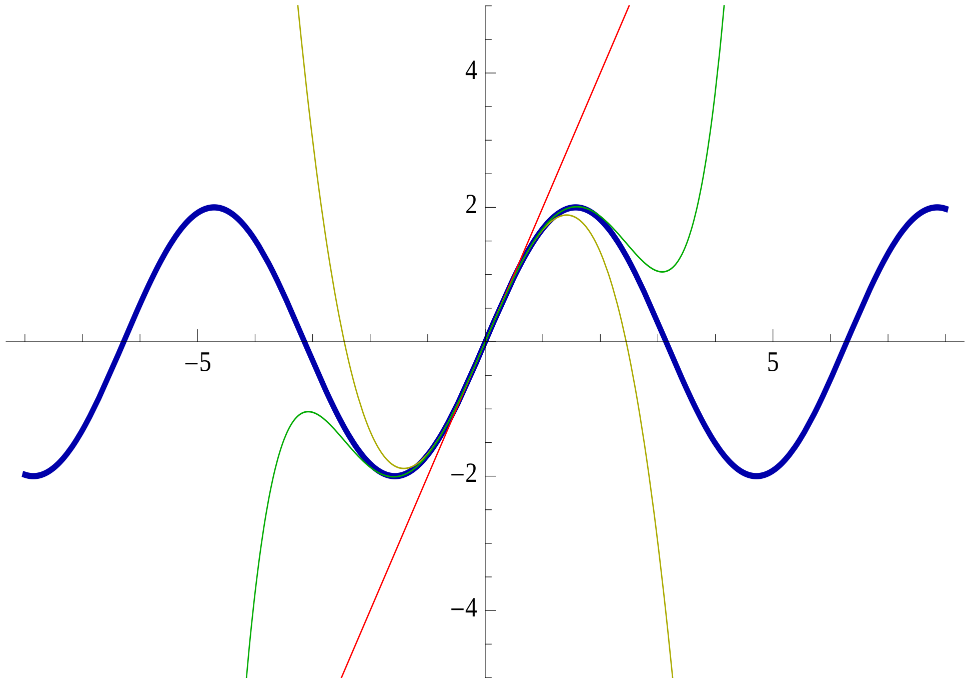

Notes on Infinite Series

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1 Infinite Series

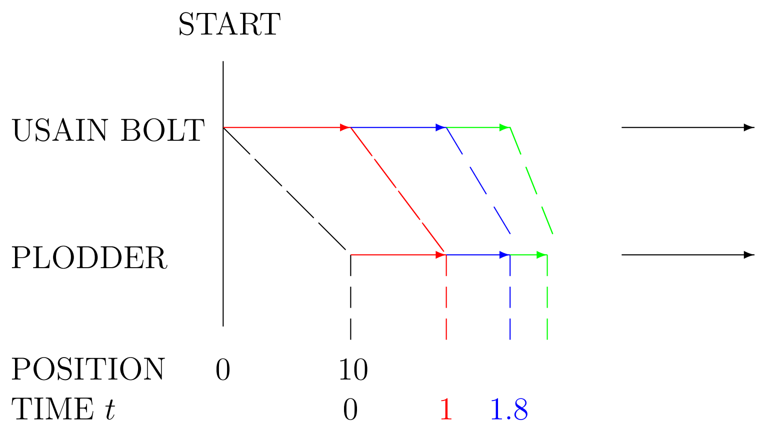
1.1 The idea of an infinite series

To motivate the ideas of this chapter, we start with an example: consider a footrace of 100 meters being run between Usain Bolt (the world-record holder in the 100m) and a plodder. Assume that Usain Bolt runs 10 meters per second and that the plodder runs 8 meters per second. However, the plodder gets a head start of 10 meters. Who wins and why?

The ancient Greeks considered problems like these. One Greek named Zeno offered the following solution: he argued that as long as the plodder keeps moving, he will win. His explanation is as follows: since the plodder has a head start, it will take Bolt some time to get to where the plodder started. By that time the plodder will have moved forward, to some point A. Then it will take Bolt more time to get to get to A; by the time Bolt gets there, the plodder will have again moved forward to point B. Repeating this procedure indefinitely allows the plodder to remain in the lead forever, so the plodder wins.

On the other hand, given what we know about physics (in particular, that distance traveled equals rate times time), we can solve for the time it takes each racer to run the 100m. Bolt, who has to travel 100 m but runs 10 m/s will finish the race in $\frac{100}{10} = 10$ sec. The plodder, who only has to run 90 m but runs only 8 m/s will take $\frac{90}{8} = 11.25$ sec to finish, so Bolt will win (and will complete the race course 1.25 sec ahead of the plodder).

This seeming contradiction between Zeno's argument and modern physics is one version of what is called *Zeno's paradox*. In fact, there is no paradox at all; Zeno simply made a logical error which is fixed through the careful study of what are called "infinite series". To find Zeno's error, let's look more closely at his argument. We will refer to the picture on the next page (the picture is explained below):



Repeating Zeno's argument, the plodder starts with a head start of 10m. Now since Bolt runs 10 m/s, it takes him 1 sec to run to the point where the plodder started. During this same 1 second, the plodder runs 8 m. These distances, which represent the amount of distance covered by each sprinter in the first second, are colored in red on the above picture (Bolt's red distance is 10 m; the plodder's is 8 m).

Now, we need to figure out how long it will take Bolt to run to where the plodder is at time 1. Since at time 1 the plodder is 8 m ahead, it will take Bolt $\frac{8}{10}$ seconds to cover that 8 m. But in that time frame, the plodder will run $8 \cdot \frac{8}{10} = \frac{8^2}{10}$ m. These distances are indicated in blue.

Repeating the same logic, Bolt will need $\frac{(8^2/10)}{10}$ seconds to cover his green distance, but in the same time the plodder will run $8 \cdot \frac{(8^2/10)}{10} = \frac{8^3}{10^2}$ m (this is the plodder's green distance). Repeating this process indefinitely, we see that the plodder stays in the lead for a distance which is equal to his head start, plus his red distance, plus his blue distance, plus his green distance, etc., which is

$$10 + 8 + \frac{8^2}{10} + \frac{8^3}{10^2} + \frac{8^4}{10^3} + ...\text{m.}$$

Notice that this is the sum of **infinitely many positive numbers**. Zeno assumed (incorrectly) that when one tries to add up infinitely many positive numbers, the sum will be infinite (and in particular that the sum is greater than 100, so the plodder would win the race). However, Zeno's assumption is wrong. It is sometimes possible to add up infinitely many numbers and give a finite sum. The numbers above are one such example: from physics, we know that Bolt's position at time t is $10t$ and the plodder's position at time t is $8t + 10$. When these are equal, Bolt catches up. So we set

$$10t = 8t + 10$$

and solve for t to get $t = 5$. At this time, both runners will be at the 50m mark of the race, so the total amount of distance the plodder covers in the lead is 50, and therefore it must be the case that

$$10 + 8 + \frac{8^2}{10} + \frac{8^3}{10^2} + \frac{8^4}{10^3} + ... = 50.$$

An **infinite series** is an attempt to add (and/or subtract) an infinite list of numbers. These lecture notes deal with the main questions regarding the study of infinite series. For now, the main questions are as follows:

Big-picture questions related to infinite series:

1. **Classification problem:** Given an infinite list of numbers, do the numbers add up to a finite sum or not?
2. **Computation problem:** Given an infinite list of numbers whose sum is finite, what is the value of that sum?

Over the course of our study of infinite series, we will discover that the first question is mostly "doable", but the second question is much harder.

For now, we give two examples of infinite series:

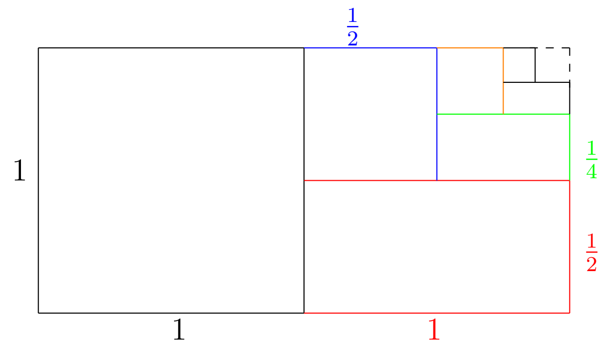
Example 1.1. *(a)* $1 + 1 + 1 + 1 + 1 + 1 + ...$

This series clearly adds up to ∞, *because as you add more* 1*s the running total increases without bound.*

(b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$

This series adds up to 2, *because of the following area calcuation. Notice that the area of the black square in the figure below is* 1*; the area of the red rectangle is* $\frac{1}{2}$, *the area of the blue square is* $\frac{1}{4}$, *etc. So the total area enclosed by these rectangles is*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$



But this area must also be 2*, since it is the area of a rectangle with width* 2 *and height* 1*. Therefore*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = 2.$$

1.1.1 The role of calculus; convergence and divergence

On the face of things, it may seem like the study of infinite series does not belong in a calculus class. After all, previously studied calculus concepts like "limit", "derivative" and "integral" have to do with functions and graphs, and neither the notion of function nor the idea of a graph seems to have anything to do with adding up an infinite list of numbers. What then, does calculus have to do with infinite series? To address this question, we think back to how the ideas of derivative and integral were developed. We will observe a pattern to the logic behind the ideas and will follow this pattern to study infinite series.

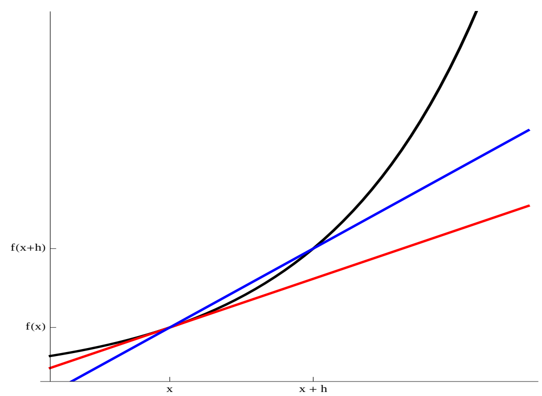
First, let's review the idea of a derivative. Recall first that derivatives were developed to solve a particular problem: **given a function f and a value x, find the slope of the tangent line to f at x.** (An equivalent formulation of the same problem is as follows: given a function which measures an object's position at time t, determine the object's velocity at any given time.) Usually, finding the slope of a line involves finding two points on the line and computing the slope by the formula $\frac{y_2 - y_1}{x_2 - x_1}$. However, one cannot find the slope of a tangent line by this procedure alone because there is only one known point on the tangent line (namely $(x, f(x))$.

