.8) Find the probability that $X \leq 5$.

   b) (4.8) Write an integral which will compute the expected value of X. Your integrand should contain no letters other than x, but otherwise does not need to be simplified or evaluated.

Solutions

1. Here is a picture of R:



   a) Integrate (top function minus bottom function) from left to right:

$$A = \int_0^1 (8x - x^3)\, dx + \int_1^2 (8 - x^3)\, dx.$$

   b) Solve the functions for x: $y = x^3$ becomes $x = \sqrt[3]{y}$ and $y = 8x$ becomes $x = \frac{y}{8}$. Now integrate the (right-most function minus left-most function) from bottom to top:

$$A = \int_0^8 \left(\sqrt[3]{y} - \frac{y}{8} \right) dy.$$

2. Note that the top function is $y = 2\sqrt{x}$ (i.e. $x = \frac{y^2}{4}$) and the bottom function is $y = 2x^4$ (i.e. $x = \sqrt[4]{y/2}$). Let Q be the region of points in the $xy-$plane located below the curve $y = 2\sqrt{x}$ and above the curve $y = 2x^4$ (these curves meet at the origin and at the point $(1, 2)$).

   a) Since the direction of integration (left to right) is parallel to the axis of revolution, use washers:

$$V = \int_a^b \left[\pi R^2 - \pi r^2 \right] dx = \int_0^1 \left[\pi(2\sqrt{x})^2 - \pi(2x^4)^2 \right] dx.$$

   b) Since the direction of integration (bottom to top) is perpendicular to the axis of revolution, use shells:

$$V = \int_c^d 2\pi r h\, dy = \int_0^2 2\pi y \left[\sqrt[4]{\frac{y}{2}} - \frac{y^2}{4} \right] dy.$$

   c) Using x as the variable, this is washers:

$$V = \int_a^b \left[\pi R^2 - \pi r^2 \right] dx = \int_0^1 \left[\pi(10 - 2x^4)^2 - \pi(10 - 2\sqrt{x})^2 \right] dx.$$

   Using y as the variable, use shells:

$$V = \int_c^d 2\pi r h\, dy = \int_0^2 2\pi(10 - y) \left[\sqrt[4]{\frac{y}{2}} - \frac{y^2}{4} \right] dy.$$

d) Since the known cross-sections are parallel to the y-axis, they are perpendicular to the x-axis and since you need the cross-sections to be perpendicular to the direction of integration, you have to use x as the variable. The cross-sectional area is

$$A(x) = (\text{side length})^2 = (2\sqrt{x} - 2x^4)^2$$

so the volume is

$$V = \int_a^b A(x)\,dx = \int_0^1 (2\sqrt{x} - 2x^4)^2\,dx.$$

3. a) $P(X \le 5) = \displaystyle\int_0^5 2e^{-2x}\,dx = \left[-e^{-2x}\right]_0^5 = 1 - e^{-10}.$

b) $EX = \displaystyle\int_a^b x f(x)\,dx = \int_0^\infty x2e^{-2x}\,dx.$

3.15 Fall 2014 Exam 3

1. a) (5.2) Define precisely what it means for a series to *converge* to a number L. (It is not sufficient to just write "$\sum a_n = L$"; I want to know what this notation *means*.)

   b) (5.2) Why is the problem of determining whether an infinite series converges or not difficult? Put another way, why do we need calculus to study infinite series?

   c) (5.2) Suppose that the N^{th} partial sum of a series $\sum a_n$ is $S_N = \frac{N+1}{N-1}$. Does the series converge or diverge (or do you not know)? Explain your answer.

2. Find the sum of each of the following series:

   a) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{5 \cdot 3^n}{2^{2n}}$

   b) (8.2) $\displaystyle\sum_{n=0}^{\infty} \frac{-1}{2^n n!}$

   c) (6.2) $20 - 5 + \dfrac{5}{4} - \dfrac{5}{16} + \dfrac{5}{64} - \dfrac{5}{4^4} + ...$

   d) (6.2) $\displaystyle\sum_{n=2}^{25} 4^n$

3. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverges.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

   b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3n^5}$

   c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{10^{(e^{-n})}}$

   d) $\displaystyle\sum_{n=2}^{\infty} \frac{4}{\sqrt[3]{n}-1}$

4. a) (8.2) Approximate $\sin\left(\frac{2}{3}\right)$ by evaluating the fourth Taylor polynomial for an appropriately chosen function.

   b) (8.2) Approximate the following integral by replacing the integrand with its fifth Taylor polynomial:

   $$\int_0^2 \arctan x^2 \, dx$$

   c) (8.2) Evaluate the following limit without using L'Hôpital's Rule (simplify your answer):

   $$\lim_{x\to 0} \frac{12x^8}{e^{x^4} - x^4 - 1}$$

5. (8.1) Find the Taylor series of each of the following functions (your answer can be in Σ notation or written out, but if you write it out you should include at least four nonzero terms). You do not need to specify the interval of x values for which the series converges.

   a) $f(x) = x^2 \cos 2x^2$

   b) $f(x) = \frac{3}{2-5x}$

Solutions

1. a) $\sum a_n$ converges to L if $\lim_{N\to\infty} S_N = L$ where S_N is the N^{th} partial sum of $\sum a_n$.

   b) Addition is defined as a binary operation (i.e. it has two inputs), so if you add a finite list of numbers two at a time you eventually run out of numbers to add. But with an infinite series, you never run out of numbers. Also, with a finite list of numbers to add, the associative and commutative properties hold, but infinite series cannot be regrouped or rearranged legally.

   c) $\displaystyle\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N+1}{N-1} = 1$, so by the definition in part (a) of this problem, $\sum a_n = 1$, i.e. the series converges.

2. a) $\displaystyle\sum_{n=0}^{\infty} \frac{5\cdot 3^n}{2^{2n}} = 5\sum_{n=0}^{\infty} \frac{3^n}{4^n} = 5\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = 5\cdot\frac{1}{1-\frac{3}{4}} = 20.$

   b) $\displaystyle\sum_{n=0}^{\infty} \frac{-1}{2^n n!} = -\sum_{n=0}^{\infty} \frac{x^n}{n!}\bigg|_{x=\frac{1}{2}} = -e^{1/2} = -\sqrt{e}.$

   c) $\displaystyle 20 - 5 + \frac{5}{4} - \frac{5}{16} + \frac{5}{64} - \frac{5}{4^4} + \dots = 20\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n = 20\cdot\frac{1}{1-\left(\frac{-1}{4}\right)} = 16.$

   d) $\displaystyle\sum_{n=2}^{25} 4^n = \sum_{n=0}^{23} 4^{n+2} = 4^2\sum_{n=0}^{23} 4^n = 4^2\left(\frac{4^{24}-1}{4-1}\right) = \frac{16}{3}\left(4^{24}-1\right).$

3. a) Use the Ratio Test (the last step below uses L'Hôpital's Rule):

$$\rho = \lim_{n\to\infty} \frac{\left|\frac{(n+1)^2}{(n+1)!}\right|}{\left|\frac{n^2}{n!}\right|} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2} = \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)n^2} = \lim_{n\to\infty}\frac{n+1}{n^2} = 0$$

   Since $\rho < 1$, the series converges absolutely.

   b) We consider $\displaystyle\sum |a_n| = \sum\left|\frac{(-1)^n}{3n^5}\right| = \sum\frac{1}{3n^5}$. This is a p-series with $p = 5 > 1$, so it converges. Therefore the original series converges absolutely.

   c) Use the n^{th}-Term Test: $\displaystyle\lim_{n\to\infty}\frac{1}{10^{(e^{-n})}} = \frac{1}{10^{e^{-\infty}}} = \frac{1}{10^0} = \frac{1}{10} \neq 0$ so the series diverges.

   d) Observe $\displaystyle\frac{4}{\sqrt[3]{n}-1} \geq \frac{4}{\sqrt[3]{n}} = \frac{4}{n^{1/3}} \geq 0$. $\sum\frac{4}{n^{1/3}}$ is a divergent p-series ($p = \frac{1}{3} \leq 1$) so by the Comparison Test, $\sum a_n$ diverges as well.

4. a) $\displaystyle\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ so $\displaystyle P_4(x) = x - \frac{x^3}{3!}$. Thus $\displaystyle\sin\left(\frac{2}{3}\right) \approx P_4\left(\frac{2}{3}\right) = \frac{2}{3} - \frac{(2/3)^3}{3!} = \frac{2}{3} - \frac{8/27}{6} = \frac{2}{3} - \frac{4}{81} = \frac{50}{81}.$

b) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ so $\arctan x^2 = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \ldots$ so $P_5(x) = x^2$. Thus

$$\int_0^2 \arctan x^2\,dx \approx \int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}.$$

c) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$ so $e^{x^4} - x^4 - 1 = 1 + x^4 + \frac{x^8}{2} + \frac{x^{12}}{3!} + \ldots - x^4 - 1 = \frac{1}{2}x^8 + \frac{x^{12}}{3!} + \ldots$. Thus

$$\lim_{x\to 0} \frac{12x^8}{e^{x^4} - x^4 - 1} \approx \lim_{x\to 0} \frac{12x^8}{\frac{1}{2}x^8 + \frac{x^{12}}{3!} + ..} = \lim_{x\to 0} \frac{12}{\frac{1}{2} + \frac{x^4}{3!} + \ldots} = \frac{12}{\frac{1}{2} + 0 + 0 + \ldots} = 24.$$

5. a) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Substitute $2x^2$ for x to get $\cos 2x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n}}{(2n)!}$ and multiply by x^2 to get

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n} x^2}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{4n+2}}{(2n)!} = x^2 - \frac{4}{2!}x^6 + \frac{16}{4!}x^{10} - \frac{4^3}{6!}x^{14} + \frac{4^4}{8!}x^{18} - \ldots$$

b) $f(x) = \frac{3}{2-5x} = \frac{3}{2}\left(\frac{1}{1-\frac{5}{2}x}\right)$. Now we know $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ so

$$\begin{aligned} f(x) = \frac{3}{2}\left(\frac{1}{1-\frac{5}{2}x}\right) &= \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{5}{2}x\right)^n = \sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{5}{2}\right)^n x^n \\ &= \frac{3}{2} + \frac{15}{4}x + \frac{75}{8}x^2 + \frac{3\cdot 5^3}{2^4}x^3 + \frac{3\cdot 5^4}{2^5}x^4 + \ldots \end{aligned}$$

3.16 Fall 2014 Final Exam

1. (2.1, 2.2) Evaluate each of the following integrals:

   a) $\displaystyle\int \frac{1}{x}\,dx$ c) $\displaystyle\int \frac{1}{x+1}\,dx$ e) $\displaystyle\int \frac{1}{(x+1)^2}\,dx$

   b) $\displaystyle\int \frac{1}{x^2}\,dx$ d) $\displaystyle\int \frac{1}{x^2+1}\,dx$ f) $\displaystyle\int \frac{1}{x^2-1}\,dx$

2. a) (2.8) Find the partial fraction decomposition of one of the following two expressions.

   $$\frac{16-12x-x^2}{x^3-2x^2-8x} \qquad \frac{3x^2+x-1}{x^2+x^3}$$

   b) (2.6) Evaluate one of the following two integrals.

   $$\int \sqrt{x}\,\ln x\,dx \qquad \int 12x^2\cos 2x\,dx$$

   c) (2.5) Evaluate one of the following two integrals.

   $$\int \tan^2 x\,dx \qquad \int \sin^2 x\,dx$$

3. Let R be the region in the $xy-$plane which lies below the graph of $y=e^x$, to the right of the y-axis, above the x-axis and to the left of the line $x=8$.

   a) (4.1) Write an expression involving one or more integrals with respect to x which gives the area of R.

   b) (4.1) Write an expression involving one or more integrals with respect to y which gives the area of R.

   c) (4.3) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the length of the curve which makes up the top of R.

   d) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the line $y=0$.

   e) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the line $x=-2$.

4. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converges absolutely, converges conditionally, or diverges.

a) $\sum_{n=0}^{\infty} \frac{7^{2n}}{20^n}$

b) $\sum_{n=0}^{\infty} \frac{4n}{n^3+2}$

c) $\sum_{n=2}^{\infty} \left(\frac{4}{n} - \frac{3}{n^2}\right)$

d) $\sum_{n=3}^{\infty} \frac{\ln n}{n!}$

5. a) (8.2) Estimate the following integral by replacing the integrand with its tenth Taylor polynomial:

$$\int_0^{1/2} \frac{1}{1-x^4}\, dx$$

   b) (8.2) Estimate $\ln 2$ by computing the fourth Taylor polynomial for an appropriately chosen function.

6. Find the sum of each of these series (they all converge):

   a) (6.2) $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

   b) (8.2) $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots$

   c) (6.2) $\sum_{n=0}^{\infty} \frac{5 \cdot 3^{2n}}{2 \cdot 11^{n+2}}$

Solutions

1. a) $\int \frac{1}{x}\,dx = \ln|x| + C$ (no penalty for a missing absolute value sign)

   b) $\int \frac{1}{x^2}\,dx = \frac{-1}{x} + C$

   c) $\int \frac{1}{x+1}\,dx = \ln|x+1| + C$ (no penalty for a missing absolute value sign)

   d) $\int \frac{1}{x^2+1}\,dx = \arctan x + C$

   e) $\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + C$

   f) Perform partial fractions: write $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$; finding common denominators and clearing denominators we get $1 = A(x+1) + B(x-1)$; substituting in $x = 1$ gives $2A = 1$ i.e. $A = \frac{1}{2}$ and substituting in $x = -1$ gives $-2B = 1$, i.e. $B = \frac{-1}{2}$. Therefore

$$\int \frac{1}{x^2-1}\,dx = \int \left(\frac{1/2}{x+1} + \frac{-1/2}{x-1}\right)\,dx = \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1) + C$$

   (there is no penalty for missing absolute value signs).