To get around this problem, what one does is *approximate* the slope of the tangent line by finding the slope of a secant line (i.e. a line connecting two points on the graph of f). First, one chooses a new constant h and moves h units to the left of x, to get to point $x + h$ on the $x-$axis. Then, one connects the points on the graph of f at

$x-$values x and $x+h$ with a secant line. The slope of this secant line is

$$\frac{f(x+h)-f(x)}{h}$$

and this slope **serves as an approximation to the quantity we set out to solve**. In particular, notice that this approximation is in terms of not only the given information of the problem (the f and the x), but also an extra variable we invented (the h). The picture below explains the approximation: we are seeking the slope of the red tangent line at x, and approximate this slope by computing the slope of the blue secant line:



The next step is to figure out how changes in the extra variable make the approximation better. With regard to derivatives, as h gets smaller, the second point $(x+h, f(x+h))$ on the secant line gets closer and closer to $(x, f(x))$, so the secant line twists toward the tangent line. Thus, **as h gets closer and closer to 0, the approximation improves**. Using the idea of "limit", we can then say that the slope of the tangent line is the limit as $h \to 0$ of the approximation. Defining this limit to be the derivative $f'(x)$, we have

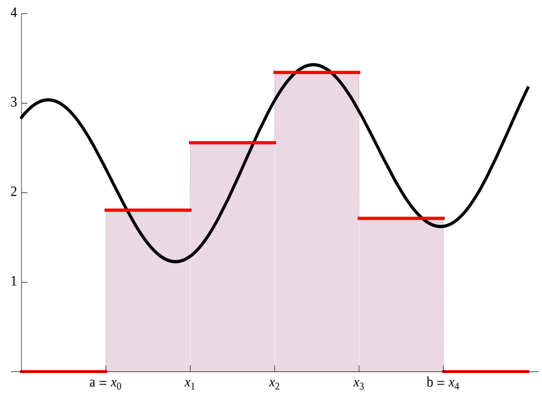
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Now let's review the definition of the definite integral. As with derivatives, integrals were developed to solve a particular problem: **find the area under the graph of f from $x = a$ to $x = b$**. If the graph is not linear or not part of a circle, formulas from high-school geometry are not sufficient to fiund the area, so again the problem is approached by *approximation*.

This time, the approximation is obtained by computing a Riemann sum: one takes a partition $\mathcal{P}$ of $[a, b]$ which divides the interval $[a, b]$ into subintervals; then rectangles are constructed over each of these subintervals so that the total area of the rectangles is roughly equal to the area under the function. Without describing all the notation, the approximation is called a Riemann sum and is denoted

$$\sum_{j=1}^{n} f(c_j)\,\Delta x_j.$$

As with derivatives, the approximation is in terms of not only the given information of the problem (the f, a and b, but also something extra we invented (the partition $\mathcal{P}$ and the test points c_j). A picture explaining the approximation is below: we approximate the area under the function from a to b by the shaded area:



The next step, as with derivatives, is to figure out how changes in the extra stuff we invented make the approximation better. Here, the approximation improves as we use rectangles of smaller and smaller width, i.e. as the norm $||\mathcal{P}||$ of the partition gets closer and closer to zero. Thus the exact area under the function is the limit as $||\mathcal{P}|| \to 0$ of the Riemann sum. Defining this area to be the definite integral of the function from a to b, we have

$$\int_a^b f(x)\,dx = \lim_{||\mathcal{P}||\to 0} \sum_{j=1}^{n} f(c_j)\,\Delta x_j.$$

Note the similarities in the reasoning that led to the definitions of the derivative and the definite integral. Both ideas follow the same five-step process, which involves answering four questions and then describing the result:

1. **What is the motivating problem?** In both cases, there is a problem which motivates the discussion (for derivatives, finding the slope of a tangent line; for integrals, finding area under the graph of a function).

2. **Why can the motivating problem not be solved without calculus?** In both cases, there is a reason the problem cannot be solved directly (for derivatives, only one point on the line is known; for integrals, no area formulas for odd-shaped regions are known from high-school geometry).

3. **How can you approximate the solution to the motivating problem?** In both cases, the solution to the problem is approximated in terms of an extra quantity (for derivatives, the approximation is the slope of the secant line and the extra quantity is h and for integrals, the approximation is a Riemann sum and the extra quantity is $\mathcal{P}$).

4. **What makes the approximation better?** In both cases, we study how the approximation improves as the extra quantity is changed (for derivatives, the approximation improves as $h \to 0$; for integrals, the approximation improves as $||\mathcal{P}|| \to 0$).

5. **Define the theoretical answer to the motivating problem.** We get the exact theoretical answer to the problem (i.e. the derivative or the integral) by taking the limit suggested in step 4 of the approximation obtained in step 3.

In both cases the quantity which solves the problem is obtained by approximation. We will see that the right way to approach infinite series also involves approximation, and thus calculus will be the appropriate context to develop the theory of infinite series. It is worth noting, however, that the theory behind the derivative and integral is not useful for actually evaluating derivatives and integrals:

- If asked to compute the derivative of a function, although the theoretical definition is
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$
we don't actually use this theoretical definition in practice. We learn rules like the Power Rule, Product Rule, Quotient Rule and Chain Rule and apply these.

- If asked to compute a definite integral, although formally we are being asked to compute the limit of a Riemann sum, we don't actually compute integrals this way. We find an antiderivative of the integrand and use the Fundamental Theorem of Calculus.

For now, we return to our big-picture problems with infinite series and see how the same pattern that was used to create the derivative and integral can be used to develop a theoretical machinery to examine series (in practice, our theoretical machinery won't be useful for solving problems, but it is important to understand so that its application makes sense).

1. **What is the motivating problem?** Given an infinite series $a_1 + a_2 + a_3 + ...$, determine whether or not the series adds up to a finite number. If it adds to a finite number, find the sum.

2. **Why can the motivating problem not be solved without calculus?** In other words, why do we need calculus to add up an infinite list of numbers?

   To explain this, we need to go back to the concept of addition. Formally speaking, addition is a *binary operation*, i.e. it is an operation that has 2 inputs and 1 output. Put another way, the process of adding 8 and 5 to get 13 can be thought of as a function where the inputs are 8 and 5 (notice that there are 2 inputs) and the output is 13 (one output).

   Now suppose you were asked to add $7 + 3 + 9$. How do you really get the answer 19? Formally speaking, since addition is a binary operation, you add the numbers two at a time. In other words, you are really doing the following sequence of operations (even if you don't bother to write them out):

$$7 + 3 + 9 = (7 + 3) + 9 = 10 + 9 = 19.$$

   The reason this logic is valid is that addition is *associative*, i.e. for any three numbers a, b and c, the following holds:

$$a + (b + c) = (a + b) + c.$$

   This property has the important theoretical consequence that you can add any three numbers by choosing two, adding them, and then adding their sum to the third number. In other words, **finite sums can be rearranged and regrouped arbitrarily without changing the sum** because of the associative property. Also, if you have a sum of finitely many numbers, say N numbers like

$$a_1 + a_2 + a_3 + a_4 + ... + a_N,$$

   you start by adding any two of the numbers (say a_1 and a_2) to get

$$(a_1 + a_2) + a_3 + a_4 + ... + a_N.$$

   Notice that (since $a_1 + a_2$ is now one number) there are now $N - 1$ numbers to add (where there were originally N numbers), i.e. the number of numbers to add has dropped by one, so the problem is "easier". If you then add a_3 to the running total, you get

$$((a_1 + a_2) + a_3) + a_4 + ... + a_N$$

(which leaves $N-2$ numbers to add). So by repeatedly adding one more term to the running sum, you eventually run out of numbers to add and get a final answer. Let's contrast this with an infinite series: if you start with an infinite series like

$$a_1 + a_2 + a_3 + a_4 + a_5 + \ldots$$

and add the first two numbers to get

$$(a_1 + a_2) + a_3 + a_4 + a_5 + \ldots,$$

you went from having to add infinitely many numbers to having to add... infinitely many numbers. In other words, your problem didn't get any easier. Put another way, if you keep adding numbers two at a time in this setting, you will always have infinitely many numbers to add, so the process will never terminate and you will never end up with a final answer.

What's just as bad is that the associative property doesn't work for infinite series, as we see in the following example:

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

If you group the terms into pairs, this series can be rewritten as

$$\begin{aligned} 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + .. &= (1-1) + (1-1) + (1-1) + (1-1) + \ldots \\ &= 0 + 0 + 0 + 0 + 0 + 0 \ldots \\ &= 0. \end{aligned}$$

But, if you group the terms differently, you can also obtain

$$\begin{aligned} 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + .. &= 1 + (-1+1) + (-1+1) + (-1+1) + \ldots \\ &= 1 + 0 + 0 + 0 + 0 + 0 + 0 \ldots \\ &= 1. \end{aligned}$$

We do not yet know if either of these calculations are valid. But what we do know is that they cannot both be valid (because that would imply $0 = 1$). Therefore **it cannot be the case than one can legally regroup or rearrange terms in an infinite series**. In other words, **the associative property is invalid for infinite series** in general. This leads to a third big picture question which we will explore later in this chapter:

> ***Rearrangement problem:*** When, if ever, can the terms of an infinite series be rearranged or regrouped without affecting the sum of the series?

To summarize, adding the terms of an infinite series cannot be done without calculus because (1) addition is defined to be a binary operation and the process of adding numbers two at a time does not terminate for an infinite series, and (2) the associative property does not hold for infinite series.

3. **How can you approximate the solution to the motivating problem?** To approximate the infinite series

$$a_1 + a_2 + a_3 + a_4 + ...,$$

a reasonable thing to do to approximate the infinite sum is to add up a large, but finite, number of terms of the series. This motivates the following definition:

Definition 1.1. *Given an infinite series $a_1 + a_2 + a_3 + a_4 + ...$, and given any index N, the N^{th}* **partial sum** *of the series, denoted S_N is*

$$S_N = a_1 + a_2 + a_3 + ... + a_N.$$

Notice that S_N is always defined, since it is a sum of finitely many numbers. We remark, that the indexing of an infinite series does not always start with the index 1. In general, the N^{th} partial sum of an infinite series is the sum of all terms in the series whose index is $\leq N$. For example, if your series is

$$a_5 + a_6 + a_7 + a_8 + a_9 + ...,$$

then the seventh partial sum of this series is $S_7 = a_5 + a_6 + a_7$. In particular, you do not necessarily add up N terms of the series to get S_N. You add up terms until you get to index N.

Example 1.2. *For each of the following series, find the second and fourth partial sums. Assume that the initial term of each series is a_1.*

(a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$

Solution: *By definition,*

$$S_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2}.$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

(b) $2-1+2-1+2-1+2-1+...$

Solution: *By definition,*

$$S_2 = a_1 + a_2 = 2 - 1 = 1.$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = 2 - 1 + 2 - 1 = 2.$$

4. **What makes the approximation better?** Notice that we have obtained an approximation to the sum of the infinite series $a_1+a_2+a_3+...$; the approximation is the partial sum S_N and depends on the extra variable N which we choose. Now, since $S_N = a_1 + a_2 + ... + a_N$, the difference between the infinite series and the partial sum S_N is the sum of the terms of the series starting at a_{N+1}, i.e.

$$\begin{aligned} a_1 + a_2 + a_3 + ... - S_N &= a_1 + a_2 + a_3 + ... - (a_1 + a_2 + ... + a_N) \\ &= a_{N+1} + a_{N+2} + a_{N+3} + ... \end{aligned}$$

   As N gets larger, there are less and less terms in this difference, so the difference between the infinite series and the N^{th} partial sum goes to zero, i.e. the N^{th} partial sum should approach the sum of the infinite series as $N \to \infty$.