2. a) i. First, factor the denominator to get

$$\frac{16-12x-x^2}{x^3-2x^2-8x} = \frac{16-12x-x^2}{x(x^2-2x-8)} = \frac{16-12x-x^2}{x(x-4)(x+2)}$$

   Now the guessed form of the decomposition is

$$\frac{A}{x} + \frac{B}{x-4} + \frac{C}{x+2}$$

   and after finding common denominators and clearing the denominators, we get

$$16 - 12x - x^2 = A(x-4)(x+2) + Bx(x+2) + Cx(x-4)$$

   Now plug in values of x:

$$\begin{aligned} x = 0 :& 16 = A(-4)(2) \Rightarrow A = -2 \\ x = 4 :& 16 - 48 - 16 = B(4)(6) \Rightarrow B = -2 \\ x = -2 :& 16 + 24 - 4 = C(-2)(-6) \Rightarrow C = 3 \end{aligned}$$

   Therefore the partial fraction decomposition is

$$\frac{16-12x-x^2}{x^3-2x^2-8x} = \frac{-2}{x} + \frac{-2}{x-4} + \frac{3}{x+2}.$$

ii. First, factor the denominator to get

$$\frac{3x^2+x-1}{x^2+x^3} = \frac{3x^2+x-1}{x^2(x+1)}$$

Now the guessed form of the decomposition is

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

and after finding common denominators and clearing the denominators, we get

$$3x^2+x-1 = A(x)(x+1) + B(x+1) + Cx^2$$

Now plug in values of x:

$$\begin{aligned} x = 0 &: -1 = B \\ x = -1 &: 3 - 1 - 1 = C \Rightarrow 1 = C \\ x = 1 &: 3 + 1 - 1 = 2A + 2B + C \Rightarrow 3 = 2A + 2(-1) + 1 \Rightarrow A = 2 \end{aligned}$$

Therefore the partial fraction decomposition is

$$\frac{3x^2+x-1}{x^2+x^3} = \frac{2}{x} + \frac{-1}{x^2} + \frac{1}{x+1}.$$

b) i. Use integration by parts: set $u = \ln x$ so that $du = \frac{1}{x}$, and let $dv = \sqrt{x}\,dx$ so that $v = \int dv = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2}$. Therefore by the parts formula,

$$\begin{aligned} \int \sqrt{x}\,\ln x\,dx = \int u\,dv = uv - \int v\,du \\ &= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\frac{1}{x}\,dx \\ &= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{1/2}\,dx \\ &= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C. \end{aligned}$$

ii. Use integration by parts twice: first, set $u = x^2$ so that $du = 2x\,dx$ and set $dv = 12\cos 2x\,dx$ so that $v = \int dv = 6\sin 2x$. Therefore by the parts formula,

$$\int 12x^2\cos 2x\,dx = \int u\,dv = uv - \int v\,du = 6x^2\sin 2x - \int 12x\sin 2x\,dx.$$

Now for the remaining integral, use parts again: set $u = x$ so that $du = dx$ and set $dv = 12\sin 2x\,dx$ so that $v = -6\cos 2x$. Therefore by the parts formula,

$$\begin{aligned} \int 12x\sin 2x\,dx = \int u\,dv = uv - \int v\,du &= -6x\cos 2x - \int -6\cos 2x\,dx \\ &= -6x\cos 2x + 3\sin 2x \end{aligned}$$

so putting all this together,

$$\begin{aligned}\int 12x^2\cos 2x\,dx &= 6x^2\sin 2x - \int 12x\sin 2x\,dx\\ &= 6x^2\sin 2x - [-6x\cos 2x + 3\sin 2x] + C\\ &= 6x^2\sin 2x + 6x\cos 2x - 3\sin 2x + C.\end{aligned}$$

c) i. Rewrite using a trig identity:

$$\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C.$$

ii. Rewrite using a trig identity:

$$\begin{aligned}\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx &= \int\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\,dx\\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.\end{aligned}$$

3. a) Integrate from the left to the right, where the integrand is the top-most function minus the bottom-most function (the bottom function in this case is $y = 0$):

$$A = \int_0^8 e^x\,dx$$

b) Solve $y = e^x$ for x to get $x = \ln y$. Then integrate from the bottom to the top, where the integrand is the right-most function minus the left-most function (you need two integrals because the left-most function is $x = 0$ from $y = 0$ to $y = 1$ and the left-most function is $x = \ln y$ from $y = 1$ to $y = 8$):

$$A = \int_0^1 8\,dy + \int_1^{e^8}(8 - \ln y)\,dy$$

c) By the arc length formula, the length is

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^8 \sqrt{1 + e^{2x}}\,dx.$$

d) Note that $y = 0$ is the x-axis. Using x as the variable, the direction of integration is parallel to the axis of revolution so you get washers (actually disks because there is no hole):

$$V = \int_0^8 \pi R^2\,dx = \int_0^8 \pi(e^x)^2\,dx.$$

(When y is the variable, you need shells (and two integrals)):

$$V = \int_0^1 2\pi y 8\,dy + \int_1^{e^8} 2\pi y(8 - \ln y)\,dy.$$

e) Using x as the variable, the direction of integration is perpendicular to the axis of revolution so you get shells:

$$V = \int_0^8 2\pi r h\, dx = \int_0^8 2\pi(x+2)e^x\, dx.$$

(When y is the variable, use washers (and two integrals)):

$$V = \int_0^1 \pi[(8+2)^2 - (0+2)^2]\, dy + \int_1^{e^8} \pi[(8+2)^2 - (\ln y + 2)^2]\, dy.$$

4. a) This series is geometric:

$$\sum_{n=0}^{\infty} \frac{7^{2n}}{20^n} = \sum_{n=0}^{\infty} \left(\frac{49}{20}\right)^n$$

Since the common ratio is at least 1, the series **diverges** by the Geometric Series Test.

b) $\sum \frac{4}{n}$ diverges (it is harmonic) and $\sum \frac{3}{n^2}$ converges (p-series, $p = 2 > 1$) so the whole series **diverges** (since diverges $\pm$ converges = diverges).

c) Note $0 \leq \frac{4n}{n^3+2} \leq \frac{4n}{n^3} = \frac{4}{n^2}$. Since $\sum \frac{4}{n^2}$ converges (p-series, $p = 2 > 1$), the given series converges by the Comparison Test. Since the series is positive, it **converges absolutely**.

d) Use the Ratio Test:

$$\begin{aligned}
\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{\ln(n+1)}{(n+1)!}}{\frac{\ln n}{n!}} &= \lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)!} \cdot \frac{n!}{\ln n} \\
&= \lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)\ln n} = \frac{\infty}{\infty} \\
&\overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln n + (n+1)\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\ln n + 1 + \frac{1}{n}} = \frac{0}{\infty + 1 + 0} = 0.
\end{aligned}$$

Since $\rho < 1$, the series **converges absolutely** by the Ratio Test.

5. a) We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Replacing x with x^4, we get the Taylor series of the integrand:

$$\frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + \ldots$$

So the tenth Taylor polynomial is $P_{10}(x) = 1 + x^4 + x^8$. Now

$$\begin{aligned}
\int_0^{1/2} \frac{1}{1-x^4}\, dx \approx \int_0^{1/2} \left(1 + x^4 + x^8\right) dx &= \left[x + \frac{x^5}{5} + \frac{x^9}{9}\right]_0^{1/2} \\
&= \frac{1}{2} + \frac{(1/2)^5}{5} + \frac{(1/2)^9}{9}.
\end{aligned}$$

b) We know $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ so the fourth Taylor polynomial of $\ln(x+1)$ is

$$P_4(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

Now to estimate $\ln 2$, let $x = 1$:

$$\ln 2 = \ln(1+1) \approx P_4(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}.$$

6. a) This series is geometric:

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{4} \cdot 2 = \frac{1}{2}.$$

b) The Taylor series of $\cos x$ is $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ so

$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - ... = \cos\pi = -1.$$

c) This series is geometric:

$$\sum_{n=0}^{\infty} \frac{5 \cdot 3^{2n}}{2 \cdot 11^{n+2}} = \frac{5}{2 \cdot 11}\sum_{n=0}^{\infty}\left(\frac{9}{11}\right)^n = \frac{5}{22} \cdot \frac{1}{1 - 9/11} = \frac{5}{22} \cdot \frac{11}{2} = \frac{5}{4}.$$

Chapter 4

Exams from 2016 to present

4.1 Spring 2016 Exam 1

1. a) Suppose a student is trying to define the function $f(x) = xe^{x-1}$ in *Mathematica*, and they execute

   ```
   f[x] = xE^x-1
   ```

   There are three things wrong with this command.

   b) Write down the *Mathematica* code which will find the exact solutions (i.e. not decimal approximations) of the equation $x^3 - 3x = 5$.

   c) Write down a *Mathematica* command (or commands) that will graph the function $f(x) = \cos 2x$ where x ranges from 0 to 6.

2. Compute each of the following integrals:

   a) $\displaystyle\int \left(4x + \frac{2}{3x} - 6\sqrt[3]{x}\right)\, dx$

   b) $\displaystyle\int \left(4\cos x - \frac{2}{e^x}\right)\, dx$

   c) $\displaystyle\int \left(\frac{\sec x \tan x}{7} + \sec^2 x - 3\right)\, dx$

3. In a previous lab assignment, we encountered two functions which you probably had never heard of: "sinh" and "cosh". Among other things, in the lab we discovered that

   - the derivative of $\sinh x$ is $\cosh x$; and
   - the derivative of $\cosh x$ is $\sinh x$.

   Use those two facts to evaluate the following integrals:

a) (2.1) $\int \cosh x \, dx$

b) (2.2) $\int \sinh(2x) \, dx$

c) (2.2) $\int \cosh(1-x) \, dx$

d) (2.6) $\int x \cosh x \, dx$

4. (2.8) Evaluate one of the following two integrals:

$$\int \frac{3x^2 - 4x - 1}{(x+1)(x^2+1)} \, dx \qquad \int \frac{-2x^2 + 7x + 4}{x(x-2)^2} \, dx$$

5. (2.6) Evaluate one of the following two integrals:

$$\int \ln^2 x \, dx \qquad \int 18x^2 \sin 3x \, dx$$

6. (2.5) Evaluate one of the following two integrals:

$$\int_1^e \frac{\ln x}{x} \, dx \qquad \int_{-1}^2 \frac{x}{\sqrt{x+2}} \, dx$$

7. (2.3) Evaluate one of the following two integrals:

$$\int \sin^2 x \, dx \qquad \int \sin^3 x \, dx$$

8. (3.3) Determine, with justification, whether the following integral converges or diverges:

$$\int_2^\infty \frac{3x}{x^4 + 7x^2} \, dx$$

9. (2.10) **(Bonus)** Evaluate $\int \frac{x^2}{x^6+1} \, dx$.

Solutions

1. a) i. The student is missing an underscore after the `x` (it should be `f[x_]`);
   ii. the student needs a space between the `x` and the `E` to multiply them;
   iii. the student needs parenthesis around the `x-1` to group them.

   The correct command should be `f[x_] = x E^(x-1)`.

   b) `Solve[x^3 - 3x == 5, x]`

   c) `Plot[Cos[2x], {x, 0, 6}]`

2. Just "do" all these integrals.

   a) $\displaystyle\int \left(4x + \frac{2}{3x} - 6\sqrt[3]{x}\right) dx = 4\frac{x^2}{2} + \frac{2}{3}\ln x - 6\frac{x^{4/3}}{4/3} + C = 2x^2 + \frac{2}{3}\ln x - \frac{9}{2}x^{4/3} + C.$

   b) $\displaystyle\int \left(4\cos x - \frac{2}{e^x}\right) dx = \int 4\cos x\, dx - \int 2e^{-x}\, dx = 4\sin x + 2e^{-x} + C.$

   c) $\displaystyle\int \left(\frac{\sec x \tan x}{7} + \sec^2 x - 3\right) dx = \frac{1}{7}\sec x + \tan x - 3x + C.$

3. Since sinh and cosh are derivatives of one another, they are also integrals of one another (since integration is the inverse operation of differentiation). Therefore, we know

$$\int \cosh x\, dx = \sinh x + C \qquad \text{and} \qquad \int \sinh x\, dx = \cosh x + C.$$

   Using these facts:

   a) $\displaystyle\int \cosh x\, dx = \sinh x + C.$

   b) Use the Linear Replacement Principle with $m = 2, b = 0$:

$$\int \sinh(2x)\, dx = \frac{1}{2}\cosh(2x) + C.$$

   c) Use the Linear Replacement Principle with $m = -1, b = 1$:

$$\int \cosh(1-x)\, dx = -\sinh(1-x) + C.$$

   d) Start with integration by parts. Let $u = x$ and $dv = \cosh x\, dx$. Then $du = dx$ and $v = \int dv = \sinh x$. By the parts formula,

$$\begin{aligned}\int x\cosh x\, dx = \int u\, dv &= uv - \int v\, du \\ &= x\sinh x - \int \sinh x\, dx \\ &= x\sinh x - \cosh x + C.\end{aligned}$$

4. a) Use partial fractions. The guess should be

$$\frac{3x^2 - 4x - 1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1};$$

after finding a common denominator and clearing denominators we get the equation

$$3x^2 - 4x - 1 = A(x^2+1) + (Bx+C)(x+1). \tag{4.1}$$

- Let $x = -1$ in equation (4.1) above; this gives $3+4-1 = A(2)+(Bx+C)(0)$, i.e. $6 = 2A$, i.e. $A = 3$.
- Let $x = 0$ and $A = 3$ in equation (4.1) above; this gives $-1 = (3)(1) + (B(0)+C)(1)$, i.e. $-1 = 3 + C$, i.e. $C = -4$.
- Let $x = 1$, $A = 3$ and $C = -4$ in equation (4.1) above; this gives $3 - 4 - 1 = 3(2) + (B-4)(2)$, i.e. $-2 = 6 + 2B - 8$, i.e. $B = 0$.

Therefore

$$\int \frac{3x^2 - 4x - 1}{(x+1)(x^2+1)}\,dx = \int \left(\frac{3}{x+1} + \frac{-4}{x^2+1}\right) dx = 3\ln(x+1) - 4\arctan x + C.$$

b) Use partial fractions. The guess should be

$$\frac{-2x^2 + 7x + 4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2};$$

after finding a common denominator and clearing denominators we get the equation

$$-2x^2 + 7x + 4 = A(x-2)^2 + Bx(x-2) + Cx. \tag{4.2}$$

- Let $x = 2$ in equation (4.2) above; this gives $-2(4) + 14 + 4 = A(0) + B(2)(0) + C(2)$, i.e. $10 = 2C$, i.e. $C = 5$.
- Let $x = 0$ in equation (4.2) above; this gives $4 = A(4) + B(0) + C(0)$, i.e. $4 = 4A$, i.e. $A = 1$.
- Let $x = 3$, $A = 1$ and $C = 5$ in equation (4.2) above; this gives $-2(9) + 21 + 4 = 1(1) + B(3)(1) + 5(3)$, i.e. $7 = 3B + 16$, i.e. $B = -3$.

Therefore

$$\begin{aligned}\int \frac{-2x^2 + 7x + 4}{x(x-2)^2}\,dx &= \int \left(\frac{1}{x} + \frac{-3}{x-2} + \frac{5}{(x-2)^2}\right) dx \\ &= \ln x - 3\ln(x-2) - \frac{5}{(x-2)} + C.\end{aligned}$$

5. a) This integral requires integration by parts twice. First, let $u = \ln^2 x$ so that $du = 2\ln x \cdot \frac{1}{x}\,dx$ (this uses the Chain Rule) and let $dv = dx$ so that $v = \int dv = x$. So by the parts formula,

$$\begin{aligned}\int \ln^2 x\,dx = \int u\,dv = uv - \int v\,du &= x\ln^2 x - \int x\left(2\ln x \cdot \frac{1}{x}\right) dx \\ &= x\ln^2 x - \int 2\ln x\,dx.\end{aligned}$$

Now do parts again on the last integral: let $u = 2\ln x$ so that $du = \frac{2}{x}dx$ and let $dv = dx$ so that $v = x$. Thus by the parts formula we have

$$\int 2\ln x\,dx = \int u\,dv = uv - \int v\,du = x(2\ln x) - \int x\cdot\frac{2}{x}\,dx$$
$$= 2x\ln x - \int 2\,dx = 2x\ln x - 2x$$

and therefore the entire answer is

$$x\ln^2 x - (2x\ln x - 2x) + C = x\ln^2 x - 2x\ln x + 2x + C.$$

b) This integral requires integration by parts twice. First, let $u = x^2$ so that $du = 2x\,dx$ and let $dv = 18\sin 3x\,dx$ so that $v = \int dv = 18\cdot\frac{1}{3}(-\cos 3x) = -6\cos 3x$. So by the parts formula,

$$\int 18x^2\sin 3x\,dx = \int u\,dv = uv - \int v\,du = x^2(-6\cos 3x) - \int 2x(-6\cos 3x)\,dx$$
$$= -6x^2\cos 3x + \int 12x\cos 3x\,dx.$$

Now do parts again on the last integral: let $u = x$ so that $du = dx$ and let $dv = 12\cos 3x$ so that $v = 12\cdot\frac{1}{3}\sin 3x = 4\sin 3x$. Thus by the parts formula we have

$$\int 12x\cos 3x\,dx = \int u\,dv = uv - \int v\,du = x(4\sin 3x) - \int 4\sin 3x\,dx$$
$$= 4x\sin 3x + \frac{4}{3}\cos 3x + C.$$

Putting this all together, we have

$$\int 18x^2\sin 3x\,dx = -6x^2\cos 3x + \int 12x\cos 3x\,dx$$
$$= -6x^2\cos 3x + 4x\sin 3x + \frac{4}{3}\cos 3x + C.$$

6. a) Think of the integral as $\int_1^e \ln x\cdot\frac{1}{x}\,dx$ and use the $u-$substitution $u = \ln x$. Thus $\frac{du}{dx} = \frac{1}{x}$ so $du = \frac{1}{x}\,dx$. The limits need to be changed as well; when $x = 1$, $u = \ln 1 = 0$ and when $x = e$, $u = \ln e = 1$. So

$$\int_1^e \frac{\ln x}{x}\,dx = \int_0^1 u\,du = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}.$$

b) Use the u-substitution $u = x + 2$. Then $\frac{du}{dx} = 1$ so $du = dx$. Furthermore, $x = u - 2$ and the limits need to be changed: when $x = -1$, $u = -1 + 2 = 1$ and when $x = 2$, $u = 2 + 2 = 4$. So the integral is

$$\begin{aligned}
\int_{-1}^{2} \frac{x}{\sqrt{x+2}}\,dx = \int_{1}^{4} \frac{u-2}{\sqrt{u}}\,du &= \int_{1}^{4} \left(u^{1/2} - 2u^{-1/2}\right)\,du \\
&= \left[\frac{2}{3}u^{3/2} - 2\cdot 2u^{1/2}\right]_1^4 \\
&= \left(\frac{2}{3}\cdot 4^{3/2} - 4(4)^{1/2}\right) - \left(\frac{2}{3}1^{3/2} - 4\cdot 1^{1/2}\right) \\
&= \frac{2}{3}(8) - 4(2) - \frac{2}{3} + 4 \\
&= \frac{2}{3}.
\end{aligned}$$

7. a) Rewrite with a power-reducing identity:

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

b) Rewrite with a Pythagorean trig identity:

$$\int \sin^3 x\,dx = \int \sin^2 x\,\sin x\,dx = \int (1-\cos^2 x)\sin x\,dx.$$

Now use the u-substitution $u = \cos x$ so that $\frac{du}{dx} = -\sin x$ and therefore $-du = \sin x\,dx$. Then the integral becomes

$$\int (1-u^2)(-1)\,du = \int (u^2-1)\,du = \frac{1}{3}u^3 - u + C = \frac{1}{3}\cos^3 x - \cos x + C.$$

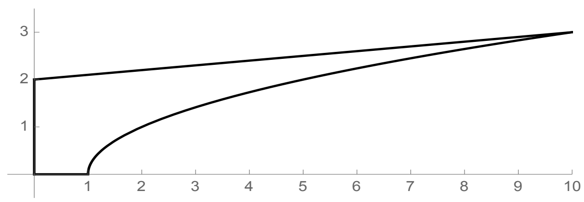
8. First, $0 \le \dfrac{3x}{x^4+7x^2} \le \dfrac{3x}{x^4} = \dfrac{3}{x^3}$ since the last fraction has smaller denominator. The integral $\displaystyle\int_2^\infty \frac{3}{x^3}\,dx$ converges (it is a p-integral with $p = 3 > 1$), so by the Comparison Test $\displaystyle\int_2^\infty \frac{3x}{x^4+7x^2}\,dx$ also converges.

9. The trick is to use the $u-$substition $u = x^3$. Then $\frac{du}{dx} = 3x^2$ so $du = 3x^2\,dx$ so $\frac{1}{3}du = x^2\,dx$. Then the integral becomes

$$\int \frac{x^2}{x^6+1}\,dx = \int \frac{x^2}{(x^3)^2+1}\,dx = \int \frac{\frac{1}{3}}{u^2+1}\,du = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan x^3 + C.$$

4.2 Spring 2016 Exam 2

1. Let Q be the region of points in the $xy-$plane located to the right of the y-axis, above the x-axis, above the curve $y = \sqrt{x-1}$ and below the line $y = \frac{1}{10}x + 2$. A picture of this region is given below; the upper-right corner point is $(10, 3)$.



   a) (4.1) Write an expression involving one or more integrals with respect to the variable x which gives the area of Q.

   b) (4.1) Write an expression involving one or more integrals with respect to the variable y which gives the area of region Q.

2. Let R be the region of points in the $xy-$plane which are located above the x-axis, below the curve $y = x^3 + 1$, to the right of the y-axis, and to the left of the line $x = 2$.

   a) (4.2) Suppose a solid is built whose base is R. If cross-sections of the solid parallel to the y-axis are squares, write an integral (with respect to whatever variable you like) which gives the volume of the solid.

   b) (4.2) Suppose R is revolved around the y-axis to produce a solid. Write an integral with respect to the variable x which gives the volume of this solid.

   c) (4.2) Write an integral with respect to y which gives the volume of the same solid as in part (b).

   d) (4.2) Suppose R is revolved around the line $y = -2$ to produce a solid. Write an integral with respect to whatever variable you like that gives the volume of this solid.

   e) (4.2) Suppose R is revolved around the line $x = 2$ to produce a solid. Write an integral with respect to whatever variable you like that gives the volume of this solid.

3. Suppose that the time, in seconds, until two particles in a nuclear reactor collide is given by a continuous random variable whose density function is

$$f(x) = \begin{cases} \frac{9}{10}x - \frac{3}{10}x^2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{else} \end{cases}$$

a) (4.8) Find the probability that the particles collide in less than 1 second.

b) (4.8) Find the expected amount of time until the particles collide.

4. (4.4) Write an integral which will give the length of the curve $y = \sin x + \cos 2x$ from $x = 0$ to $x = \pi$.

Solutions

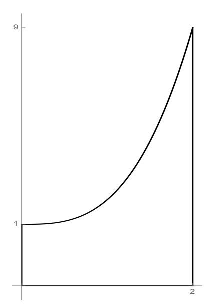
1. a) Integrate from left to right; the integrand is the top function minus the bottom function; the bottom function changes at $x = 1$ so you need two integrals:

$$A = \int_0^1 \left[\frac{1}{10}x + 2\right]\, dx + \int_1^{10} \left[\frac{1}{10}x + 2 - \sqrt{x-1}\right]\, dx$$

b) First, you need to solve the equations for x in terms of y: $y = \sqrt{x-1}$ becomes $x = y^2 + 1$; $y = \frac{1}{10}x + 2$ becomes $x = 10y - 20$. Integrate from the bottom to the top; the integrand is the right-hand function minus the left-hand function; the left-hand function changes at $y = 2$ so you need two integrals:

$$A = \int_0^2 \left[y^2 + 1 - 0\right]\, dy + \int_2^3 \left[y^2 + 1 - (10y - 20)\right]\, dy$$

2. First, here is a picture of R:



Notice that the upper-right corner point is $(2, 9)$ because when you plug $x = 2$ into $y = x^3 + 1$, you get $y = 2^3 + 1 = 9$.

a) Integrate with respect to x; the cross-sections are areas with side length $x^3 + 1$. Therefore the volume is

$$V = \int_0^2 (x^3 + 1)^2\, dx.$$

b) Again, integrate with respect to x; cross-sections are shells with $r = x$ and $h = x^3 + 1$. Therefore the volume is

$$V = \int_0^2 2\pi x(x^3 + 1)\, dx.$$

c) You need to solve the equation for x in terms of y: $y = x^3 + 1$ gives $x = \sqrt[3]{y-1}$. Then, integrating with respect to y, the cross-sections are washers (or circles). Importantly, you need two integrals because the left-most function of the region changes at $y = 1$ (this makes the r of the washer formula change at $y = 1$). So the volume is

$$V = \int_0^1 \left[\pi(2)^2 - \pi(0)^2\right]\, dy + \int_1^9 \left[\pi(2)^2 - \pi(\sqrt[3]{y-1})^2\right]\, dy.$$

d) Integrate with respect to x (so we only have to write one integral). This makes the cross-sections washers with $R = x^3+1-(-2) = x^3+3$ and $r = 0-(-2) = 2$ so the volume is
$$V = \int_0^2 \left[\pi(x^3+3)^2 - \pi(2)^2\right]\, dx.$$

e) Integrate with respect to x (so we only have to write one integral). This makes the cross-sections shells with $r = 2-x$ and $h = x^3+1$ so the volume is
$$V = \int_0^2 2\pi(2-x)(x^3+1)\, dx.$$

3. a) $P(X \leq 1) = \int_0^1 f(x)\, dx = \int_0^1 \left[\frac{9}{10}x - \frac{3}{10}x^2\right]\, dx = \left[\frac{9}{20}x^2 - \frac{1}{10}x^3\right]_0^1 = \frac{9}{20} - \frac{1}{10} = \frac{7}{20}$.

b) $EX = \int_0^2 xf(x)\, dx = \int_0^2 x\left[\frac{9}{10}x - \frac{3}{10}x^2\right]\, dx = \int_0^2 \left[\frac{9}{10}x^2 - \frac{3}{10}x^3\right]\, dx = \left[\frac{3}{10}x^3 - \frac{3}{40}x^4\right]_0^2 = \frac{3}{10}(8) - \frac{3}{40}(16) = \frac{24}{10} - \frac{48}{40} = \frac{6}{5}$.

4. By the arc length formula, this is $s = \int_a^b \sqrt{1+[f'(x)]^2}\, dx = \int_0^\pi \sqrt{1+[\cos x - 2\sin 2x]^2}\, dx$.

4.3 Spring 2016 Exam 3

1. Throughout this problem, consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

   a) (5.2) Find the fifth term of this series.

   b) (5.2) Find the third partial sum of the series (simplify your answer).

   c) (5.5) Rewrite the series so that its initial index is $n = 0$.

   d) (5.7) Does the series converge or diverge?

2. Compute the sum of five of the following six series:

   a) (6.2) $\displaystyle\sum_{n=3}^{\infty} 5\left(\frac{3}{4}\right)^n$

   b) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{3n+1}}$

   c) (8.2) $\displaystyle\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \ldots$

   d) (6.2) $\displaystyle 18 - 3 + \frac{1}{2} - \frac{1}{12} + \frac{1}{72} - \ldots$

   e) (8.2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$

   f) (6.2) $\displaystyle\sum_{n=0}^{2016} \frac{1}{7^n}$

3. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2}{n^4 + 7n^3 + 2}$

   b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n^4} + \frac{1}{2n^2} + \frac{5}{6n^{9/2}}\right)$

   c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n 8^n n!}{(2n)!}$

   d) $\displaystyle\sum_{n=1}^{\infty} \frac{4 + \sin(n^2)}{\sqrt{n}}$

4. a) (8.2) Estimate the following integral by replacing the integrand with its third Taylor polynomial (simplify your answer).

   $$\int_0^1 e^{-x^2}\,dx$$

   b) (8.1) Find a power series representation of the function $f(x) = x^2 \arctan(x^4)$. Write your series in Σ-notation, with initial index $n = 0$.

   c) (8.2) Approximate $\cos\left(\frac{1}{3}\right)$ by evaluating the fifth Taylor polynomial for an appropriately chosen function.

5. (7.3) Why do we care whether or not a series *converges absolutely*, as opposed to just knowing whether or not it *converges*?

Solutions

1. a) $a_5 = \dfrac{1}{2(5)-1} = \dfrac{1}{9}$.

   b) $S_3 = a_1 + a_2 + a_3 = \dfrac{1}{2(1)-1} + \dfrac{1}{2(2)-1} + \dfrac{1}{2(3)-1} = 1 + \dfrac{1}{3} + \dfrac{1}{5} = \dfrac{23}{15}$.

   c) Replace the n in the series with $n+1$ to get $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2(n+1)-1} = \sum_{n=0}^{\infty} \frac{1}{2n+1}$.

   d) This series is harmonic, so it diverges.