5. **What is the theoretical answer to the problem?** Based on our work above, the sum of the infinite series should be $\lim_{N\to\infty} S_N$.

This logic leads to the following definitions. As with improper integrals, we use the word "converges" to connote a situation where the answer exists and is finite, and we use the word "diverges" to connote a situation where the answer does not exist or is infinite. More precisely:

Definition 1.2. *Let $a_1 + a_2 + a_3 + a_4 + ...$ be an infinite series. For each N, let $S_N = a_1 + a_2 + ... + a_N$ be the N^{th} partial sum of the series. Then:*

1. *If L is a real number such that $\lim_{N\to\infty} S_N = L$, then we say that the infinite series $a_1 + a_2 + a_3 + ...$* **converges (to** L**)** *and write*

$$a_1 + a_2 + a_3 + ... = L.$$

   In this setting L is called the **sum** *of the series.*

2. *If $\lim_{N\to\infty} S_N = \pm\infty$ or if $\lim_{N\to\infty} S_N$ DNE, then we say that the infinite series $a_1 + a_2 + a_3 + ...$* **diverges**.

This gives us the theoretical answer to the question "does an infinite series add to a finite number". As with derivatives and integrals, this theoretical definition is not useful in practice; we learn rules to study series in the same way that we calculate derivatives and integrals without using the definitions of these concepts. An understanding of the definition is, however, crucial for developing an understanding of the theory of infinite series.

At this point, we have three "big picture" questions with infinite series we want to solve:

Big-picture questions related to infinite series:

1. **Classification problem:** Given an infinite series $a_1 + a_2 + a_3 + ...$, does the series converge or diverge?
2. **Computation problem:** Given an infinite series that converges, what is the sum of the series?
3. **Rearrangement problem:** When, if ever, can the terms of an infinite series be rearranged and/or regrouped without affecting the series' convergence?

We will begin to address these questions in Section 1.3. For now, we discuss some notation associated to infinite series:

1.2 Notation associated to infinite series

To avoid writing expressions like

$$a_1 + a_2 + a_3 + a_4 + ...$$

over and over, mathematicians have developed notation to write infinite series which takes up less space. In particular, the above sum can be written as

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \Sigma_{n=1}^{\infty} a_n.$$

This notation has several ingredients to it. First, the Greek letter sigma (Σ) is used to represent the concept of "summation" (since sigma and summation both start with the letter "s"). The letter n is called the **variable of summation**; the number 1 is called the **starting index** or **lower index** of the sum, and ∞ is called the **upper index** or **finishing index** of the sum. The a_n, which are numbers depending on n, are called the **terms** of the series (these are the numbers being added). For a fixed n, a_n is called the n^{th} **term** of the series.

For example, the series

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

has variable of summation n, starting index 1, and has terms as follows:

n	n^{th} term (i.e. a_n)
1	the first term is $a_1 = \frac{1^2}{3^1} = \frac{1}{3}$
2	the second term is $a_2 = \frac{2^2}{3^2} = \frac{4}{9}$
3	the third term is $a_3 = \frac{3^2}{3^3} = \frac{1}{3}$
4	the fourth term is $a_4 = \frac{4^2}{3^4} = \frac{16}{81}$
$\vdots$	$\vdots$
12	the twelfth term is $a_{12} = \frac{12^2}{3^{12}} = \frac{144}{3^{12}}$
$\vdots$	$\vdots$
n	the n^{th} term is $a_n = \frac{n^2}{3^n}$

Written out, this series is

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{1}{3} + \frac{4}{9} + \frac{1}{3} + \frac{16}{81} + ...$$

There is no reason why the starting index of a series has to be $n = 1$; it could be any positive or negative integer (or zero). However, no matter what the starting index is, we always assume that the variable of the summation counts upward by 1 to get from term to term (i.e. if the variable is n, n must be an integer and cannot be $\frac{1}{2}$ or π or -2.68). For example, the series

$$\sum_{k=3}^{\infty} \frac{6}{n^4}$$

means the following expression, written out:

$$\frac{6}{3^4} + \frac{6}{4^4} + \frac{6}{5^4} + \frac{6}{6^4} + ...$$

Note: We will see that some of the things we want to say about series do not depend on the starting index of the series. In this setting, we will just write $\sum a_n$ to represent the series. If you see $\sum a_n$ (without the upper and lower indices indicated), this means one of two things:

1. The starting index was given earlier in the problem and is being omitted solely for the sake of brevity (while this is "legal", it is not recommended that you do this), or
2. Some property of the series is being described which does not depend on the starting index of the series (so the starting index is irrelevant to the context).

Example 1.3. *Given each infinite series, give the second term of the series, the seventh term of the series, and the third partial sum of the series:*

(a) $\sum_{n=1}^{\infty} \frac{2}{2n^2-3n}$

Solution: *The general form for the n^{th} term is $a_n = \frac{2}{2n^2-3n}$; the second term is when $n = 2$, i.e $a_2 = \frac{2}{2\cdot 2^2-3\cdot 2} = 1$. The seventh term is $a_7 = \frac{2}{2\cdot 7^2-3\cdot 7} = \frac{2}{77}$. The third partial sum is the sum of all the terms up to the third term, which is in this case*

$$S_3 = a_1 + a_2 + a_3 = \frac{2}{-1} + 1 + \frac{2}{9} = \frac{-7}{9}.$$

(b) $\sum_{n=0}^{\infty} \frac{n-1}{n+1}$

Solution: *The general form for the n^{th} term is $a_n = \frac{n-1}{n+1}$; the second term is when $n = 2$, i.e $a_2 = \frac{2-1}{2+1} = \frac{1}{3}$. The seventh term is $a_7 = \frac{7-1}{7+1} = \frac{3}{4}$. The third partial sum is the sum of all the terms up to the third term, which is in this case*

$$S_3 = a_0 + a_1 + a_2 + a_3 = -1 + 0 + \frac{1}{3} + \frac{1}{2} = \frac{-1}{6}.$$

Notice that to get the third partial sum of a series, you do not necessarily add up the first three terms. You add up all the terms until you get to the $n = 3$ term (and exactly how many terms that is depends on what the starting index is).

A useful skill is writing series that are written out in Σ notation. There are no "one-size fits all" rules for doing this, but the following examples illustrate some guidelines:

Example 1.4. *Write each of the following series in $\Sigma-$notation:*

(a) $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots$

Solution: *This is fairly straightforward. If we think of n as counting upward from 5 by 1s, then this series can be written as*

$$\sum_{n=5}^{\infty}\frac{1}{n}.$$

(b) $\frac{2}{5}+\frac{2}{11}+\frac{2}{17}+\frac{2}{23}+\frac{2}{29}+\ldots$

Solution: *This is similar to (a) except that the denominators count upwards by 6s rather than by 1s.* ***Here is a general principle:*** *when a quantity counts upward by steps of size V, then what should go in the addend for that quantity is some expression of the form $Vn+C$, where C is some constant that depends on your choice of starting index. Here $V=6$. Let's arbitrarily choose a starting index, say $n=1$. Then the term of the series corresponding to $n=1$is $\frac{2}{5}$, so the denominator should be of the form $6n+C$ where C chosen so that when $n=1$, $6n+C=5$. Thus $6\cdot 1+C=5$ so $C=-1$. Thus the denominators should be $6n-1$ and the series is therefore*

$$\sum_{n=1}^{\infty}\frac{2}{6n-1}.$$

Remark: *there are multiple correct answers, because the formula you get depends on what you choose the starting index to be (you could choose it to be any integer). For example,*

$$\sum_{n=0}^{\infty}\frac{2}{6n+5}.$$

gives the same series.

(c) $\frac{2}{5^3}+\frac{4}{5^4}+\frac{6}{5^5}+\frac{8}{5^6}+\frac{10}{5^7}+\ldots$

Solution: *Proceeding as in Example (b), we first identify quantities which change from term to term and see how much they go up by. The numerators go up by 2 each time, so they should be of the form $2n+C$ where C is a constant. The exponent in the denominator goes up by 1 each time, so it should be of the form $n+D$ where D is a constant. Let's choose the starting index to be $n=1$;*

then when $n = 1$ the numerator should be $2(1) + C = 2$ so $C = 0$ and the denominator should be $1 + D = 3$ so $D = 3$. Putting this together, we have

$$\sum_{n=1}^{\infty} \frac{2n}{5^{n+1}}.$$

(d) $\frac{1}{4} - \frac{1}{9} + \frac{1}{14} - \frac{1}{19} + \ldots$

Solution: *This series has something new: the terms alternate between addition and subtraction. To generate this, insert a $(-1)^n$ or $(-1)^{n+1}$ into the summand, because whenever n is odd, $(-1)^n = -1$ (generating a $-$ sign) and whenever n is even, $(-1)^n = 1$ (generating a $+$ sign).*

In this series, the denominators increase by 5 each time so they should be of the form $5n + C$. If we choose starting index $n = 1$, then $5 \cdot 1 + C = 4$ so $C = -1$. Since our series alternates positive and negative terms, we need either a $(-1)^n$ or a $(-1)^{n+1}$ in the terms. Since the initial term is when $n = 1$ but is positive, we should use $(-1)^{n+1}$ so that when $n = 1$, we get a $+$ sign (if we used $(-1)^n$, all the signs would be backwards). Therefore the sum in $\Sigma-$notation is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n-1}.$$

1.2.1 Elementary properties of convergence and divergence

In this section we begin by restating the definition of convergence and divergence that was derived in the previous section, this time using $\Sigma-$notation:

Definition 1.3. *Let $\sum a_n$ be an infinite series. For each N, let S_N be the N^{th} partial sum of the series; this is defined to be the sum of all the a_n for which $n \leq N$. Then:*

1. *If there is a real number L such that $\lim_{N\to\infty} S_N = L$, then we say the series $\sum a_n$* **converges to** *L and we write*

$$\sum a_n = L.$$

   In this situation L is called the **sum** *of the series.*

2. *If $\lim_{N\to\infty} S_N = \pm\infty$ or if $\lim_{N\to\infty}$ DNE, then we say the series $\sum a_n$* **diverges**.