2. a) Rewrite the series so that it starts at zero; then use the geometric series formula:

$$\sum_{n=3}^{\infty} 5\left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty} 5\left(\frac{3}{4}\right)^{n+3} = 5\left(\frac{3}{4}\right)^3 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = 5\left(\frac{3}{4}\right)^3 \frac{1}{1-\frac{3}{4}} = 20\left(\frac{3}{4}\right)^3.$$

   b) Use exponent rules to rewrite the series; then use the geometric series formula:

$$\sum_{n=0}^{\infty} \frac{1}{2^{3n+1}} = \sum_{n=0}^{\infty} \frac{1}{2^{3n}2^1} = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{4}{7}.$$

   c) Identify this as the power series of e^x with $x = 4$:

$$\sum_{n=0}^{\infty} \frac{4^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}\bigg|_{x=4} = e^x\big|_{x=4} = e^4.$$

   d) This is a geometric series, which can be seen after factoring out the initial term:

$$\begin{aligned} 18 - 3 + \frac{1}{2} - \frac{1}{12} + \frac{1}{72} - \ldots &= 18\left[1 - \frac{1}{6} + \frac{1}{36} - \frac{1}{216} + \ldots\right] \\ &= 18\left[1 + \left(\frac{-1}{6}\right) + \left(\frac{-1}{6}\right)^2 + \left(\frac{-1}{6}\right)^3 + \ldots\right] \\ &= \frac{18}{1 - \frac{-1}{6}} = \frac{108}{7}. \end{aligned}$$

   e) This is the power series for $\ln(1+x)$, with $x = \frac{1}{3}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}\bigg|_{x=\frac{1}{3}} = \ln(1+x)\big|_{x=\frac{1}{3}} = \ln\left(1 + \frac{1}{3}\right) = \ln\left(\frac{4}{3}\right).$$

   f) This uses the finite sum formula for a geometric series:

$$\sum_{n=0}^{2016} \frac{1}{7^n} = \sum_{n=0}^{2016} \left(\frac{1}{7}\right)^n = \frac{1 - (1/7)^{2017}}{1 - (1/7)}.$$

3. a) Observe $0 \leq \frac{5n^2}{n^4+7n^3+2} \leq \frac{5n^2}{n^4} = \frac{5}{n^2}$. Since $\sum \frac{5}{n^2}$ is a convergent p-series ($p = 2 > 1$), the series $\sum a_n$ converges by the Comparison Test. Since the series is positive, it **converges absolutely**.

   b) This is the sum of three convergent p-series, hence converges (and **converges absolutely** since it is a positive series).

   c) Use the Ratio Test, since the series contains factorials:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}8^{n+1}(n+1)!}{(2(n+1))!}\right|}{\left|\frac{(-1)^n 8^n n!}{(2n)!}\right|}$$
$$= \lim_{n\to\infty} \frac{8^{n+1}(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{8^n n!}$$
$$= \lim_{n\to\infty} \frac{8n}{(2n+1)(2n+2)} = 0$$

   (The limit is zero because the power in the denominator is 2, and the power in the numerator is only 1; you could also compute this with L'Hôpital's Rule.) Since $\rho < 1$, the series **converges absolutely** by the Ratio Test.

   d) Notice that $\frac{4+\sin(n^2)}{\sqrt{n}} \geq \frac{4+(-1)}{\sqrt{n}} = \frac{3}{\sqrt{n}} \geq 0$. Since $\sum \frac{3}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \leq 1$), the series $\sum a_n$ **diverges** by the Comparison Test.

4. a) We know $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$; by replacing x with $-x^2$ we see that $e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + ...$ Thus the third Taylor polynomial of e^{-x^2} is

$$P_3(x) = 1 - x^2$$

   and the integral is therefore

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 P_3(x)\,dx = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}.$$

   b) We know $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$; replacing x with x^4 gives $\arctan(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{2n+1}$ and then multiplying by x^2 gives

$$x^2 \arctan x^4 = \sum_{n=0}^{\infty} x^2 \frac{(-1)^n (x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+6}}{2n+1}.$$

   c) We know $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ so its fifth Taylor polynomial is $P_5(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$. Therefore $\cos\left(\frac{1}{3}\right) \approx P_5\left(\frac{1}{3}\right) = 1 - \frac{(1/3)^2}{2} + \frac{(1/3)^4}{4!}$.

5. If a series converges absolutely, it can be rearranged legally without changing its sum. If it only converges conditionally, then rearrangement and/or regrouping is illegal, because the sum of the series might change.

4.4 Spring 2016 Final Exam

1. (2.10) Evaluate each of the following six integrals:

a) $\displaystyle\int \left(\frac{1}{x} + 4\right) dx$

b) $\displaystyle\int \frac{1}{x+4}\, dx$

c) $\displaystyle\int \frac{1}{x-4}\, dx$

d) $\displaystyle\int \frac{1}{x^2-4}\, dx$

e) $\displaystyle\int \frac{1}{x^2+4}\, dx$

f) $\displaystyle\int \frac{x}{x^2+4}\, dx$

2. a) (2.6) Evaluate one of the following two integrals. If it is not clear from your work which integral you are trying, please circle the integral you want graded (I will only grade work from one of the two integrals):

$$\int 8x \sin 2x \, dx \qquad\qquad \int \arctan x \, dx$$

b) (2.5) Evaluate one of the following two integrals. If it is not clear from your work which integral you are trying, please circle the integral you want graded (I will only grade work from one of the two integrals):

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx \qquad\qquad \int_1^3 \frac{x-2}{\sqrt{x+1}}\, dx$$

c) (4.3) Determine, with justification, whether or not one of the following two improper integral converges or diverges.

$$\int_1^\infty x e^{-x}\, dx \qquad\qquad \int_3^\infty \frac{x + \sin x}{x^3}\, dx$$

3. Let R be the region in the $xy-$plane which lies above the x-axis, to the right of the curve $y = x^3$, and to the left of the vertical line $x = 3$.

a) (4.1) Write an expression involving one or more integrals (with respect to the variable x) which gives the area of R.

b) (4.1) Write an expression involving one or more integrals (with respect to the variable y) which gives the area of R.

c) (4.4) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the perimeter of R.

d) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the $x-$axis.

e) (4.2) Write an expression involving one or more integrals (with respect to whatever variable you like) which gives the volume of the solid generated when R is revolved around the line $x = -1$.

4. (7.4) Choose three of the following four series, and for the series you choose, determine with justification whether each of the following series converge absolutely, converge conditionally, or diverge.

   a) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{2016^n}$

   b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{17^n}{3^{3n}} + \frac{1}{2n\sqrt{n}}\right)$

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3}{10n^2+4n+1}$

   d) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^5+7n^2}$

5. a) (8.2) Compute $\displaystyle\lim_{x\to 0} \frac{e^{x^5} - 1 - x^5}{x^{10}}$ without using L'Hôpital's Rule.

   b) (8.2) Estimate $\displaystyle\int_0^{1/2} \ln(1+x^2)\,dx$ by replacing the integrand with its fourth Taylor polynomial.

   c) (8.2) Estimate $\arctan \dfrac{1}{4}$ using the fourth Taylor polynomial of an appropriately chosen function.

   d) (8.1) Let $f(x) = x\cos(2x^2)$. Find the Taylor series of f.

   e) (8.2) Use your answer to part (d) to find $f^{(101)}(0)$, where f is as in part (d).

6. Compute the sum of each series:

   a) (6.2) $\displaystyle\sum_{n=1}^{\infty} 3\left(\frac{2}{7}\right)^n$

   b) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{2\cdot 3^{n-3}}{11\cdot 2^{2n}5^{n+1}}$

   c) (8.2) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n!}$

Solutions

1. a) $\displaystyle\int \left(\frac{1}{x}+4\right)\,dx = \ln x + 4x + C.$

   b) $\displaystyle\int \frac{1}{x+4}\,dx = \ln(x+4) + C.$

   c) $\displaystyle\int \frac{1}{x-4}\,dx = \ln(x-4) + C.$

   d) First use partial fractions on the integrand. The integral works out to be

$$\int \frac{1}{x^2-4}\,dx = \int \left(\frac{1/4}{x-2} + \frac{-1/4}{x+2}\right)\,dx = \frac{1}{4}\ln(x-2) - \frac{1}{4}\ln(x+2) + C.$$

   e) "Just do it": $\displaystyle\int \frac{1}{x^2+4}\,dx = \frac{1}{2}\arctan\frac{x}{2} + C.$

   f) Use the u-substitution $u = x^2+4$. Then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$ so $\frac{1}{2}du = x\,dx$ so after substitution, you get

$$\int \frac{x}{x^2+4}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(x^2+4) + C.$$

2. a) For the first integral, use parts with $u = x$; $dv = 8\sin 2x\,dx$. Then $du = 1\,dx = dx$ and $v = \int dv = -4\cos 2x$. Thus the integral becomes

$$\begin{aligned}\int 8x\sin 2x\,dx = \int u\,dv = uv - \int v\,du &= -4x\cos 2x - \int -4\cos 2x\,dx\\ &= -4x\cos 2x + 2\sin 2x + C.\end{aligned}$$

   For the second integral, use parts with $u = \arctan x$ and $dv = dx$. Then $du = \frac{1}{1+x^2}$ and $v = x$ so by the parts formula, we have

$$\int \arctan x\,dx = \int u\,dv = uv - \int v\,du = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

   To do the remaining integral, use a u-substitution $u = 1+x^2$, so that $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$. Then we have

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln u + C = \frac{1}{2}\ln(1+x^2) + C$$

   so the entire integral is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

b) For the first integral, use the u-substitution $u = \sqrt{x}$. Then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ so $2du = \frac{1}{\sqrt{x}}\,dx$. When $x = 1$, $u = \sqrt{1} = 1$ and when $x = 4$, $u = \sqrt{4} = 2$ so the integral becomes

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 2e^u\,du = 2e^u\big|_1^2 = 2e^2 - 2e^1 = 2e^2 - 2e.$$

For the second integral, use the u-substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$ so $x - 2 = u - 3$. When $x = 1$, $u = 1 + 1 = 2$ and when $x = 3$, $u = 3 + 1 = 4$ so the integral becomes

$$\begin{aligned}\int_1^3 \frac{x-2}{\sqrt{x+1}}\,dx = \int_2^4 \frac{u-3}{\sqrt{u}}\,du &= \int_2^4 \left(u^{1/2} - 3u^{-1/2}\right)du \\ &= \left[\frac{2}{3}u^{3/2} - 6u^{1/2}\right]_2^4 \\ &= \frac{2}{3}(8) - 6(2) - \frac{2}{3}(2^{3/2}) + 6\sqrt{2} \\ &= \frac{-20}{3} + \frac{14}{3}\sqrt{2}.\end{aligned}$$

c) For the first integral, use parts with $u = x$, $dv = e^{-x}\,dx$. Then $du = dx$ and $v = -e^{-x}\,dx$ so

$$\begin{aligned}\int_1^\infty xe^{-x}\,dx &= \lim_{b\to\infty}\int_1^b xe^{-x}\,dx \\ &= \lim_{b\to\infty}\left(-xe^{-x}\big|_1^b - \int_1^b -e^{-x}\,dx\right) \\ &= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \\ &= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + 1e^{-1} - e^{-1}\right] \\ &= 2e^{-1} + \lim_{b\to\infty}\frac{-b-1}{e^b} = 2e^{-1} + \frac{\infty}{\infty} \\ &\overset{L}{=} 2e^{-1} + \lim_{b\to\infty}\frac{-1}{e^b} \\ &= 2e^{-1} + 0 = 2e^{-1}.\end{aligned}$$

Therefore the integral **converges** to $2e^{-1}$.

For the second integral, note that $\sin x \le 1$ so $\frac{x+\sin x}{x^3} \le \frac{x+1}{x^3} = \frac{1}{x^2} + \frac{1}{x^3}$. We know $\int_3^\infty \left(\frac{1}{x^2} + \frac{1}{x^3}\right)dx$ converges (it is the sum of two convergent p-integrals), so by the Comparison Test, the original integral **converges** as well.

3. R is a triangular-shaped region with corner points $(0,0)$, $(3,0)$ and $(3,27)$. The curve defining the left-side of the triangle can be thought of as $y = x^3$ or $x = \sqrt[3]{y}$.

a) $A = \displaystyle\int_0^3 x^3\,dx$.

b) $A = \displaystyle\int_0^{27} (3 - \sqrt[3]{y})\;dy$.

c) The perimeter of the "triangle" is the length of the bottom line segment (which is 3) plus the length of the right-hand vertical line segment (which is 27) plus the length of the curve $y = x^3$ from $x = 0$ to $x = 3$. This length is $s = \int_0^3 \sqrt{1 + [f'(x)]^2}\,dx = \int_0^3 \sqrt{1 + (3x^2)^2}\,dx$. So the perimeter of R is

$$3 + 27 + \int_0^3 \sqrt{1 + (3x^2)^2}\,dx = 30 + \int_0^3 \sqrt{1 + 9x^4}\,dx.$$

d) If integrating with respect to x, use the disc method (cross-sections are circles with radius x^3). So the volume is $V = \int_0^3 \pi (x^3)^2\,dx$.

If integrating with respect to y, use the shell method ($r = y - 0 = y$ and $h = 3 - \sqrt[3]{y}$) to obtain $V = \int_0^{27} 2\pi y(3 - \sqrt[3]{y})\,dy$.

e) If integrating with respect to x, use the shell method (cross-sections are cylinders with $r = x - (-1) = x + 1$ and $h = x^3$) to get $V = \int_0^3 2\pi(x + 1)x^3\,dx$.

If integrating with respect to y, use the washer method (with $R = 3 - (-1) = 4$ and $r = \sqrt[3]{y} - (-1) = \sqrt[3]{y} + 1$) to obtain

$$V = \int_0^{27} \left[\pi(4)^2 - \pi(\sqrt[3]{y} + 1)^2\right]\,dy.$$

4. a) Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{((n+1)!)^2}{2016^{n+1}}\right|}{\left|\frac{(n!)^2}{2016^n}\right|} = \lim_{n\to\infty} \frac{(n+1)!^2}{2016^{n+1}} \cdot \frac{2016^n}{(n!)^2}$$
$$= \lim_{n\to\infty} \frac{(n+1)^2}{2016} = \infty.$$

Since $\rho > 1$, the series **diverges** by the Ratio Test.

b) $\displaystyle\sum \frac{17^n}{3^{3n}} = \sum \left(\frac{17}{27}\right)^n$ is a convergent geometric series; $\displaystyle\sum \frac{1}{2n\sqrt{n}} = \sum \frac{1/2}{n^{3/2}}$ is a convergent p-series. Therefore the entire series converges (since converges + converges = converges), and **converges absolutely** since it is a positive series.

c) $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n^2+3}{10n^2+4n+1} = \frac{2}{10} \neq 0$ so the series **diverges** by the N^{th}-term test.

d) Consider $\sum |a_n| = \sum \frac{1}{n^5+7n^2}$. Since $\frac{1}{n^5+7n^2} \leq \frac{1}{n^5}$ and $\sum \frac{1}{n^5}$ is a convergent p-series, we know $\sum |a_n|$ converges by the Comparison Test. Therefore $\sum a_n$ **converges absolutely** by definition.