Important remark: There is a big difference between saying "$\sum a_n$ converges" and saying "a_n converges". The first expression refers to the infinite series, which is an attempt to *add* the numbers $a_1, a_2, \ldots$. The second expression has no reference to adding in it (because it lacks the Σ). In fact, it means something else which we will not discuss in Calculus II. In particular, **you should never omit the Σ when describing whether or not an infinite series converges.**

Here are the big-picture questions outlined earlier:

Big-picture questions related to infinite series:

1. **Classification problem:** Given an infinite series $\sum a_n$, does the series converge or diverge?

2. **Computation problem:** Given an infinite series that converges, what is the sum of the series?

3. **Rearrangement problem:** When, if ever, can the terms of an infinite series be rearranged and/or regrouped without affecting the series' convergence?

To get started addressing these issues, we begin with some initial properties of convergence and divergence. The first set of results deals with addition/subtraction of series and multiplication of series by constants; they are collectively called "linearity" properties:

Theorem 1.1 (Linearity Part I)**.** *Suppose $\sum a_n$ is an infinite series that converges to L and $\sum b_n$ is an infinite series that converges to M. Then:*

1. *The series $\sum(a_n + b_n)$ converges to $L + M$.*

2. *The series $\sum(a_n - b_n)$ converges to $L - M$.*

3. *For any constant k, the series $\sum(k\, a_n)$ converges to kL.*

Theorem 1.2 (Linearity Part II). *Suppose $\sum a_n$ is an infinite series that converges to L and $\sum b_n$ is an infinite series that diverges. Then:*

1. *The series $\sum(a_n + b_n)$ diverges.*
2. *The series $\sum(a_n - b_n)$ diverges.*
3. *For any constant $k \neq 0$, the series $\sum(k\,b_n)$ diverges.*

Theorem 1.3 (Linearity Part III). *Suppose $\sum a_n$ is an infinite series that diverges and $\sum b_n$ is an infinite series that diverges. Then:*

1. *The series $\sum(a_n + b_n)$ might converge or diverge.*
2. *The series $\sum(a_n - b_n)$ might converge or diverge.*

Essentially, the content of these theorems can be summed up in the following phrases:

Linearity Part I:	"convergent $\pm$ convergent $=$ convergent"
	"constant times convergent $=$ convergent"
Linearity Part II:	"convergent $\pm$ divergent $=$ divergent"
	"nonzero constant times divergent $=$ divergent"
Linearity Part III:	"divergent $\pm$ divergent $=$ unknown".

These are useful because if you are given a series like

$$\sum \left(\frac{3}{2^n} - \frac{4}{n^2} \right),$$

then if you could somehow show that $\sum \left(\frac{1}{2^n}\right)$ converges and if you can show that $\sum \frac{1}{n^2}$ converges, then the original series must converge by linearity. More generally, the linearity properties suggest that you can study series by breaking them into pieces and studying each piece individually.

Here is a brief explanation as to why these linearity properties are true: to say that a series converges means the terms of that series add up to a finite number. If you have two finite numbers added together, then the sum is a finite number. Thus the sum or difference of two convergent series, being the sum or difference of two finite numbers, is finite, hence convergent. The other linearity properties are explained similarly. For example, if $\sum a_n$ converges but $\sum b_n$ diverges, then $\sum(a_n + b_n)$ is an attempt to add a finite number to something that doesn't exist. This gives you something that doesn't exist, hence $\sum(a_n + b_n)$ diverges (i.e. "convergent + divergent = divergent"). But if both $\sum a_n$ and $\sum b_n$ diverge, then neither of these sums exist. That doesn't tell you anything about $\sum(a_n \pm b_n)$, because it is possible to add two things that don't exist and have their nonexistence "cancel" (so to speak). Thus "divergent $\pm$ divergent is unknown".

A second important idea to keep in mind is the following result, which says that if you change the starting index of a series, then that does not change whether or not the series converges or diverges:

Theorem 1.4 (Starting Index is Irrelevant)**.** *Suppose $\sum_{n=K}^{\infty} a_n$ is an infinite series. Then, so long as all the terms are defined, for any constant M we have:*

1. *The series $\sum_{n=M}^{\infty} a_n$ converges if and only if $\sum_{n=K}^{\infty} a_n$ converges.*
2. *The series $\sum_{n=M}^{\infty} a_n$ diverges if and only if $\sum_{n=K}^{\infty} a_n$ diverges.*

For example, if you know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, then this theorem tells you that the series $\sum_{n=3}^{\infty} \frac{1}{n^2+1}$, $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ and $\sum_{n=-8}^{\infty} \frac{1}{n^2+1}$ all converge. **What this theorem does not say is that these series converge to the same sum.** (We will talk about how the value of the sum is affected when one changes the starting index one paragraph further down.)

Here is an explanation as to why this theorem is true. Suppose $K > M$. Then, if you consider the series

$$\begin{aligned}\sum_{n=M}^{\infty} a_n &= a_M + a_{M+1} + a_{M+2} + a_{M+3} + \dots \\ &= a_M + a_{M+1} + a_{M+2} + a_{M+3} + \dots + a_{K-1} + a_K + a_{K+1} + \dots \\ &= [a_M + a_{M+1} + a_{M+2} + \dots + a_{K-1}] + \sum_{n=K}^{\infty} a_n.\end{aligned}$$

Thus the difference between the two series $\sum_{n=M}^{\infty} a_n$ and $\sum_{n=K}^{\infty} a_n$ is only the finite sum $[a_M + a_{M+1} + a_{M+2} + \dots + a_{K-1}]$. Therefore either both series (the one starting at M and the one starting at K) are finite (so they both converge) or they are both non-finite (so they both diverge).

Continuing with this idea, consider the following example. Suppose you are given that

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{2}.$$

Given this, we can compute the sum of other infinite series obtained from this one by changing the starting index. For example, if we want to figure out $\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n$, we think of this series as "the series, starting at $n = 3$". This is the same as the series, starting at $n = 1$, with the $n = 1$ and $n = 2$ terms removed (i.e. subtracted). So

$$\begin{aligned}
\sum_{n=3}^{\infty} \left(\frac{1}{3}\right)^n &= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^1 - \left(\frac{1}{3}\right)^2 \\
&= \frac{1}{2} - \frac{1}{3} - \frac{1}{9} \\
&= \frac{1}{18}.
\end{aligned}$$

Similarly, $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ is the same series starting at $n = 0$. This is the series, starting at $n = 1$, with the $n = 0$ term added on. So

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n &= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{1}{3}\right)^0 \\
&= \frac{1}{2} + 1 \\
&= \frac{3}{2}.
\end{aligned}$$

Here are some more examples illustrating this concept:

Example 1.5. *Throughout this example, you are to assume that*

$$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = 3.$$

Using this fact, find the sum of each of the following series:

(a) $\sum_{n=4}^{\infty} \left(\frac{3}{4}\right)^n$

Solution: *Proceeding as above, we see we need to subtract the* $n = 1$, $n = 2$

and $n = 3$ terms from what we are given.

$$\begin{aligned}\sum_{n=4}^{\infty}\left(\frac{3}{4}\right)^n &= \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n - \left(\frac{3}{4}\right)^1 - \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 \\ &= 3 - \frac{3}{4} - \frac{9}{16} - \frac{27}{64} \\ &= \frac{81}{64}.\end{aligned}$$

(b) $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

Solution: *This problem has two steps: first, factor out the 2. Then, to get to the series that is left, we need to add in the $n = 0$ term:*

$$\begin{aligned}\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n &= 2\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \\ &= 2\left[\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n + \left(\frac{3}{4}\right)^0\right] \\ &= 2\left[3+1\right] \\ &= 8.\end{aligned}$$

(c) $\sum_{n=1}^{\infty} \frac{3^{n+2}}{4^n}$

Solution: *Here the starting index is unchanged, but we need to do a bit of algebra to sum the series:*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{3^{n+2}}{4^n} &= \sum_{n=1}^{\infty}\frac{3^n 3^2}{4^n} \\ &= 9\sum_{n=1}^{\infty}\frac{3^n}{4^n} \\ &= 9\cdot 3 \\ &= 27.\end{aligned}$$

(d) $\sum_{n=2}^{\infty} \frac{3^{n+3}}{4^{n-1}}$

Solution: *We start as in Example (c) above, factoring out terms. Then because the starting index is $n = 2$, we need to subtract the $n = 1$ term from the series we were given:*

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{3^{n+3}}{4^{n-1}} &= \sum_{n=2}^{\infty} \frac{3^n 3^3}{4^n 4^{-1}} \\
&= \frac{3^3}{4^{-1}} \sum_{n=2}^{\infty} \frac{3^n}{4^n} \\
&= 27 \cdot 4 \left[\sum_{n=1}^{\infty} \frac{3^n}{4^n} - \left(\frac{3}{4}\right)^1 \right] \\
&= 27 \cdot 4 \left[3 - \frac{3}{4} \right] \\
&= 27 \cdot 4 \cdot \frac{13}{4} = 351.
\end{aligned}$$

1.2.2 Changing indices

To motivate the ideas of this section, consider these two series:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n+1}.$$

At first glance, these series look different: after all, they have a different starting index and a different formula for the terms. But, if you write these series out, you obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

and

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...,$$

i.e. these series are adding exactly the same numbers (so they are the same series)! We say that these are the "same series, but with a different starting index".

An important technique to master when working with series is the ability to change the starting index of a series. For example, suppose we are given the series

$$\sum_{n=3}^{\infty} \frac{(n+2)^2}{2^n},$$

and we want to write this series as a series with starting index 0. To do this, we will change the indexing variable of the series from n to some other letter (I'll choose k, but the letter you choose doesn't matter). I want to rewrite this series in terms of k where the initial term corresponds to $k = 0$. To do this, I will first find an equation that relates k and n. If the values of n which I am adding over are $n = 3, 4, 5, 6, 7, 8, ...$ and the values of k I want to add over are $0, 1, 2, 3, ...$, then a reasonable equationrelating k and n is $k = n - 3$. So let's start by letting $k = n - 3$.

Then, by solving the equation $k = n - 3$ for n, we see that $n = k + 3$. At this point, we can take the series we are given and replace every n in the series (including the one under the Σ with $k + 3$. This rewrites the original series as

$$\sum_{k+3=3}^{\infty} \frac{((k+3)+2)^2}{2^{k+3}}.$$

Now the expression under the Σ can be rewritten as $k = 0$ (by subtracting 3 from both sides), and some simplification can be done inside the Σ to obtain

$$\sum_{k=0}^{\infty} \frac{(k+5)^2}{2^{k+3}}.$$

This is the series, rewritten with starting index 0.

Warning: The only kinds of index-changing substitutions that are allowed in the world of infinite series are substitutions of the form $k = n\pm$ a constant. You are not allowed to set $k = 2n$ or $k = n^2$, etc., because the ks have to count upward by 1 as the ns increase by 1 (otherwise you introduce new terms to the series or delete some terms that were previously there, changing the sum).

The previous example is pretty easy to understand given the equation $k = n - 3$. Unfortunately, mathematicians often use the same letter for both the old index of the series and the new index. For example, you might see the following thing written, without any justification:

$$\sum_{n=2}^{\infty} \frac{4(n-1)^2}{n^8 6^{n-5}} = \sum_{n=0}^{\infty} \frac{4(n+1)^2}{(n+2)^8 6^{n-3}} \tag{1.1}$$

How did the person who wrote this go from the sum on the left to the sum on the right? First of all, it is important to understand that the two series above are in fact the same series, because if you write them out, you will find that their terms are the same. But, the n on the left-hand side is not the same as the n on the right-hand side. The n on the right-hand side is something like the k of the previous example.

The way you go from the sum on the left to the sum on the right is as follows:

First, let $k = n - 2$. Then $n = k + 2$ so

$$\sum_{n=2}^{\infty} \frac{4(n-1)^2}{n^8 6^{n-5}} = \sum_{k+2=2}^{\infty} \frac{4((k+2)-1)^2}{(k+2)^8 6^{(k+2)-5}} = \sum_{k=0}^{\infty} \frac{4(k+1)^2}{(k+2)^8 6^{k-3}}.$$

If you replace the ks on the right with ns, then you get the series written in Equation (1.1).

Another way of thinking about the same example is to think of the n on the left-hand side of Equation (1.1) as "old n" and the n on the right-hand side of Equation (1.1) as "new n". You obtain the right-hand side from the left by the substitution "new n = old n - 2". I think it is easier to use a different letter, however.

Here are some more examples of this technique:

Example 1.6. *For each series, rewrite the series so that it begins with the specified starting index:*

(a) $\sum_{n=5}^{\infty} \frac{2^{n-3}}{3^{2n-3}}$*; starting index* 2

Solution: *Since we want the starting index to go from* 5 *to* 2 *(which is* 3 *less than* 5*), we let* $k = n - 3$*. Then* $n = k + 3$ *so our series is*

$$\sum_{k+3=5}^{\infty} \frac{2^{k+3-3}}{3^{2(k+3)-3}} = \sum_{k=2}^{\infty} \frac{2^k}{3^{2k+3}}.$$

(b) $\sum_{n=0}^{\infty} \frac{3}{n^4+2n}$*; starting index* 3

Solution: *Since we want the starting index to go from* 0 *to* 3 *(which is* 3 *more than* 0*), we let* $k = n + 3$*. Then* $n = k - 3$ *so our series is*

$$\sum_{k-3=0}^{\infty} \frac{3}{(k-3)^4 + 2(k-3)} = \sum_{k=3}^{\infty} \frac{3}{(k-3)^4 + 2k - 6}.$$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$*; starting index* 0

Solution: *Since we want the starting index to go from* 1 *to* 0 *(which is* 1 *less than* 1*), we let* $k = n - 1$*. Then* $n = k + 1$ *so our series is*

$$\sum_{k+1=1}^{\infty} \frac{(-1)^{k+1+1}}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+2}}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^2}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}.$$

1.3 Geometric series

The most important class of infinite series are *geometric series*:

Definition 1.4. *A series* $\sum a_n$ *is called* **geometric** *if there exists a real number* r *such that* $a_{n+1} = ra_n$ *for all* n*. The number* r *is called the* **common ratio** *of the series.*

The reason for using the term "common ratio" is that for a geometric series $\sum a_n$, the equation $a_{n+1} = ra_n$ can be rewritten as $\frac{a_{n+1}}{a_n} = r$; therefore, the following ratios are all equal (i.e. "common"):

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = r.$$

Suppose the initial index of the geometric series is $n = 1$. Then, by repeated application of the formula $a_{n+1} = ra_n$, we see:

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= ra_1 \\
a_3 &= ra_2 = r(ra_1) = r^2 a_1 \\
a_4 &= ra_3 = r(ra_2) = r(r(ra_1)) = r^3 a_1 \\
&\vdots \\
a_n &= r^{n-1} a_1 \\
&\vdots
\end{aligned}$$

Therefore, the geometric series $\sum_{n=1}^{\infty} a_n$ can be rewritten as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n &= a_1 + a_2 + a_3 + a_4 + \ldots \\
&= a_1 + ra_1 + r^2 a_1 + r^3 a_1 + \ldots \\
&= a_1 \left(1 + r + r^2 + r^3 + r^4 + \ldots\right) \\
&= a_1 \sum_{n=0}^{\infty} r^n. \\
&= \sum_{n=0}^{\infty} a_1 r^n.
\end{aligned}$$

If we change notation and call the number a_1 just "a", we get the following important characterization of geometric series:

Characterization of geometric series: Every infinite geometric series can be written in the standard form

$$\sum_{n=0}^{\infty} ar^n$$

where a is the initial term of the series and r is the common ratio of the series.

In other words, **every geometric series is the sum of a constant times all the nonnegative powers of the common ratio**.

Notice that if you are given a geometric series, irrespective of what the starting index of the series was, you can always rewrite the series in the standard form given in the yellow box above, where the starting index is $n = 0$. To study a geometric series, your first step should almost always be to rewrite the series in this standard form. To do this, simply write the terms out and factor out the initial term of the series, as is done in these examples:

Example 1.7. *For each series, determine if the series is geometric. If it is, write the series in standard form and identify the common ratio of the series.*

(a) $\sum_{n=3}^{\infty} \frac{1}{3^n}$.

Solution: *Write the series out to get*

$$\sum_{n=3}^{\infty} \frac{1}{3^n} = \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} + \ldots$$

Now factor out the initial term of the series (in this case $\frac{1}{3^3}$) to get

$$\frac{1}{3^3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\right)$$

and think of all the fractions as powers of the same number:

$$\frac{1}{3^3}\left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots\right]$$

Writing this in $\Sigma-$notation, we get

$$\sum_{n=0}^{\infty} \frac{1}{3^3} \left(\frac{1}{3}\right)^n.$$

Thus this is geometric (because it is of the form $\sum_{n=0}^{\infty} ar^n$) and the common ratio is $r = \frac{1}{3}$.

(b) $72 - 36 + 18 - 9 + \frac{9}{2} - \frac{9}{4} + \frac{9}{8} - \frac{9}{16} + ...$

Solution: *As with the previous example, factor out the initial term of the series and see if what is left consists of all powers of the same number:*

$$\begin{aligned}
&72 - 36 + 18 - 9 + \frac{9}{2} - \frac{9}{4} + \frac{9}{8} - \frac{9}{16} + ... \\
&= 72\left[1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ...\right] \\
&= 72\left[1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^4 + \left(\frac{-1}{2}\right)^5 + ...\right] \\
&= 72\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \\
&= \sum_{n=0}^{\infty} 72\left(\frac{-1}{2}\right)^n.
\end{aligned}$$

Thus this series is geometric and the common ratio is $r = \frac{-1}{2}$.

(c) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...$

Solution: *As with the previous examples, factor out the initial term. This gives:*

$$1\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ...\right].$$

Now if this series is geometric, what appears inside the bracket has to be (once the initial term is factored out) $1+r+r^2+r^3+r^4+...$. Therefore the only possible value of r is $\frac{1}{4}$. But the next term added is $\frac{1}{9}$, which is not $\left(\frac{1}{4}\right)^2$. Therefore this series is not geometric.

Recall that our "big picture" questions with infinite series include determining which infinite series converge, and finding the sum of a convergent infinite series. We can answer these questions now for geometric series; the content of the answers is what is called the Geometric Series Test:

Theorem 1.5 (Geometric Series Test)**.** *Consider a geometric series written in standard form $\sum_{n=0}^{\infty} ar^n$. Then:*

1. *The series converges if and only if $|r| < 1$ (or if $a = 0$).*
2. *The series diverges if and only if $|r| \geq 1$.*

Furthermore, if the series converges, its sum is $\frac{a}{1-r}$.

Proof of the Geometric Series Test: We will prove this result by considering several different cases:

First of all, if $a = 0$, then the series is just

$$\sum_{n=0}^{\infty} 0r^n = \sum_{n=0}^{\infty} = 0 + 0 + 0 + 0 + ... = 0 = \frac{0}{1-r},$$

so the theorem is true in this case. (Admittedly, this is a dumb situation.) Henceforth, we assume $a \neq 0$.

Second, assume $a \neq 0$ and $r = 1$. Then the series is

$$\sum_{n=0}^{\infty} a1^n = \sum_{n=0}^{\infty} a$$

and the N^{th} partial sum of the series is $S_N = a + a + a + ... + a = a(N+1)$ (because in the N^{th} partial sum, if the indexing starts at $n = 0$, there are $N + 1$ terms being added). Thus $\lim_{N\to\infty} S_N = \lim_{N\to\infty} a(N+1) = a \cdot \infty$. Since the limit of the partial sums does not exist, the geometric series diverges if $r = 1$ and $a \neq 0$.

Third, assume $a \neq 0$ and $r = -1$. Then the series is

$$\sum_{n=0}^{\infty} a(-1)^n = a - a + a - a + a - a + a - a + a...$$

and the N^{th} partial sum of this series is 0 if N is even (since there are the same number of as added as subtracted) and the N^{th} partial sum of this series is a if N is odd (since there is one more a being added than subtracted). Thus the partial sums

S_N oscillate between 0 and a, so $\lim_{N\to\infty} S_N$ does not exist, so the geometric series diverges if $r = -1$ and $a \neq 0$.

Fourth, assume $a \neq 0$ and $|r| \neq 1$. Then we directly compute the N^{th} partial sum of the series:

$$S_N = a_0 + a_1 + a_2 + ... + a_N \tag{1.2}$$

$$S_N = ar^0 + ar^1 + ar^2 + ar^3 + ... + ar^N \tag{1.3}$$

Now for a trick: multiply through both sides of the equation by r to get

$$r\,S_N = r\left[ar^0 + ar^1 + ar^2 + ar^3 + ... + ar^N\right] \tag{1.4}$$

$$r\,S_N = ar^1 + ar^2 + ar^3 + ar^4 + ... + ar^N + ar^{N+1}. \tag{1.5}$$

Now take Equation (1.3) and subtract Equation (1.5) to get

$$S_N - r\,S_N = ar^0 + ar^1 + ar^2 + ... + ar^N - \left(ar^1 + ar^2 + ar^3 + ... + ar^N + ar^{N+1}\right).$$

On the left-hand side, factor out S_N. On the right-hand side, all the terms cancel except for ar^0 (which is added but not subtracted) and ar^{N+1} (which is subtracted but not added. We obtain

$$S_N(1-r) = ar^0 - ar^{N+1} = a(1-r^{N+1}).$$

Divide through by $1-r$ (recall that we are now assuming that $r \neq 1$ so we are definitely not doing the illegal operation of dividing by zero here) to obtain

$$S_N = \frac{a(1-r^{N+1})}{1-r}. \tag{1.6}$$

Finally, to determine convergence or divergence of the series, take the limit of S_N as $N \to \infty$. We see that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} = \frac{a}{1-r}\lim_{N\to\infty}\left[1-r^{N+1}\right].$$

Now if $|r| < 1$, then as $N \to \infty$, $r^{N+1} \to 0$ so $\lim_{N\to\infty} S_N = \frac{a}{1-r}$, i.e. the series converges to $\frac{a}{1-r}$ as desired. On the other hand, if $|r| > 1$, then as $N \to \infty$, $r^{N+1} \to \infty$ so $\lim_{N\to\infty} S_N$ does not exist, and by definition the series diverges.