5. a) Write a power series for the numerator:

$$\begin{aligned}\lim_{x\to 0}\frac{e^{x^5}-1-x^5}{x^{10}} &= \lim_{x\to 0}\frac{\left[1+x^5+\frac{(x^5)^2}{2!}+\frac{(x^5)^3}{3!}+...\right]-1-x^5}{x^{10}}\\ &= \lim_{x\to 0}\frac{\frac{x^{10}}{2!}+\frac{x^{15}}{3!}+...}{x^{10}}\\ &= \lim_{x\to 0}\left[\frac{1}{2!}+\frac{x^5}{3!}+...\right]\\ &= \frac{1}{2}+0+0+...=\frac{1}{2}.\end{aligned}$$

b) We know $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...$ so by replacing x with x^2, we get $\ln(1+x^2)=x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+....$ Thus $P_4(x)=x^2-\frac{x^4}{2}$ so

$$\begin{aligned}\int_0^{1/2}\ln(1+x^2)\,dx \approx \int_0^{1/2}\left[x^2-\frac{x^4}{2}\right]dx &= \left[\frac{x^3}{3}-\frac{x^5}{10}\right]_0^{1/2}\\ &= \frac{(1/2)^3}{3}-\frac{(1/2)^5}{10}=\frac{37}{960}.\end{aligned}$$

c) $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$ so $P_4(x)=x-\frac{x^3}{3}$ so

$$\arctan\frac{1}{4}\approx P_4\left(\frac{1}{4}\right)=\frac{1}{4}-\frac{(1/4)^3}{3}=\frac{47}{192}.$$

d) $\cos x=\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$; replace x with $2x^2$ and then multiply in front by x to get

$$f(x)=x\cos(2x^2)=\sum_{n=0}^{\infty}\frac{x(-1)^n(2x^2)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^n 4^n x^{4n+1}}{(2n)!}.$$

e) The coefficient on the x^{101} term in the answer to part (d) occurs when $4n+1=101$, i.e. when $n=25$. This coefficient is

$$\frac{(-1)^{25}4^{25}}{(2\cdot 25)!}=\frac{-4^{25}}{50!}.$$

But by uniqueness of power series, this coefficient must be equal to $\frac{f^{(101)}(0)}{101!}$, so $\frac{f^{(101)}(0)}{101!}=\frac{-4^{25}}{50!}$ and by multiplying through by $101!$, we can conclude that $f^{(101)}(0)=101!\cdot\frac{-4^{25}}{50!}$.

6. a) $$\sum_{n=1}^{\infty}3\left(\frac{2}{7}\right)^n=\sum_{n=0}^{\infty}3\left(\frac{2}{7}\right)^{n+1}=3\cdot\frac{2}{7}\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n=3\cdot\frac{2}{7}\cdot\frac{1}{1-\frac{2}{7}}=\frac{6}{5}.$$

b) $\displaystyle\sum_{n=0}^{\infty} \frac{2 \cdot 3^{n-3}}{11 \cdot 2^{2n} 5^{n+1}} = \sum_{n=0}^{\infty} \frac{2 \cdot 3^n \cdot 3^{-3}}{11 \cdot 4^n 5^n 5} = \sum_{n=0}^{\infty} \frac{2}{3^3 \cdot 55} \left(\frac{3}{20}\right)^n = \frac{2}{27 \cdot 55} \cdot \frac{1}{1 - \frac{3}{20}} = \frac{8}{5049}$.

c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n!} = \sum_{n=0}^{\infty} \frac{(1/2)^n}{n!} = e^{1/2}$.

4.5 Spring 2023 Exam 1

1. In each part of this problem, write *Mathematica* code which would accomplish the given task:

   a) Define the function $f(x) = e^{3x}$.

   b) Sketch the graph of $g(x) = \tan 4x$, where x ranges from 0 to π.

   c) Solve the equation $x^4 - 3x^2 = 10$.

2. Compute each integral:

   a) (2.1) $\displaystyle\int \left(\frac{1}{4\sqrt{x}} + \frac{2}{3x} \right) dx$

   b) (2.2) $\displaystyle\int_0^{\pi/3} 6 \cos \frac{x}{2}\, dx$

   c) (2.2) $\displaystyle\int \left(8e^{1-4x} \right) dx$

3. (2.10) For each of the given integrals, determine what the **best method** to compute the integral by hand.

   - If the best method is to just write the answer, write "JUST DO IT" and write the answer. (This includes use of the Linear Replacement Principle.)
   - If the best method is to rewrite the integrand, write "REWRITE" and describe how the integrand should be rewritten.
   - If the best method is to use a $u-$substitution, write "U-SUB" and write what the u would be in your $u-$sub.
   - If the best method is integration by parts, write "PARTS" and write what you would assign to be r and ds.
   - If the best method is partial fractions, write "PARTIAL FRACTIONS" and write your 'guessed' form of the decomposition.

a) $\displaystyle\int \frac{1}{x+4}\,dx$

b) $\displaystyle\int \frac{1}{(x+4)^2}\,dx$

c) $\displaystyle\int \frac{1}{x^2-4}\,dx$

d) $\displaystyle\int \frac{1}{x^2+4}\,dx$

e) $\displaystyle\int \frac{x}{x^2-4}\,dx$

f) $\displaystyle\int \frac{x^2-4}{x}\,dx$

4. Determine, with justification, whether each integral converges or diverges:

   a) (3.3) $\displaystyle\int_5^\infty \left(\frac{12}{x^3} - \frac{8}{\sqrt[3]{x}}\right) dx$

   b) (3.3) $\displaystyle\int_2^\infty \frac{x^3}{x^7+5x+1}\,dx$

   c) (3.1) $\displaystyle\int_1^\infty 2^{-x}\,dx$

5. a) (2.8) Compute the partial fraction decomposition of $\dfrac{7x+29}{x^2+8x+15}$.

   b) (2.4) Compute $\displaystyle\int x^4 \sec^2 x^5\,dx$.

6. (2.5) Evaluate one of the following two integrals:

$$\int \frac{x(x+5)}{x-8}\,dx \qquad\qquad \int \cos^3 x\,dx$$

7. (2.6) Evaluate one of the following two integrals:

$$\int x^4 \ln x\,dx \qquad\qquad \int 4xe^{x/2}\,dx$$

8. **(Bonus)** Compute $\displaystyle\int x^{1/3}\,\sin x^{2/3}\,dx$.

Solutions

1. a) `f[x_] = E^(3x)`

   b) `Plot[Tan[4x], {x,0,Pi}]`

   c) `Solve[x^4 - 3x^2 == 10, x]`

2. a) Split and then use the Power Rule on the first term: $\int \left(\frac{1}{4\sqrt{x}} + \frac{2}{3x} \right) dx =$ $\frac{1}{4} \int x^{-1/2}\, dx + \frac{2}{3} \int \frac{1}{x}\, dx = \frac{1}{4} \cdot \frac{1}{\frac{1}{2}} x^{1/2} + \frac{2}{3} \ln x + C = \boxed{\frac{1}{2}\sqrt{x} + \frac{2}{3} \ln x + C}$.

   b) Use the Linear Replacement Principle with $m = \frac{1}{2}$ to get $\int_0^{\pi/3} 6 \cos \frac{x}{2}\, dx =$ $6 \cdot \frac{1}{\frac{1}{2}} \sin \frac{x}{2} \Big|_0^{\pi/3} = 12 \sin \frac{x}{2} \Big|_0^{\pi/3} = 12 \sin \frac{\pi}{6} - 12 \sin 0 = 12 \left(\frac{1}{2} \right) - 0 = \boxed{6}$.

   c) Use the Linear Replacement Principle with $m = -4$ to get $\int \left(8e^{1-4x} \right) dx =$ $8 \cdot \frac{1}{-4} e^{1-4x} + C = \boxed{-2e^{1-4x} + C}$.

3. a) **JUST DO IT:** $\int \frac{1}{x-4}\, dx = \boxed{\ln(x-4) + C}$.

   b) **JUST DO IT** (with the Lin. Rep. Principle): $\int \frac{1}{(x-4)^2}\, dx = \boxed{-(x-4)^{-1} + C}$

   c) **PARTIAL FRACTIONS:** $\frac{1}{x^2 - 4} = \boxed{\frac{A}{x-2} + \frac{B}{x+2}}$.

   d) **JUST DO IT:** $\int \frac{1}{x^2+4}\, dx = \boxed{\frac{1}{2} \arctan \frac{x}{2} + C}$ (see Theorem 2.7 in the notes).

   e) u-**SUB:** $\boxed{u = x^2 - 4}$.

   f) **REWRITE:** $\int \frac{x^2 - 4}{x}\, dx = \int \left(\frac{x^2}{x} - \frac{4}{x} \right) dx = \boxed{\int \left(x - \frac{4}{x} \right) dx}$.

4. a) $\int_5^{\infty} \left(\frac{12}{x^3} - \frac{8}{\sqrt[3]{x}} \right) dx = 12 \int_5^{\infty} \frac{1}{x^3} - 8 \int_5^{\infty} \frac{1}{x^{1/3}}\, dx$. The first integral converges (p-integral, $p = 3 > 1$) but the second does not (p-integral, $p = \frac{1}{3} \leq 1$), so this is a convergent integral minus a divergent one, which $\boxed{\textbf{diverges}}$.

   b) Note $0 \leq \frac{x^3}{x^7 + 5x + 1} \leq \frac{x^3}{x^7} = \frac{1}{x^4}$. Also, $\int_2^{\infty} \frac{1}{x^4}\, dx$ converges (p-integral, $p = 4 > 1$), so by the Comparison Test $\int_2^{\infty} \frac{x^3}{x^7 + 5x + 1}\, dx$ $\boxed{\textbf{converges}}$.

c) $\displaystyle\int_1^\infty 2^{-x}\,dx = -\frac{1}{\ln 2}2^{-x}\Big|_1^\infty = -\frac{1}{\ln 2}2^{-\infty} + \frac{1}{\ln 2}2^{-1} = 0 + \frac{1}{2\ln 2} = \frac{1}{2\ln 2} < \infty$, so this integral $\boxed{\textbf{converges}}$.

5. a) Factor the denominator as $(x+3)(x+5)$; then guess the decomposition as $\dfrac{7x+29}{x^2+8x+15} = \dfrac{A}{x+3} + \dfrac{B}{x+5}$. Multiply through by $x^2+8x+15$ to clear denominators; this gives the equation $7x+29 = A(x+5)+B(x+3)$. Now,

- by letting $x=-5$ in this equation, we get $-6 = B(-2)$ so $B=3$.
- by letting $x=-3$ in this equation, we get $8 = A(2)$ so $A=4$.

All together, we have $\dfrac{7x+29}{x^2+8x+15} = \boxed{\dfrac{4}{x+3} + \dfrac{3}{x+5}}$.

b) Use the u-sub $u = x^5$. Then $du = 5x^4\,dx$ so $\frac{1}{5}du = x^4\,dx$. Substituting in the integral, we get $\displaystyle\int x^4 \sec^2 x^5\,dx = \int \frac{1}{5}\sec^2 u\,du = \frac{1}{5}\tan u + C = \boxed{\frac{1}{5}\tan x^5 + C}$.

6. a) For the first integral, let $u = x-8$ so that $du = dx$ and $x = u+8$. Substituting in, we get

$$\begin{aligned}
\int \frac{x(x+5)}{x-8}\,dx &= \int \frac{(u+8)(u+8+5)}{u}du \\
&= \int \frac{(u+8)(u+13)}{u}\,du \\
&= \int \frac{u^2+21u+104}{u}\,du \\
&= \int \left(u+21+\frac{104}{u}\right)\,du \\
&= \frac{1}{2}u^2 + 21u + 104\ln u + C \\
&= \boxed{\frac{1}{2}(x-8)^2 + 21(x-8) + 104\ln(x-8) + C}.
\end{aligned}$$

b) For the second integral, rewrite it as

$$\int \cos^3 x\,dx = \int \cos^2 x\,\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx$$

and then use the u-sub $u = \sin x$, $du = \cos x\,dx$ to rewrite it as

$$\int (1-u^2)\,du = u - \frac{1}{3}u^3 + C = \boxed{\sin x - \frac{1}{3}\sin^3 x + C}.$$

7. a) For the first integral, use parts by setting $r = \ln x$ and $ds = x^4\,dx$. Then $dr = \frac{1}{x}\,dx$ and $s = \frac{1}{5}x^5$. Therefore

$$\begin{aligned}\int x^4 \ln x\,dx = \int r\,ds = rs - \int s\,dr &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^5 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4\,dx \\ &= \boxed{\frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C}.\end{aligned}$$

b) For the second integral, use parts by setting $r = 4x$ and $ds = e^{x/2}\,dx$. Then $dr = 4\,dx$ and $s = 2e^{x/2}$. Therefore

$$\begin{aligned}\int 4xe^{x/2}\,dx = \int r\,ds = rs - \int s\,dr &= 4x\left(2e^{x/2}\right) - \int 2e^{x/2}(4)dx \\ &= 8xe^{x/2} - \int 8e^{x/2}\,dx \\ &= \boxed{8xe^{x/2} - 16e^{x/2} + C}.\end{aligned}$$

8. For the bonus question, start with the u-sub $u = x^{2/3}$. Therefore $du = \frac{2}{3}x^{-1/3}\,dx$, so $dx = \frac{3}{2}x^{1/3}\,du$. Since $u = x^{2/3}$, we also can compute $x^{1/3} = (x^{2/3})^{1/2} = u^{1/2}$. Substituting into our equation for dx, we get $dx = \frac{3}{2}u^{1/2}\,du$. Now, we can substitute into the integral to get

$$\int x^{1/3} \sin x^{2/3}\,dx = \int u^{1/2} \sin u \left(\frac{3}{2}u^{1/2}\,du\right) = \int \frac{3}{2}u \sin u\,du.$$

This integral can now be done with parts. Set $r = \frac{3}{2}u$ and $ds = \sin u\,du$, which implies $dr = \frac{3}{2}du$ and $s = -\cos u$. By the parts formula,

$$\begin{aligned}\int u \sin u\,du = \int r\,ds = rs - \int s\,dr &= -\frac{3}{2}u \cos u - \int -\frac{3}{2}\cos u\,du \\ &= -\frac{3}{2}u\cos u + \frac{3}{2}\sin u + C.\end{aligned}$$

Finally, back-substitute to get $\boxed{-\frac{3}{2}x^{2/3}\cos x^{2/3} + \frac{3}{2}\sin x^{2/3} + C}$.