This completes the proof of the Geometric Series Test. □

If $|r| < 1$, then the Geometric Series Test says that $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$; factoring out a from both sides and then dividing by a yields the formula

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \text{ if } |r| < 1.$$

Notice that in order for this equation to hold, the starting index of the geometric series must be $n = 0$. The content of the Geometric Series Test can therefore be restated as follows:

Geometric Series Test (restated):

$$\sum_{n=0}^{\infty} r^n \begin{cases} = \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

In the context of proving the Geometric Series Test, we proved a formula (given in Equation (1.6) for the partial sums of a geometric series. We restate this as a theorem, as this result will be used in examples that follow. In particular, this formula holds in any situation where $r \neq 1$ (even if $|r| > 1$):

Theorem 1.6 (Finite Sum Formula for a Geometric Series)**.** *Consider a geometric series written in standard form $\sum_{n=0}^{\infty} ar^n$ where $r \neq 1$. Then the N^{th} partial sum satisfies*

$$a[1 + r + r^2 + r^3 + ... + r^N] = \frac{a(1 - r^{N+1})}{1 - r}.$$

In particular, if $a = 1$, the above formula reduces to

$$1 + r + r^2 + r^3 + ... + r^N = \frac{1 - r^{N+1}}{1 - r}.$$

Note: The finite and infinite sum formulas for geometric series are only valid if the starting index of the series is $n = 0$. If the starting index is nonzero, then you first need to do some algebra and perform an index change as in Section 1.2 to write the series in the appropriate form. For example, suppose you are given the series

$$\sum_{n=3}^{\infty} ar^n.$$

By letting "new n" = "old $n - 3$", we can change indices to rewrite this as

$$\sum_{n=0}^{\infty} ar^{n+3}.$$

Now use exponent rules to rewrite the series as

$$\sum_{n=0}^{\infty} (ar^3)r^n.$$

The expression inside the parentheses (i.e. ar^3) is the "a" being referred to in the geometric series summation formula. This sum therefore sums to

$$\frac{ar^3}{1-r}.$$

In general, it turns out that if $|r| < 1$, then $\sum_{n=K}^{\infty} ar^n = \frac{ar^K}{1-r}$. You can either memorize this formula or rewrite series one at a time (I tend to do the latter in examples).

Let us now see how these ideas are applied in purely mathematical examples:

Example 1.8. *For each finite or infinite geometric series, find the sum (if the series diverges, say so):*

(a) $\frac{4}{3} + \frac{4}{27} + \frac{4}{3^5} + \frac{4}{3^7} + ...$

Solution: *Start by writing the series in standard form. Factor out the first term to get*

$$\frac{4}{3}\left(1 + \frac{1}{9} + \frac{1}{3^4} + \frac{1}{3^6} + ...\right)$$

and then rewrite the terms inside the parentheses to get

$$\frac{4}{3}\left[1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + ...\right]$$

so this series, written in standard form, is

$$\sum_{n=0}^{\infty} \frac{4}{3} \cdot \left(\frac{1}{9}\right)^n.$$

Thus $r = \frac{1}{9}$; since $|r| < 1$ this series converges by the Geometric Series Test and its sum is

$$\frac{a}{1-r} = \frac{\frac{4}{3}}{1-\frac{1}{9}} = \frac{\frac{4}{3}}{\frac{8}{9}} = \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}.$$

(b) $125 - 25 + 5 - 1 + \frac{1}{5} - \frac{1}{25} + \frac{1}{125} - ...$

Solution: *Proceed as in the preceding example:*

$$\begin{aligned}
125 - 25 + 5 - 1 + \frac{1}{5} - \frac{1}{25} + ... &= 125\left[1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + ...\right] \\
&= 125\left[1 + \left(\frac{-1}{5}\right) + \left(\frac{-1}{5}\right)^2 + \left(\frac{-1}{5}\right)^3 + ...\right] \\
&= \sum_{n=0}^{\infty} 125\left(\frac{-1}{5}\right)^n.
\end{aligned}$$

Thus $r = \frac{-1}{5}$ *so since* $|r| < 1$ *this infinite series converges by the Geometric series test; the sum is*

$$\frac{a}{1-r} = \frac{125}{1-\left(-\frac{1}{5}\right)} = \frac{125}{\frac{6}{5}} = \frac{125 \cdot 6}{5} = 150.$$

(c) $2 + 6 + 18 + 54 + ...$

Solution: *Proceed as in examples (a) and (b):*

$$\begin{aligned}
2 + 6 + 18 + 54 + ... &= 2\left[1 + 3 + 9 + 27 + 81 + ...\right] \\
&= \sum_{n=0}^{\infty} 2 \cdot 3^n.
\end{aligned}$$

This time, since $r = 3$, $|r| > 1$ *so the series diverges by the Geometric Series Test.*

(d) $\sum_{n=1}^{\infty} \frac{5 \cdot 26^{n+3}}{3 \cdot 7^{2n-1}}$

Solution: *Use some algebra to rewrite the series in standard form:*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{5\cdot 26^{n+3}}{3\cdot 7^{2n-1}} &= \frac{5}{3}\sum_{n=1}^{\infty}\frac{26^3 26^n}{7^{2n}7^{-1}}\\ &= \frac{5\cdot 26^3}{3\cdot 7^{-1}}\sum_{n=1}^{\infty}\frac{26^n}{49^n}\\ &= \frac{5\cdot 26^3\cdot 7}{3}\sum_{n=1}^{\infty}\left(\frac{26}{49}\right)^n\\ &= \frac{35\cdot 26^3}{3}\sum_{n=0}^{\infty}\left(\frac{26}{49}\right)^{n+1}\\ &= \frac{35\cdot 26^3}{3}\left(\frac{26}{49}\right)\sum_{n=0}^{\infty}\left(\frac{26}{49}\right)^n.\end{aligned}$$

Now we have a geometric series with initial index $n = 0$ and common ratio $r = \frac{26}{49}$. Thus by the Geometric Series Test this series converges and the sum is

$$\frac{35\cdot 26^3}{3}\left(\frac{26}{49}\right)\left(\frac{1}{1-\frac{26}{49}}\right) = \frac{35\cdot 26^4}{3\cdot 49}\left(\frac{49}{23}\right) = \frac{35\cdot 26^4}{3\cdot 23}.$$

(e) $\sum_{n=3}^{19}\left(\frac{3}{8}\right)^n$

Solution: *Notice first that this is a* ***finite*** *geometric series; therefore the words "converge" and "diverge" don't really apply. This expression, being a finite series, will definitely work out to be a number. To evaluate this, first rewrite the series using some algebra and a change of index so that the first term corresponds to $n = 0$:*

$$\sum_{n=3}^{19}\left(\frac{3}{8}\right)^n = \sum_{n=0}^{16}\left(\frac{3}{8}\right)^{n+3} = \sum_{n=0}^{16}\left(\frac{3}{8}\right)^3\left(\frac{3}{8}\right)^n.$$

The sum is then evaluated by the Finite Sum Formula for Geometric Series, with $a = \left(\frac{3}{8}\right)^3$, $N = 16$ and $r = \frac{3}{8}$:

$$\sum_{n=0}^{16}\left(\frac{3}{8}\right)^3\left(\frac{3}{8}\right)^n = a\frac{1-r^{N+1}}{1-r} = \left(\frac{3}{8}\right)^3\frac{1-\left(\frac{3}{8}\right)^{17}}{1-\frac{3}{8}} = \left(\frac{3}{8}\right)^3\left(\frac{8}{5}\right)\left(1-\left(\frac{3}{8}\right)^{17}\right).$$

(f) $\sum_{n=1}^{12}(-3)^n$

Solution: Again, this is a finite geometric series. Rewrite the series so that the first index is $n = 0$, and then apply the Finite Sum Formula for Geometric Series:

$$\sum_{n=1}^{12}(-3)^n = \sum_{n=0}^{11}(-3)^{n+1} = \sum_{n=0}^{11}(-3)(-3)^n = (-3)\frac{1-(-3)^{12}}{1-(-3)} = \frac{-3}{4}\left[1-(-3)^{12}\right].$$

1.3.1 Applications of geometric series

Geometric series are by far the most important class of infinite series, because they appear naturally in many different situations, both inside and outside mathematics. We touch on some various applications of the ideas expressed in the previous section here.

Example 1.9 (Finance)**.** *Suppose you invest* \$100 *at the start of each year into an account which pays* 7% *interest, compounded annually. How much do you have in the account after* 20 *years (before you make the deposit at the start of the* 21^{st} *year)?*

Solution: *To answer this question, we begin with a side problem:*

Side question: *Suppose you invest $\$P$ once into an account paying interest rate R, compounded once per time period. How much do you have after n time periods?*

Answer to side question: *Initially, you have $\$P$ in the account. After one time period, you have the principal $\$P$ plus the interest earned, which is PR. So after one time period, you have $P + PR = (1 + R)P$. Now after two time periods, you have the amount you had after one time period plus the interest earned on that money, which is*

$$(1 + R)P + R[(1 + R)P] = (1 + R)(1 + R)P = (1 + R)^2P.$$

Hopefully, at this point you can see the pattern. Every time period, the amount of money in the account gets multiplied by $1 + R$, so after n time periods, you have

$$(1 + R)^n\, P$$

dollars in the account.