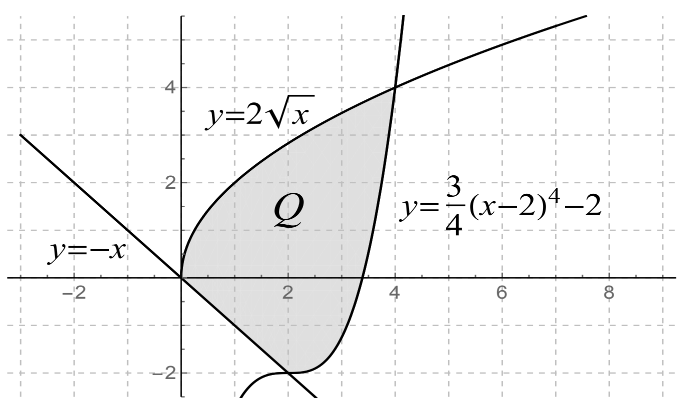
4.6 Spring 2023 Exam 2

1. (4.4) Write an integral which gives the length of the curve $y = 2e^{4x}$ between the points $(0, 2)$ and $(5, 2e^{20})$.

2. (4.3) In general, if a machine's efficiency is constant, then its output is given by multiplying the efficiency by the time the machine works. Now suppose that a certain machine's efficiency is *nonconstant* and that at time t, its efficiency is $E(t) = 4 + \cos t$. What is the machine's output between times 0 and $\frac{3\pi}{2}$?

3. (4.8) Suppose that the time, in days, until the next major solar flare is a continuous random variable with density function
$$f(x) = \begin{cases} \frac{1}{96}x^3e^{-x/2} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$
   a) Write an integral which gives the probability that the next major solar flare will occur at least 5 days from now.
   b) Compute the expected amount of time until the next major solar flare.

4. In this problem, Q refers to the region of points in the xy-plane pictured below:



   a) (4.1) Write an expression involving one or more integrals with respect to the variable x that gives the area of Q.
   b) (4.1) Write an expression involving one or more integrals with respect to the variable y that gives the area of Q.
   c) (4.2) Write an expression involving one or more integrals with respect to any variable you like that gives the volume of the solid whose base is Q, where cross-sections parallel to the y-axis are squares.

5. (4.7) Let R be the region of points in the xy-plane enclosed by the graphs of $y = 4x^2$ and $y = 4x$.

a) Compute the moment of inertia of R about the x-axis.

b) Compute the moment of inertia of R about the y-axis.

6. (4.6) Compute the centroid of the triangle whose vertices are $(0,0)$, $(2,0)$ and $(2,4)$.

7. (4.2) In this problem, let S be the region of points in the xy-plane above the graph of $y = \frac{1}{2}(x+1)$ and below the graph of $y = 4^x$, between $x = -1$ and $x = 3$.

   a) Write an expression involving one or more integrals with respect to any variable you like that would give the volume of the solid obtained when S is revolved around the line $x = -2$.

   b) Write an expression involving one or more integrals with respect to any variable you like that would give the volume of the solid obtained when S is revolved around the line $y = -4$.

Solutions

1. If $f(x) = 2e^{4x}$, then $f'(x) = 8e^{4x}$. Therefore the arc length is
$$s = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \boxed{\int_0^5 \sqrt{1+[8e^{4x}]^2}\,dx}.$$

2. By the general principle of applications of integration, the output is
$$\int_a^b E(t)\,dt = \int_0^{3\pi/2} (4+\cos t)\,dt = [4t+\sin t]_0^{3\pi/2} = \left[4\left(\frac{3\pi}{2}\right)-1\right] - [0+0] = \boxed{6\pi - 1}.$$

3. a) $P(X \geq 5) = \displaystyle\int_5^\infty f(x)\,dx = \boxed{\int_5^\infty \frac{1}{96}x^3e^{-x/2}\,dx}$.

   b) This integral uses the Gamma Integral Formula:
$$EX = \int_{-\infty}^\infty x f(x)\,dx = \int_0^\infty \frac{1}{96}x^4e^{-x/2}\,dx = \frac{1}{96}\cdot\frac{4!}{\left(\frac{1}{2}\right)^{4+1}} = \frac{2^5\cdot 4!}{96} = \frac{4!}{3} = \boxed{8 \text{ days}}.$$

4. a) The area is $\displaystyle\int_l^r [t(x)-b(x)]\,dx$. Since the bottom function changes at $x = 2$, you need two integrals:
$$\boxed{\int_0^2 \left[2\sqrt{x}-(-x)\right]\,dx + \int_2^4 \left[2\sqrt{x}-\left(\frac{3}{4}(x-2)^4-2\right)\right]\,dx}$$

   b) First, you need to solve all the equations for x in terms of y:
$$y = -x \Rightarrow x = -y$$
$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \rightarrow x = \frac{1}{4}y^2$$
$$y = \frac{3}{4}(x-2)^4 - 2 \Rightarrow y+2 = \frac{3}{4}(x-2)^4 \Rightarrow \frac{4}{3}(y+2) = (x-2)^4 \Rightarrow x = \sqrt[4]{\frac{4}{3}(y+2)}+2$$
   Then the area is $\displaystyle\int_b^t [r(y)-l(y)]\,dy$. Since the left function changes at $y = 0$, you need two integrals:
$$\boxed{\int_{-2}^0 \left[\left(\sqrt[4]{\frac{4}{3}(y+2)}+2\right)-(-y)\right]\,dy + \int_0^4 \left[\left(\sqrt[4]{\frac{4}{3}(y+2)}+2\right)-\left(\frac{1}{4}y^2\right)\right]\,dy}$$

   c) Since the cross-sections are parallel to the y-axis, you have to integrate with respect to x. The area of the cross-section at x is $A(x) = (\text{side length})^2 = (t(x)-b(x))^2$; as in part (a), you need two integrals:
$$V = \boxed{\int_0^2 \left[2\sqrt{x}-(-x)\right]^2\,dx + \int_2^4 \left[2\sqrt{x}-\left(\frac{3}{4}(x-2)^4-2\right)\right]^2\,dx}$$

5. First, set the equations equal to one another to find intersection points: $4x^2 = 4x$ gives $x = 0$ and $x = 1$, and the corresponding y-values are $y = 0$ and $y = 4$.

   a) In this problem, we need to solve the equations for x: $y = 4x^2$ gives $x = \frac{1}{2}\sqrt{y}$ and $y = 4x$ gives $x = \frac{1}{4}y$. Now, by the formula for moment of inertia about the x-axis,

$$\begin{aligned} I_x = \int_b^t y^2\left[r(y) - l(y)\right]\,dy &= \int_0^4 y^2\left[\frac{1}{2}\sqrt{y} - \frac{1}{4}y\right]\,dy \\ &= \int_0^4 \left(\frac{1}{2}y^{5/2} - \frac{1}{4}y^3\right)\,dy \\ &= \left[\frac{1}{7}y^{7/2} - \frac{1}{16}y^4\right]_0^4 = \frac{4^{7/2}}{7} - \frac{1}{16}(4^4) = \boxed{\frac{16}{7}}. \end{aligned}$$

   b) By the formula for moment of inertia about the y-axis,

$$\begin{aligned} I_y = \int_l^r x^2\left[t(x) - b(x)\right]\,dx = \int_0^1 x^2(4x - 4x^2)\,dx &= \int_0^1 (4x^3 - 4x^4)\,dx \\ &= \left[x^4 - \frac{4}{5}x^5\right]_0^1 \\ &= \left[1 - \frac{4}{5}\right] - [0] = \boxed{\frac{1}{5}}. \end{aligned}$$

6. This region goes from $x = 0$ to $x = 2$ and has top function $t(x) = 2x$ (this is the line passing through $(0,0)$ and $(2,4)$) and bottom function $b(x) = 0$. Since we are computing a centroid, the density can be assumed to be $\rho(x) = 1$. Therefore, by the formulas for centers of mass:

$$\begin{aligned} M &= \int_a^b \left[t(x) - b(x)\right]\rho(x)\,dx = \int_0^2 2x\,dx = x^2\Big|_0^2 = 4; \\ M_y &= \int_a^b x\left[t(x) - b(x)\right]\rho(x)\,dx = \int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3}; \\ M_x &= \int_a^b \left[\frac{1}{2}(t(x))^2 - \frac{1}{2}(b(x))^2\right]\,dx = \int_0^2 \frac{1}{2}(2x)^2\,dx = \int_0^2 2x^2\,dx = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3}. \end{aligned}$$

   Finally, the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{\frac{16}{3}}{4}, \frac{\frac{16}{3}}{4}\right) = \boxed{\left(\frac{4}{3}, \frac{4}{3}\right)}$.

7. a) If you use x as your variable of the integral, this is shells with $r = x - (-2) = x + 2$ and $h = t(x) - b(x) = 4^x - \frac{1}{2}(x+1)$. This gives

$$V = \int_a^b 2\pi r h\,dx = \boxed{\int_{-1}^3 2\pi(x+2)\left[4^x - \frac{1}{2}(x+1)\right]\,dx}.$$

b) If you use x as your variable of the integral, this is washers with $R = t(x) - (-4) = 4^x + 4$ and $r = b(x) - (-4) = \frac{1}{2}(x+1) + 4$. This gives

$$V = \int_a^b \left[\pi R^2 - \pi r^2\right]\, dx = \boxed{\int_{-1}^{3} \left[\pi\left(4^x + 4\right)^2 - \pi\left(\frac{1}{2}(x+1) + 4\right)^2\right]\, dx}$$

Problems 7 (a) and (b) can also be done with respect to y, but they are much harder that way (you need three integrals added together).

4.7 Spring 2023 Exam 3

1. In each part of this problem you are given an infinite series, or some information about an infinite series.

   - If you can conclude the series converges or diverges based on the given information, write "converges" or "diverges" (no further justification is necessary).
   - If the given information is insufficient to conclude whether the series converges or diverges, write "not enough info".

   a) (6.2) The series is $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n}$.

   b) (6.2) The series is $\displaystyle\sum_{n=K}^{\infty} \frac{3}{4^n}$, but you don't know what K is.

   c) (5.7) The series is $\displaystyle\sum_{n=1}^{\infty} \frac{8}{\sqrt{5n}}$.

   d) (6.2) The terms of the series are less than the terms of a geometric series with $r = 2$.

   e) (5.7) The terms of the series are less than the terms of a p-series with $p = 2$.

   f) (5.4) The series is the sum of a convergent series and a divergent series.

   g) (7.3) The series is $\sum a_n$, and you know $\sum |a_n|$ converges.

   h) (7.1) The series is $\sum a_n$, and $\displaystyle\lim_{n\to\infty} |a_n| = 0$.

2. Throughout this problem, you may assume $\displaystyle\sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n = 30$.

   a) (5.5) Compute and simplify $\displaystyle\sum_{n=2}^{\infty} n^2 \left(\frac{2}{3}\right)^n$.

   b) (5.5) Compute and simplify $\displaystyle\sum_{n=0}^{\infty} n^2 \frac{2^n}{3^{n-3}}$.

   c) (7.3) If you rearranged and/or regrouped the terms of this series, must the series you end up with sum to 30? Why or why not?

3. (7.4) Choose three of the following four series. For each series you choose, determine (with appropriate justification) whether the series converges absolutely, converges conditionally or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{7^n + n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^{-n}}$

c) $\displaystyle\sum_{n=2}^{\infty} \frac{3}{\sqrt[3]{n-1}}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,2023^n}$

4. a) (8.2) Estimate $\cos\frac{1}{8}$ by using the second-order Taylor polynomial for an appropriately chosen function.

   b) (8.2) Consider a function f which is a solution of the differential equation $f''(x) - x\,f(x) = 0$. If $f(0) = 1$ and $f'(0) = 2$, estimate $f(1)$ by computing the fourth Taylor polynomial of f.

5. a) (8.2) Estimate $\displaystyle\int_0^1 \ln(1 + x^4)\,dx$ by replacing the integrand with a Taylor polynomial that has two nonzero terms.

   b) (8.2) Compute the following limit without using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\arctan(x^8) - \sin(x^8)}{x^{24}}$$

6. Compute the sum of each given series; except in part (d), completely simplify each answer.

   a) (6.2) $\displaystyle\sum_{n=3}^{\infty} 5\left(\frac{2}{7}\right)^n$

   b) (8.2) $1 + \frac{2}{1} + \frac{4}{2} + \frac{8}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \ldots$

   c) (6.2) $\displaystyle\sum_{n=0}^{\infty} \frac{6 \cdot 4^{2n+1}}{3^{3n+2}}$

   d) (6.2) $\displaystyle\sum_{n=0}^{38} 5^{-n}$

   e) **(Bonus)** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$

Solutions

1. a) This a geometric series with $r = \dfrac{1}{4}$. Since $|r| < 1$, this series $\boxed{\textbf{converges}}$.

   b) Since the starting index is irrelevant to convergence or divergence, this series (which is the same as the one in (a) with a different starting index) also $\boxed{\textbf{converges}}$.

   c) This is a p-series with $p = \dfrac{1}{2} \leq 1$, so it $\boxed{\textbf{diverges}}$.

   d) Being less than a divergent series doesn't tell you anything: $\boxed{\textbf{not enough info}}$.

   e) Being less than a convergent series tells you the series $\boxed{\textbf{converges}}$ by the Comparison Test.

   f) convergent + divergent gives a series that $\boxed{\textbf{diverges}}$.

   g) By the Triangle Inequality, $\sum a_n$ $\boxed{\textbf{converges}}$ (absolutely, in fact).

   h) This is $\boxed{\textbf{not enough info}}$ (the series could be a p-series with any value of p, for example).

2. a) Subtract the $n = 0$ and $n = 1$ terms:

$$\sum_{n=2}^{\infty} n^2 \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} n^2 \left(\frac{2}{3}\right)^n - 0^2 \left(\frac{2}{3}\right)^0 - 1^2 \left(\frac{2}{3}\right) 1 = 30 - 0 - \frac{2}{3} = \boxed{\frac{88}{3}}.$$

   b) Use exponent rules and pull out the constants:

$$\sum_{n=0}^{\infty} n^2 \frac{2^n}{3^{n-3}} = \sum_{n=0}^{\infty} n^2 \frac{2^n}{3^n 3^{-3}} = \frac{1}{3^{-3}} \sum_{n=0}^{\infty} n^2 \frac{2^n}{3^n} = 27\,(30) = \boxed{810}.$$

   c) This series is positive, so since it converges, it converges absolutely. Therefore the series $\boxed{\textbf{can}}$ be rearranged without affecting its sum.

3. a) Notice $0 \leq \frac{5^n}{7^n+n} \leq \frac{5^n}{7^n} = \left(\frac{5}{7}\right)^n$. The series $\sum \left(\frac{5}{7}\right)^n$ converges (it is geometric with $|r| = \frac{5}{7} < 1$), so $\sum \frac{5^n}{7^n+n}$ $\boxed{\textbf{converges absolutely}}$ by the Comparison Test.

   b) $\lim\limits_{n\to\infty} \left|\frac{1}{10^{-n}}\right| = \lim\limits_{n\to\infty} \frac{1}{10^{-\infty}} = \frac{1}{0} = \infty \neq 0$, so $\sum \frac{1}{10^{-n}}$ $\boxed{\textbf{diverges}}$ by the N^{th}-term Test.

   c) Notice $0 \leq \frac{3}{\sqrt[3]{n}} \leq \frac{3}{\sqrt[3]{n-1}}$. The series $\sum \frac{3}{\sqrt[3]{n}} = \sum \frac{3}{n^{1/3}}$ diverges (p-series with $p = \frac{1}{3} \leq 1$), so by the Comparison Test $\sum \frac{3}{\sqrt[3]{n-1}}$ $\boxed{\textbf{diverges}}$.

d) Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(2(n+1))!}{(n+1)!\,2023^{n+1}}\right|}{\left|\frac{(2n)!}{n!\,2023^{n}}\right|}$$
$$= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!\,2023^{n+1}} \cdot \frac{n!\,2023^n}{(2n)!}$$
$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)\,2023}$$
$$= \lim_{n\to\infty} \frac{2(n+1)(2n+1)}{(n+1)\,2023}$$
$$= \lim_{n\to\infty} \frac{2(2n+1)}{2023} = \infty.$$

Since $\rho > 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{n!\,2023^n}$ $\boxed{\textbf{diverges}}$ by the Ratio Test.

4. a) $\cos x = 1 - \frac{x^2}{2!} + ...$, so

$$\cos\frac{1}{8} \approx P_2\left(\frac{1}{8}\right) = 1 - \frac{\left(\frac{1}{8}\right)^2}{2!} = 1 - \frac{\frac{1}{64}}{2} = 1 - \frac{1}{128} = \boxed{\frac{127}{128}}.$$

b) Let the function be represented by the Taylor series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$

Since $f(0) = 1$, $a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$ and since $f'(0) = 2$, $a_1 = \frac{f'(0)}{1!} = \frac{2}{1} = 2$. Therefore

$$f(x) = 1 + 2x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$

Now we consider the equation $f''(x) - x\,f(x) = 0$. Take two derivatives of this Taylor series to get

$$f''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + ...$$

and multiply $f(x)$ by x to get

$$x\,f(x) = x + 2x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + ...$$

Since $f''(x) - x\,f(x) = 0$, we can subtract the previous two power series together term-by-term to get

$$2a_2 + (6a_3 - 1)x + (12a_4 - 2)x^2 + ... = 0,$$

which means

$$\begin{cases} 2a_2 = 0 & \Rightarrow a_2 = 0 \\ 6a_3 - 1 = 0 & \Rightarrow a_3 = \frac{1}{6} \\ 12a_4 - 2 = 0 & \Rightarrow a_4 = \frac{1}{6} \end{cases}$$

Finally, the fourth Taylor polynomial of f is

$$\begin{aligned} P_4(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\ &= 1 + 2x + \frac{1}{6}x^3 + \frac{1}{6}x^4, \end{aligned}$$

so $f(1) \approx P_4(1) = 1 + 2(1) + \frac{1}{6}(1^3) + \frac{1}{6}(1^4) = \boxed{\frac{10}{3}}$.

5. a) Start with the Taylor series $\ln(1+x) = x - \frac{x^2}{2} + ...$ Next, replace each x with x^4 to get $\ln(1+x^4) = x^4 - \frac{(x^4)^2}{2} + ... = x^4 - \frac{1}{2}x^8 + ...$ Therefore

$$\int_0^1 \ln(1+x^4)\,dx \approx \int_0^1 \left[x^4 - \frac{1}{2}x^8\right]\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18}\right]_0^1 = \frac{1}{5} - \frac{1}{18} = \boxed{\frac{13}{90}}.$$

b) Start with $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ...$; replace x with x^8 to get $\arctan x^8 = x^8 - \frac{x^{24}}{3} + \frac{x^{40}}{5} - ....$ Next, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$; replace x with x^8 to get $\sin x^8 = x^8 - \frac{x^{24}}{3!} + \frac{x^{40}}{5!} - ....$ Now, plug both these series into the limit:

$$\begin{aligned} \lim_{x\to 0} \frac{\arctan(x^8) - \sin(x^8)}{x^{24}} &= \lim_{x\to 0} \frac{x^8 - \frac{x^{24}}{3} + \frac{x^{40}}{5} - ... - \left[x^8 - \frac{x^{24}}{3!} + \frac{x^{40}}{5!} - ...\right]}{x^{24}} \\ &= \lim_{x\to 0} \frac{\left(-\frac{1}{3} + \frac{1}{3!}\right)x^{24} + \text{ higher power terms}}{x^{24}} \\ &= \lim_{x\to 0} \left(-\frac{1}{3} + \frac{1}{6}\right) + \text{ positive powers of } x \\ &= -\frac{1}{6} + 0 = \boxed{-\frac{1}{6}}. \end{aligned}$$

6. a) Change indices so that the series starts at 0, and then use the sum formula for a geometric series:

$$\begin{aligned} \sum_{n=3}^{\infty} 5\left(\frac{2}{7}\right)^n = \sum_{n=0}^{\infty} 5\left(\frac{2}{7}\right)^{n+3} &= 5\left(\frac{2}{7}\right)^3 \sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n \\ &= 5\left(\frac{2}{7}\right)^3 \frac{1}{1-\frac{2}{7}} \\ &= 5 \cdot \frac{2^3}{7^3} \cdot \frac{7}{5} = \boxed{\frac{8}{49}}. \end{aligned}$$

b) This is the Taylor series for e^x, with $x = 2$:

$$1 + \frac{2}{1} + \frac{4}{2} + \frac{8}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + ... = \sum_{n=0}^{\infty} \frac{2^n}{n!} = \boxed{e^2}.$$

c) This series is geometric:

$$\sum_{n=0}^{\infty} \frac{6 \cdot 4^{2n+1}}{3^{3n+2}} = \sum_{n=0}^{\infty} \frac{6 \cdot 16^n \cdot 4}{27^n \cdot 9} = \frac{24}{9} \sum_{n=0}^{\infty} \left(\frac{16}{27}\right)^n$$
$$= \frac{24}{9} \left(\frac{1}{1 - \frac{16}{27}}\right) = \frac{8}{3} \cdot \frac{27}{11} = \boxed{\frac{72}{11}}.$$

d) Use the finite geometric sum formula:

$$\sum_{n=0}^{38} 5^{-n} = \sum_{n=0}^{38} \left(\frac{1}{5}\right)^n = \frac{1 - \left(\frac{1}{5}\right)^{38+1}}{1 - \frac{1}{5}} = \boxed{\frac{5}{4}\left[1 - \left(\frac{1}{5}\right)^{39}\right]}.$$

e) We know the Taylor series $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$. Differentiate both sides of this formula to get $\frac{1}{(1-x)^2} = \sum\limits_{n=1}^{\infty} nx^{n-1}$, which is exactly the series we are given in this problem with $x = \frac{3}{4}$. Therefore

$$\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^{n-1} = \frac{1}{\left(1 - \frac{3}{4}\right)^2} = \frac{1}{\frac{1}{16}} = \boxed{16}.$$

4.8 Spring 2023 Final Exam

1. Compute each integral:

   a) (2.2) $\displaystyle\int_{-1}^{0} (x+3)^8\,dx$

   b) (2.2) $\displaystyle\int e^{1-4x}\,dx$

   c) (2.1) $\displaystyle\int \left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)\,dx$

   d) (3.1) $\displaystyle\int_{1}^{\infty} \frac{4}{x^9}\,dx$

   e) (3.4) $\displaystyle\int_{0}^{\infty} x^5 e^{-3x}\,dx$

2. Evaluate four of the following five integrals:

   a) (2.1) $\displaystyle\int \cos x\,dx$

   b) (2.2) $\displaystyle\int \cos 2x\,dx$

   c) (2.1) $\displaystyle\int 2\cos x\,dx$

   d) (2.3) $\displaystyle\int \cos^2 x\,dx$

   e) (2.9) $\displaystyle\int \cos x^2\,dx$

3. (2.6) Evaluate one of the following two integrals:

$$\int 4x\,e^{-2x}\,dx \qquad\qquad \int \ln^2 x\,dx$$

4. (2.5) Evaluate one of the following two integrals:

$$\int_0^{\pi/4} 24\tan^2 x \sec^2 x\,dx \qquad\qquad \int_1^2 \frac{3x+2}{2x-1}\,dx$$

5. (2.8) Evaluate one of the following two integrals:

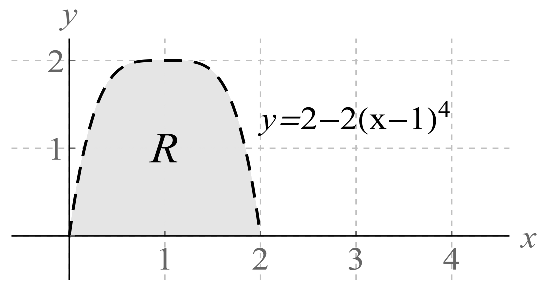
$$\int \frac{12}{x^2-4x-5}\,dx \qquad\qquad \int \frac{3x^2+1}{x^4-x^3}\,dx$$

6. (4.1) Let E be the region of points in the (x, y)-plane located above the graph of $y = \dfrac{x^2}{3}$ and below the graph of $y = 2x$.

   a) Write an integral with respect to the variable x that gives the area of E.

   b) Write an integral with respect to the variable y that gives the area of E.

7. Let R be the region pictured below:



For each given quantity, write a formula involving one or more integrals (with respect to whatever variable you want) that, if computed, would produce the quantity.

a) (4.2) The volume of the solid formed when R is revolved around the x-axis.

b) (4.2) The volume of the solid formed when R is revolved around the y-axis.

c) (4.7) The moment of inertia of R about the y-axis.

d) (4.4) The length of the curve in the picture indicated with the thick dashed markings.

e) (4.6) The y-coordinate of the center of mass of planar slab that has shape R, where the density of the lamina at point (x, y) is $\rho(x) = (x^2 + 2)$.

8. (4.8) Suppose X is a continuous random variable with density function

$$f(x) = \begin{cases} c\,x & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases},$$

where c is an unknown constant.

a) Determine the value of c.

b) Compute the probability that X is between 0 and 2.

9. (7.4) Choose three of the following four series and determine, with appropriate justification, whether those series converge absolutely, converge conditionally, or diverge. (If it isn't clear which problem you don't want graded, draw an X through that work. I will only grade three of these four problems.)

a) $\sum \frac{(-1)^n}{n^{3/4}}$

b) $\sum \left(\frac{1}{3^n} + \frac{(-1)^n}{4^n} \right)$

c) $\displaystyle\sum_{k=0}^{\infty} \frac{4}{k^2+3}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 7^n}{(2n)!}$

10. (6.2, 8.2) Compute the sum of each series (you may assume without proof that all these series converge):

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,4^n}$

b) $1 - \dfrac{1}{2} + \dfrac{1}{4!} - \dfrac{1}{6!} + \dfrac{1}{8!} - \dfrac{1}{10!} + \ldots$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{n+2}}{5^{2n+1}}$

d) $\displaystyle\sum_{n=-2}^{\infty} 4\left(\frac{-3}{5}\right)^n$

11. a) (8.2) Compute $\displaystyle\lim_{x\to 0} \frac{e^{2x^7} - 2x^7 - 1}{x^{10}}$ without using L'Hôpital's rule.

b) (8.2) Estimate $\displaystyle\int_0^2 x^2 e^{-x^2}\,dx$ by replacing the integrand with its sixth Taylor polynomial.

c) (8.2) Suppose g is an unknown function with $g(0) = 2$, $g'(0) = -1$, $g''(0) = 0$, $g'''(0) = 3$ and $g^{(4)}(0) = 6$. Compute the fourth Taylor polynomial of g, and use this polynomial to estimate $g(2)$.

d) (6.3) A ball is dropped from a height of 24 ft. Every time the ball bounces, it rebounds to a height that is $\dfrac{2}{3}$ of the height it reached on the previous bounce. Compute the total amount of vertical distance travelled by the ball before it comes to rest.

Solutions

1. a) Use the Linear Replacement Principle:

$$\int_{-1}^{0} (x+3)^8\,dx = \left[\frac{1}{9}(x+3)^9\right]_{-1}^{0} = \boxed{\frac{1}{9}\left(3^9 - 2^9\right)}.$$

b) Use the Linear Replacement Principle: $\int e^{1-4x}\,dx = \boxed{-\frac{1}{4}e^{1-4x} + C}$.

c) Integrate term-by-term with the Power Rule:

$$\int \left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)\,dx = \int \left(x^{1/2} + 2x^{-1/2}\right)\,dx = \boxed{\frac{2}{3}x^{3/2} + 4x^{1/2} + C}.$$

d) $\displaystyle\int_1^\infty \frac{4}{x^9}\,dx = \int_1^\infty 4x^{-9}\,dx = \frac{-1}{2}x^{-8}\Big|_1^\infty = \frac{-1}{2}(0) - \frac{-1}{2}(1) = \boxed{\frac{1}{2}}$.

e) Use the Gamma Integral Formula: $\displaystyle\int_0^\infty x^5 e^{-3x}\,dx = \boxed{\frac{5!}{3^6}} = \frac{40}{243}$.

2. a) $\displaystyle\int \cos x\,dx = \boxed{\sin x + C}$.

b) Use the Linear Replacement Principle: $\displaystyle\int \cos 2x\,dx = \boxed{\frac{1}{2}\sin 2x + C}$.

c) $\displaystyle\int 2\cos x\,dx = \boxed{2\sin x + C}$.

d) Use a power-reducing identity, then the Linear Replacement Principle:

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\,dx = \boxed{\frac{x}{2} + \frac{1}{4}\sin 2x + C}.$$

e) $\displaystyle\int \cos x^2\,dx$ is not doable with our methods (you should not have chosen to try this problem).