Returning to the original problem, we solve the problem by breaking the money in the account into pieces depending on when the money gets deposited. For example, your initial investment of \$100 *will earn interest at a rate of* .07 20 *times, so after* 20 *years this investment will be worth* $100(1 + .07)^{20}$. *The* \$100 *you put into the account one year later will earn the interest only* 19 *times, so this money will eventually be worth* $100(1 + .07)^{19}$. *To summarize:*

$$\begin{array}{rccccl} & & & & & \textit{amount in account} \\ & & & & & \underline{\textit{after } 20 \textit{ years}} \\ \textit{initial investment:} & \$100 & - & 20 \textit{ years} & \to & \$100(1+.07)^{20} \\ \textit{investment after year 1:} & \$100 & - & 19 \textit{ years} & \to & \$100(1+.07)^{19} \\ \textit{investment after year 2:} & \$100 & - & 18 \textit{ years} & \to & \$100(1+.07)^{18} \\ & \vdots & & \vdots & & \vdots \\ \textit{investment after year 20:} & \$100 & - & 1 \textit{ year} & \to & \$100(1+.07)^{1} \end{array}$$

To get the total amount of money in the account, add the numbers in the right-hand column above to get

$$\begin{aligned} 100(1.07) + 100(1.07)^2 + ... + 100(1.07)^{20} &= \sum_{n=1}^{20} 100(1.07)^n \\ &= 100(1.07) \sum_{n=0}^{19} (1.07)^n \\ &= 100(1.07) \frac{1-(1.07)^{20}}{1-1.07} \\ &\approx \$4386.52. \end{aligned}$$

Notice that you actually only deposited $\$100 \cdot 20 = \$2000$ *into the account, so over the* 30 *years you more than doubled your money.*

Example 1.10 (Repeating Decimals)**.** *Write the repeating decimal* 3.10454545... *as a fraction in lowest terms.*

Solution: *Notice that we can rewrite this decimal as the sum of infinitely many fractions as follows:*

$$\begin{aligned} 3.10454545... &= 3.10 + .0045 + .000045 + .00000045 + ... \\ &= \frac{31}{10} + \frac{45}{10^4} + \frac{45}{10^6} + \frac{45}{10^8} + ... \\ &= \frac{31}{10} + \sum_{n=2}^{\infty} \frac{45}{10^{2n}} \\ &= \frac{31}{10} + 45 \sum_{n=2}^{\infty} \frac{1}{100^n} \\ &= \frac{31}{10} + 45 \frac{1}{100^2} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n \end{aligned}$$

The series on the right converges by the Geometric Series Test to $\frac{1}{1-\frac{1}{100}} = \frac{100}{99}$*, so the entire expression evaluates to*

$$\frac{31}{10} + 45\frac{1}{100^2}\left(\frac{100}{99}\right) = \frac{31}{10} + \frac{45}{9900} = \frac{31 \cdot 990 + 45}{9900} = \frac{683}{220}.$$

Example 1.11 (Pharmacokinetics)**.** *Suppose a patient takes* 150 *mg of a drug each day. Suppose that in any one day, the patient's bodily functions removes* 85% *of the drug from the patient's system. How much of the drug will the patient have in her system after 30 days?*

Solution: *If the patient's bodily functions remove* 85% *of the drug from the patient's system, then* 15% *of the drug remains daily. Let's see how this can be used to keep track of how much of the drug is in the patient's body.*

After one day, the patient has taken 150 *mg of the drug, and* 15% *of it remains at the end of the day. So there is* $(.15)(150)$ *mg in the patient's body.*

At the start of the next day, the patient takes her dose of medicine, so she starts the next day with $(.15)(150)+150$ *mg of the drug in her body. At the end of the next day, there is* $.15[(.15)(150)+150] = (.15)^2(150)+(.15)(150)$ *mg of the drug in her body.*

Over the course of each day, the amount of the drug in the patient's body first is increased by 150 *(when she takes her dose), and then is multiplied by* .15 *(to account for the removal of much of the drug by her body functions). So after the third day, she has*

$$\begin{aligned} .15[(\textit{amount she had after day 2})+150] &= .15[(.15)^2(150)+(.15)(150)+150] \\ &= (.15)^3(150)+(.15)^2(150)+(.15)(150). \end{aligned}$$

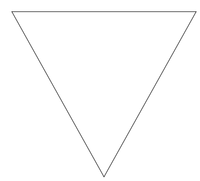
Hopefully you can see the pattern; after 30 *days the patient will have*

$$(.15)^{30}(150)+(.15)^{29}(150)+\ldots+(.15)^2(150)+(.15)(150) = \sum_{n=1}^{30}(.15)^n(150).$$

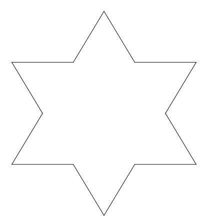
This sum evaluates to

$$\begin{aligned} \sum_{n=1}^{30}(.15)^n(150) &= 150\sum_{n=1}^{30}(.15)^n \\ &= 150(.15)\sum_{n=0}^{29}(.15)^n \\ &= 150(.15)\frac{1-(.15)^{30}}{1-.15} \approx 26.47 \textit{ mg}. \end{aligned}$$

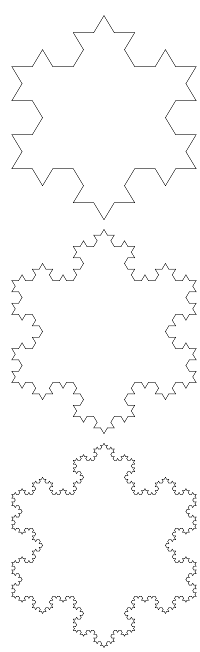
Example 1.12 (Fractal Geometry)**.** *Consider the* **Koch snowflake***, a figure constructed by the following iterative process: first, start with an equilateral triangle of side length* 1*;*



Second, divide each side of the trangle into three segments of equal length and attach an equilateral triangle to the middle of each segment. Then erase the middle of each segment. After doing this, you get the following figure:



Third, repeat this procedure indefinitely. This means that at each stage, you take each side of the figure, divide it into thirds, attach an equilateral triangle to the middle third of each segment, and the erase the middle of each previous segment. If you carry out this procedure, you get the following sequence of figures in the next three steps:



Repeating this procedure infinitely many times produces a figure called the Koch snowflake. The question is to find the perimeter of the Koch snowflake, and find the area enclosed by the Koch snowflake.

Solution: *We start by finding the perimeter. The perimeter of the original equilateral triangle is* 3 *(since it has three sides, each of length* 1*). Notice that at all stages of the iterative process that generates the Koch snowflake, every side of the figure has the same side. Suppose that after some stage this side length is* s*. At the next step, one third of this side will be deleted (removing length* $\frac{1}{3}s$*), but two new sides are added, each of length* $\frac{1}{3}s$*. So the length of each side goes from* s *to* $s - \frac{1}{3}s + 2\left(\frac{1}{3}s\right) = \frac{4}{3}s$*, i.e. the perimeter of the figure is multiplied by* $\frac{4}{3}$ *at each step. So:*

Initially:	*the perimeter is* 3
After 1 stage:	*the perimeter is* $\frac{4}{3} \cdot 3$
After 2 stages:	*the perimeter is* $\frac{4}{3}\left(\frac{4}{3} \cdot 3\right) = \left(\frac{4}{3}\right)^2 3$
After 3 stages:	*the perimeter is* $\left(\frac{4}{3}\right)^2 3$
$\vdots$	$\vdots$
After n *stages:*	*the perimeter is* $\left(\frac{4}{3}\right)^{n-1} 3$
$\vdots$	$\vdots$
$n \to \infty$	$n \to \infty$
$\vdots$	$\vdots$
Koch snowflake:	*perimeter is* $\lim_{n\to\infty} \left(\frac{4}{3}\right)^{n-1} 3 = \infty$.

Thus the perimeter of the Koch snowflake is infinite.

Now for the area. At each stage of the construction, we keep track of the amount of area being added to the Koch snowflake that wasn't previously there. To do this, we count the number of triangles added at each stage (this is equal to the number of edges of the figure at the previous stage), then figure the side length of each new triangle added, and then figure the new area added at each step by applying the area formula for an equilateral triangle ($A = \frac{s^2\sqrt{3}}{4}$ *where* s *is the side length):*

$$\begin{aligned} \textit{new area added} &= (\#\ \textit{of new } \Delta s \textit{ added}) \cdot (\textit{area of each new } \Delta \textit{ added}) \\ &= (\#\ \textit{of new } \Delta s \textit{ added}) \cdot \left(\frac{s^2\sqrt{3}}{4}\right) \\ &\qquad \textit{where } s = \textit{side length of new } \Delta \textit{ added} \end{aligned}$$

	# of new each Δ added	*# of edges in figure*	*side length of Δ added*	*total area added*
initially	*1*	*3*	*1*	$\frac{1^2\sqrt{3}}{4} = \frac{\sqrt{3}}{4}$
after step 1	*3*	*12*	$\frac{1}{3}$	$3 \cdot \frac{(1/3)^2\sqrt{3}}{4}$
after step 2	*12*	*48*	$\frac{1}{9}$	$12 \cdot \frac{(1/9)^2\sqrt{3}}{4}$
after step 3	*48*	$48 \cdot 4$	$\frac{1}{27}$	$48 \cdot \frac{(1/27)^2\sqrt{3}}{4}$
after step 4	$48 \cdot 4$	$48 \cdot 4 \cdot 4$	$\frac{1}{81}$	$(48 \cdot 4) \cdot \frac{(1/81)^2\sqrt{3}}{4}$
⋮	⋮	⋮	⋮	⋮

Therefore we get the total area of the Koch snowflake by adding up the numbers in the right-most column above:

$$\begin{aligned}
\textit{total area} &= \frac{\sqrt{3}}{4} + 3 \cdot \frac{(1/3)^2\sqrt{3}}{4} + 12 \cdot \frac{(1/9)^2\sqrt{3}}{4} + 48 \cdot \frac{(1/27)^2\sqrt{3}}{4} + \ldots \\
&= \frac{\sqrt{3}}{4} + 3 \cdot \left(\frac{1}{3}\right)^2 \frac{\sqrt{3}}{4} + 3(4) \cdot \left(\frac{1}{9}\right)^2 \frac{\sqrt{3}}{4} + 3(4^2) \cdot \left(\frac{1}{27}\right)^2 \frac{\sqrt{3}}{4} + \ldots \\
&= \frac{\sqrt{3}}{4} + \sum_{n=0}^{\infty} 3(4^n) \cdot \left(\frac{1}{3^{n+1}}\right)^2 \frac{\sqrt{3}}{4} \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} (4^n) \cdot \frac{1}{3^{2n+2}} \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \frac{1}{9} \left(\frac{4}{9}\right)^n \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \left[\frac{1}{1 - \frac{4}{9}}\right] \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \left[\frac{9}{5}\right] = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}.
\end{aligned}$$

Interestingly, the Koch snowflake is an example of a curve of infinite length that *surrounds a region of finite area.*

1.4 Determining whether or not a positive series converges

As stated earlier, one of our "big picture" questions with infinite series is to determine, given some infinite series, whether or not that series converges or diverges. The next four sections are devoted to this topic; over these four sections we will develop various "tests" which allow us to draw conclusions about the convergence/divergence of an infinite series. The general principle at work is this: given a series $\sum a_n$, you first need to determine an appropriate "test" for the series, then apply that test to find out whether or not $\sum a_n$ converges.