3. a) Use parts with $r = x$, $ds = 4e^{-2x}\,dx$. Therefore $dr = dx$ and $s = \int ds = -2e^{-2x}$. This makes the integral

$$\begin{aligned}\int 4x\,e^{-2x}\,dx = \int r\,ds = rs - \int s\,dr &= -2xe^{-2x} - \int -2e^{-2x}\,dx \\ &= \boxed{-2xe^{-2x} - e^{-2x} + C}.\end{aligned}$$

b) Use parts with $r = \ln^2 x$, $ds = dx$. Therefore $dr = 2\ln x \cdot \frac{1}{x}$ and $s = \int ds = x$. This makes the integral

$$\int \ln^2 x\,dx = \int r\,ds = rs - \int s\,dr = x\,\ln^2 x - \int x\left(\frac{2\ln x}{x}\right)\,dx$$
$$= 2\ln x - \int 2\ln x\,dx$$

Now use parts again on the second integral with $r = 2\ln x$, $ds = dx$. Therefore $dr = \frac{2}{x}\,dx$ and $s = x$, so the second integral is

$$\int 2\ln x\,dx = \int r\,ds = rs - \int s\,dr = 2x\ln x - \int x\left(\frac{2}{x}\right)\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x.$$

Substituting this into our answer from above, we get

$$x\,\ln^2 x - \int 2\ln x\,dx = 2\ln x - (2x\ln x - 2x) + C = \boxed{x\,\ln^2 x - 2x\ln x + 2x + C}.$$

4. a) Use the u-sub $u = \tan x$, $du = \sec^2 x\,dx$ to rewrite the integral (note the change in the limits) as

$$\int_0^{\pi/4} 24\tan^2 x\sec^2 x\,dx = \int_0^1 24u^2\,du = \left[8u^3\right]_0^1 = \boxed{8}.$$

b) Use the u-sub $u = 2x - 1$. That makes $x = \frac{u+1}{2}$ and $du = 2\,dx$ so $dx = \frac{1}{2}\,du$ so the integral becomes (note the change in the limits)

$$\begin{aligned}\int_1^2 \frac{3x+2}{2x-1}\,dx &= \int_1^3 \left(\frac{3\left(\frac{u+1}{2}\right)+2}{u}\right)\frac{1}{2}\,du\\ &= \int_1^3 \left(\frac{\frac{3}{2}u + \frac{7}{2}}{u}\right)\frac{1}{2}\,du\\ &= \int_1^3 \left(\frac{3}{4} + \frac{7}{4}u^{-1}\right)\,du\\ &= \left[\frac{3}{4}u + \frac{7}{4}\ln u\right]_1^3\\ &= \left[\frac{3}{4}(3) + \frac{7}{4}\ln 3\right] - \left[\frac{3}{4}(1) + \frac{7}{4}(0)\right] = \boxed{\frac{3}{2} + \frac{7}{4}\ln 3}.\end{aligned}$$

5. a) Use partial fractions: the denominator factors as $(x+1)(x-5)$, so we guess the decomposition as

$$\frac{12}{(x+1)(x-5)} = \frac{A}{x+1} + \frac{B}{x-5} \quad\Rightarrow\quad A(x-5) + B(x+1) = 12.$$

Substituting in $x = 5$, we get $6B = 12$, i.e. $B = 2$. Substituting in $x = -1$, we get $-6A = 12$ so $A = -2$. This makes the integral

$$\int \frac{12}{x^2 - 4x - 5}\,dx = \int \left(\frac{-2}{x+1} + \frac{2}{x-5}\right) dx = \boxed{-2\ln(x+1) + 2\ln(x-5) + C}.$$

b) Use partial fractions: the denominator factors as $x^3(x-1)$, so we guess the decomposition as

$$\frac{3x^2+1}{x^4 - x^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1},$$

and after clearing denominators we get

$$3x^2 + 1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3. \tag{4.3}$$

Substitute $x = 0$ to get $-C = 1$, i.e. $C = -1$. Substitute $x = 1$ to get $4 = D$. Now, substituting in $C = -1$ and $D = 4$, and expanding out (4.3) and combining like terms gives

$$3x^2 + 1 = (4 + A)x^3 + (B - A)x^2 + (-1 - B)x + 1$$

so $4 + A = 0$, i.e. $A = -4$, and $B - A = 3$, i.e. $B = -1$. All together, this makes the integral

$$\begin{aligned}\int \frac{3x^2+1}{x^4 - x^3}\,dx &= \int \left(\frac{-4}{x} - \frac{1}{x^2} - \frac{1}{x^3} + \frac{4}{x-1}\right) dx \\ &= \boxed{-4\ln x + \frac{1}{x} + \frac{1}{2}x^{-2} + 4\ln(x-1) + C}.\end{aligned}$$

6. a) First, find the intersection points of the graphs by setting the two equations equal to each other:

$$\begin{aligned}\frac{x^2}{3} &= 2x \\ x^2 &= 6x \\ x^2 - 6x &= 0 \\ x(x-6) &= 0 \Rightarrow x = 0, x = 6\end{aligned}$$

The corresponding y-values are $y = 0$ and $y = 12$, so the corner points are $(0,0)$ and $(6,12)$. Thus the area of E is

$$\int_{left}^{right} [top(x) - bot(x)]\,dx = \boxed{\int_0^6 \left[2x - \frac{x^2}{3}\right] dx}.$$

b) Working with respect to y, we have to solve the equations for y: $y = \dfrac{x^2}{3}$ becomes $x = \sqrt{3y}$ and $y = 2x$ becomes $x = \dfrac{1}{2}y$. So the area of E is

$$\int_{bottom}^{top} [right(y) - left(y)]\,dy = \boxed{\int_0^{12} \left[\sqrt{3y} - \frac{1}{2}y\right]\,dy}.$$

7. a) Use the disk method: $V = \displaystyle\int_{left}^{right} \pi R^2\,dx = \boxed{\int_0^2 \pi\left[2 - 2(x-1)^4\right]^2\,dx}$.

b) Use the shell method: $V = \displaystyle\int_{left}^{right} 2\pi r h\,dx = \boxed{\int_0^2 2\pi x\left[2 - 2(x-1)^4\right]\,dx}$.

c) $I_y = \displaystyle\int_{left}^{right} x^2\left[top(x) - bot(x)\right]\,dx = \boxed{\int_0^2 x^2\left[2 - 2(x-1)^4\right]\,dx}$.

d) Use the arc length formula:

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_0^2 \sqrt{1 + [-8(x-1)^3]^2}\,dx = \boxed{\int_0^2 \sqrt{1 + 64(x-1)^6}\,dx}.$$

(e) The center of mass of planar slab that has shape R, where the density of the lamina at point (x, y) is $\rho(x) = (x^2 + 2)$.

$$\overline{y} = \frac{M_x}{M} = \frac{\displaystyle\int_{left}^{right} \frac{1}{2}\left[top^2(x) - bot^2(x)\right]\,dx}{\displaystyle\int_{left}^{right} \rho(x)[top(x) - bot(x)]\,dx} = \boxed{\frac{\displaystyle\int_0^2 \frac{1}{2}\left[2 - 2(x-1)^4\right]^2\,dx}{\displaystyle\int_0^2 x^2[2 - 2(x-1)^4]\,dx}}$$

8. a) We need $1 = \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^4 c\,x\,dx = \left[\frac{1}{2}cx^2\right]_0^4 = \frac{1}{2}c(4^2) = 8c$, so $c = \boxed{\dfrac{1}{8}}$.

b) $P(0 < X < 2) = \displaystyle\int_0^2 f(x)\,dx = \int_0^2 \frac{1}{8}x\,dx = \left[\frac{1}{16}x^2\right]_0^2 = \boxed{\frac{1}{4}}$.

9. a) $\displaystyle\sum \frac{(-1)^n}{n^{3/4}}$ converges by the AST (the series alternates, $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$ and $|a_{n+1}| \le |a_n|$), but $\sum |a_n| = \displaystyle\sum \frac{1}{n^{3/4}}$ which diverges (p-series with $p = \frac{3}{4} \le 1$), so $\displaystyle\sum \frac{(-1)^n}{n^{3/4}}$ $\boxed{\textbf{converges conditionally}}$.

b) $\sum \frac{1}{3^n}$ converges absolutely (it is geometric with $|r| = \frac{1}{3} < 1$), $\sum \frac{(-1)^n}{4^n} = \sum \left(\frac{-1}{4}\right)^n$ also converges absolutely (it is geometric with $|r| = \frac{1}{4} < 1$), so this is the sum of two absolutely convergent series, which must $\boxed{\textbf{converge absolutely}}$.

c) Notice $0 \leq \frac{4}{k^2+3} \leq \frac{4}{k^2}$ and $\sum \frac{4}{k^2}$ is a convergent p-series ($p = 2 > 1$). Therefore by the Comparison Test, $\displaystyle\sum_{k=0}^{\infty} \frac{4}{k^2+3}$ $\boxed{\textbf{converges absolutely}}$.

d) Use the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{7^{n+1}}{(2(n+1))!}}{\frac{7^n}{(2n)!}} = \lim_{n\to\infty} \frac{7^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{7^n}$$
$$= \lim_{n\to\infty} \frac{7}{(2n+2)(2n+1)} = 0.$$

Since $\rho < 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 7^n}{(2n)!}$ $\boxed{\textbf{converges absolutely}}$ by the Ratio Test.

10. a) This is the Taylor series for $\ln(1+x)$, with $x = \frac{1}{4}$ plugged in:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,4^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}\bigg|_{x=1/4} = \ln(1+x)|_{x=1/4} = \ln\left(1+\frac{1}{4}\right) = \boxed{\ln\frac{5}{4}}.$$

b) This is the power series for $\cos x$, with $x = 1$ plugged in:

$$1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \frac{1}{10!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\bigg|_{x=1} = \cos x|_{x=1} = \boxed{\cos 1}.$$

c) This series is geometric:

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{5^{2n+1}} = \sum_{n=0}^{\infty} \frac{3^n 3^2}{5^{2n} 5} = \frac{9}{5} \sum_{n=0}^{\infty} \left(\frac{3}{25}\right)^n = \frac{9}{5}\left(\frac{1}{1-\frac{3}{25}}\right) = \frac{9}{5}\left(\frac{25}{22}\right) = \boxed{\frac{45}{22}}.$$

d) The series is geometric; first, change indices by replacing n with $n-2$ to get $\displaystyle\sum_{n=2}^{\infty} 4\left(\frac{-3}{5}\right)^n = \sum_{n=0}^{\infty} 4\left(\frac{-3}{5}\right)^{n-2}$. Then

$$\sum_{n=0}^{\infty} 4\left(\frac{-3}{5}\right)^{n-2} = 4\left(\frac{-3}{5}\right)^{-2} \sum_{n=0}^{\infty} \left(\frac{-3}{5}\right)^n = 4\left(\frac{25}{9}\right) \frac{1}{1-(\frac{-3}{5})} = \frac{4 \cdot 25}{9} \cdot \frac{5}{8} = \boxed{\frac{125}{18}}.$$

11. a) $e^x = 1 + x + \frac{x^2}{2} + ...$, so by substitution $e^{2x^7} = 1 + 2x^7 + \frac{(2x^7)^2}{2} + ... = 1 + 2x^7 + 2x^{14} + ...$. Substituting this into the limit, we get

$$\begin{aligned}\lim_{x\to 0} \frac{e^{2x^7} - 2x^7 - 1}{x^{10}} &= \lim_{x\to 0} \frac{1 + 2x^7 + 2x^{14} + ... - 2x^7 - 1}{x^{10}} \\ &= \lim_{x\to 0} \frac{2x^{14} + \text{higher power terms}}{x^{10}} \\ &= \lim_{x\to 0} \left(2x^4 + \text{higher power terms}\right) \\ &= \boxed{0}.\end{aligned}$$

b) $e^x = 1+x+\frac{x^2}{2}+...$, so by replacing x with $-x^2$ we get $e^{-x^2} = 1-x^2+\frac{x^4}{2}-....$ Multiplying through by x^2 gives $x^2e^{-x^2} = x^2-x^4+\frac{1}{2}x^6-...$, and stopping at the sixth power term gives $P_6(x) = x^2 - x^4 + \frac{1}{2}x^6$. This makes the integral

$$\begin{aligned}\int_0^2 x^2e^{-x^2}\,dx \approx \int_0^2 P_6(x)\,dx &= \int_0^2 \left[x^2 - x^4 + \frac{1}{2}x^6\right]\,dx \\ &= \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7\right]_0^2 = \boxed{\frac{2^3}{3} - \frac{2^5}{5} + \frac{2^7}{7}}. = \frac{1072}{105}.\end{aligned}$$

c) By theory, we know that if $g(x) = \sum_{n=0}^{\infty} a_nx^n$, then $a_n = \frac{g^{(n)}(0)}{n!}$. Therefore

$$a_0 = g(0) = 2; \quad a_1 = g'(0) = -1; \quad a_2 = \frac{g''(0)}{2!} = 0;$$

$$a_3 = \frac{g'''(0)}{3!} = \frac{3}{6} = \frac{1}{2}; \quad a_4 = \frac{g^{(4)}(0)}{4!} = \frac{6}{24} = \frac{1}{4}.$$

Thus $g(x) = 2-x+\frac{1}{2}x^3+\frac{1}{4}x^4+...$, so stopping at the fourth power term, we get $P_4(x) = 2 - x + \frac{1}{2}x^3 + \frac{1}{4}x^4$. Finally,

$$g(2) \approx P_4(2) = 2 - 2 + \frac{1}{2}(8) + \frac{1}{4}(16) = \boxed{8}.$$

d) **(Bonus)** The amount the ball travels is the initial drop plus twice the size of each rebound (because on each rebound, it goes up and the back down). This produces the geometric series

$$\begin{aligned}
&24 + 2\left(\frac{2}{3}\right)24 + 2\left(\frac{2}{3}\right)^2 24 + 2\left(\frac{2}{3}\right)^3 24 + \ldots \\
&= 24 + \sum_{n=1}^{\infty} 2\left(\frac{2}{3}\right)^n 24 \\
&= 24 + 48\left(\frac{2}{3}\right)\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \\
&= 24 + 32\left(\frac{1}{1-\frac{2}{3}}\right) \\
&= 24 + 32(3) = \boxed{120 \text{ ft}}.
\end{aligned}$$