The most important thing to take into account when determining what test is most appropriate for studying a given series is the signs of the individual terms of the series. This section deals with tests which apply only to series where all the terms are positive. As such, we start with a definition which describes these series:

Definition 1.5. *An infinite series is called* **positive** *if all its terms are positive; more precisely $\sum a_n$ is* **positive** *if $a_n \geq 0$ for all n.*

Definition 1.6. *An infinite series is called* **negative** *if all its terms are negative; more precisely $\sum a_n$ is* **negative** *if $a_n \leq 0$ for all n.*

Example 1.13. *Classify the following series as positive, negative, or neither:*

(a) $\sum_{n=2}^{\infty} \frac{3}{n^2+3n}$

Solution: *Since $\frac{3}{n^2+3n} \geq 0$ for all $n \geq 2$, this series is positive.*

(b) $\sum \frac{-3}{2+\cos n}$

Solution: *Since $-1 \leq \cos n \leq 1$, $2 + \cos n \geq 1 > 0$ for all n. Since the denominator of each term is positive but the numerator of each term is negative, each term is negative so the series is negative.*

(c) $\sum \frac{(-1)^n}{n^8+2^n}$

Solution: *The denominator of each term is positive, but the numerator is sometimes positive and sometimes negative (depending on whether n is even or odd). Thus this series has some positive terms and some negative terms, so the series is neither positive nor negative.*

We now turn to tests which allow us to determine whether or not a positive series converges. An important remark is that negative series can be treated as positive series by factoring out a (-1) from the series. For example, to determine whether the negative series

$$\sum_{n=1}^{\infty} \frac{-1}{n}$$

converges, we could write it as

$$\sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}.$$

Now if the positive series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges, so does $\sum_{n=1}^{\infty} \frac{-1}{n}$ (by linearity), and if the positive series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{-1}{n}$ (again, by linearity). The punchline is:

General technique to study convergence/divergence of a negative series: Given a negative series, factor out -1 from the series, i.e. write

$$\sum(-a_n) = -\sum a_n.$$

The series $\sum a_n$ is a positive series; use one of the tests outlined in this section to determine whether or not it converges. By linearity, the series $\sum(-a_n)$ has the same behavior.

1.4.1 The Integral Test and $p-$series Test

Our first test for convergence illustrates a connection between trying to add up terms of a positive series and integrating a function over an unbounded interval:

Theorem 1.7 (Integral Test)**.** *Suppose* $\sum_{n=0}^{\infty} a_n$ *is a positive series and* f *is a function with the following properties:*

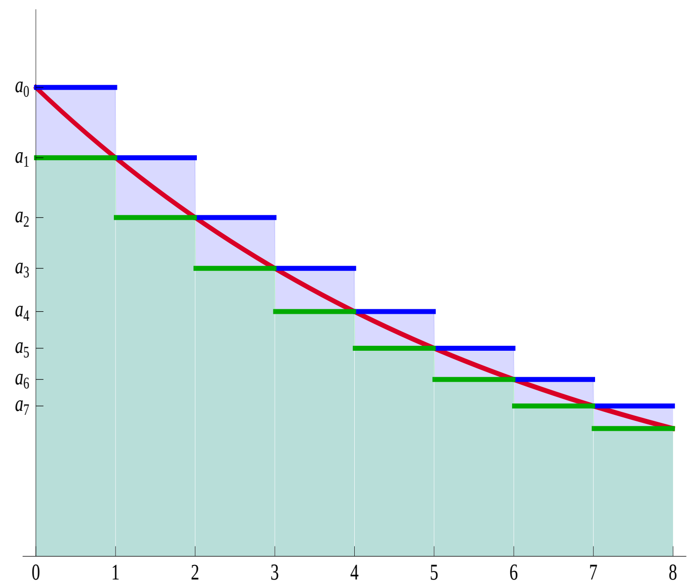
1. $f(n) = a_n$ *for all values of* n*; and*
2. f *is decreasing.*

Then:

$$\text{if } \int_0^\infty f(x)\,dx \text{ converges, then } \sum_{n=0}^{\infty} a_n \text{ converges, and}$$

$$\text{if } \int_0^\infty f(x)\,dx \text{ diverges, then } \sum_{n=0}^{\infty} a_n \text{ diverges.}$$

Proof of the Integral Test: Suppose we have a function f which is positive and decreasing, such that $f(n) = a_n$ for $n = 0, 1, 2, 3, ...$ Consider the following picture, which shows the graph of f in red:



It is clear from the picture that

$$\text{green area} \leq \int_0^\infty f(x)\,dx \leq \text{green area} + \text{blue area}. \tag{1.7}$$

Now let's figure the area of the green shaded region. This can be subdivided into rectangles of width 1 by drawing vertical line segments from the $x-$axis up to the top of the green area at each integer. If you do this, you will find that (by reading the heights of the rectangles off of the scale on the $y-$axis)

- the area of the first green rectangle is its height times its width, i.e. is $a_1 \cdot 1 = a_1$;
- the area of the second green rectangle is $a_2 \cdot 1 = a_2$;
- the area of the third green rectangle is $a_3 \cdot 1 = a_3$; etc.

Thus the total green area is $a_1 + a_2 + a_3 + ... = \sum_{n=1}^{\infty} a_n$.

Now let's figure the total area of the blue and green regions. As with just the green regions, the combined blue and green regions can be divided into rectangles of width 1. This time, however,

- the area of the left-most rectangle (whose bottom part is green but whose top part is blue) is its height times its width, i.e. is $a_0 \cdot 1 = a_0$;
- the area of the second rectangle is $a_1 \cdot 1 = a_1$;
- the area of the third rectangle is $a_2 \cdot 1 = a_2$; etc.

Thus the total combined blue and green area is $a_0 + a_1 + a_2 + ... = \sum_{n=0}^{\infty} a_n$.

Plugging these computations into equation (1.7), we see that

$$\sum_{n=1}^{\infty} a_n \leq \int_0^{\infty} f(x)\, dx \leq \sum_{n=0}^{\infty} a_n.$$

From this inequality we can prove the theorem. First, assume that $\int_0^\infty f(x)\, dx$ converges. This means that the green shaded area, being less than the finite number $\int_0^\infty f(x)\, dx$, is also finite, i.e. $\sum_{n=1}^{\infty} a_n$ converges. Since the starting index of a series is irrelevant to whether or not it converges, $\sum_{n=0}^{\infty} a_n$ converges as well.

Now assume that $\int_0^\infty f(x)\, dx$ diverges, i.e. that the area under the red function is infinite. This means that the combined green and blue shaded area, being greater than the area under the function $f(x)$ (which is $\int_0^\infty f(x)\, dx$), must also be infinite. Therefore $\sum_{n=0}^{\infty} a_n$ diverges. This completes the proof of the Integral Test. $\square$

Important remarks about the Integral Test:

1. The Integral Test can only be applied to positive series, i.e. series whose terms are all positive. It cannot be used to study series which contain both addition and subtraction.

2. The easiest way to check that a function f is decreasing is to show that the derivative f' is negative.

3. In order to apply the Integral Test to study the convergence of some series $\sum a_n$, you need to be able to define a function f whose domain is all real numbers greater than any index in the sum and which "agrees" with the formula for a_n,

i.e. if you are given $\sum \frac{1}{n}$, your function would be $f(x) = \frac{1}{x}$, and if your given sum is $\sum \frac{1}{n^3+n^2+n+1}$, your function would be $f(x) = \frac{1}{x^3+x^2+x+1}$.

This means that if your series contains terms like $n!$, the Integral Test cannot be used (because you cannot define a function with an $x!$ in it whose domain contains non-integers).

4. Although the statement of the Integral Test above uses the starting index 0 for both the integral and the series, this is really not important (since the starting index of a series / lower limit of an improper integral over a horizontally unbounded region is irrelevant to whether the series/integral converges). One can draw the same conclusions using any starting index for either the series or integral.

5. Suppose f is a decreasing, positive function with $f(n) = a_n$ for all n. It is important to know that while the Integral Test ensures that the series $\sum a_n$ and the integral $\int_0^\infty f(x)\,dx$ either both converge or both diverge, in the situation where both converge **the series and the integral do not generally converge to the same number**. For example, if you compute that $\int_0^\infty f(x)\,dx = 3$, that does not mean that $\sum_{n=0}^\infty a_n = 3$ (it does mean that the sum converges to **something**).

6. Despite the fact that this is the first test presented in these lecture notes, the Integral Test should be used only as a last resort. Usually, there are other techniques that do not involve the computation of some improper integral.

Here are some examples which illustrate the use of the Integral Test:

Example 1.14. *Determine whether or not the following series converge or diverge:*

(a) $\sum_{n=1}^\infty \frac{3}{n^2+1}$

Solution: *Let* $f(x) = \frac{3}{x^2+1}$. *Notice that* $f(n) = a_n$ *for* $n = 1, 2, 3, \ldots$. *Since* $f'(x) = 3(x^2+1)^{-2} \cdot 2x = \frac{-6x}{(x^2+1)^2} < 0$ *for all* $x > 0$, f *is decreasing. Now*

$$\begin{aligned}\int_1^\infty \frac{3}{x^2+1}\,dx &= \lim_{b\to\infty} \int_1^b \frac{3}{x^2+1}\,dx \\ &= \lim_{b\to\infty} [3\arctan x]_1^b \\ &= \lim_{b\to\infty} [3\arctan b - 3\arctan 1] \\ &= 3 \cdot \frac{\pi}{2} - 3 \cdot \frac{\pi}{4} < \infty.\end{aligned}$$

Since this integral evaluates to a finite number, it converges. Therefore, by the Integral Test, $\sum_{n=1}^{\infty} \frac{3}{n^2+1}$ *converges.*

Notice that the solution to this example contains **much more than simply the statement that the series converges**. It contains a complete explanation of the logic necessary to draw the conclusion that the series converges. In this case, since we are using the Integral Test, such logic includes a verification of all the hypotheses of the Integral Test (that f is decreasing, that $f(n) = a_n$ for all appropriate n, and that the integral $\int_1^{\infty} f(x)\,dx$ converges).

A problem which asks you to determine whether or not a series converges or diverges is called a *classification problem.* Here are some general principles that describe what your solution to a classification problem should look like:

General remarks on how to present arguments as to whether a series converges or diverges: If you are asked to determine whether a series $\sum a_n$ converges or diverges, your solution should have the following characteristics:

1. The last line of your argument should be a concluding statement, which states the answer to the question and the name of the test used (i.e. "$\sum a_n$ converges by the Integral Test").

2. You must explicitly show that you verified/checked all the hypotheses of the test you are using to draw the conclusion. For example, if using the Integral Test, you need to verify that the function f is decreasing and that the improper integral of f converges (or diverges). Every test carries with it its own hypotheses that need to be checked.

You should carefully read the solutions to the examples in these lecture notes, taking careful note of the way the solutions are presented. Try to present your solutions in the same fashion.

Here are some more examples applying the Integral Test:

Example 1.15. *(b)* $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution: *Let* $f(x) = \frac{1}{x \ln x}$. *We have* $f'(x) = \frac{-1}{x \ln x} \cdot (\ln x + x \cdot 1x) = \frac{-(\ln x + 1)}{x \ln x} <$

0 *so* f *is decreasing. Now*

$$\begin{aligned}\int_2^\infty \frac{1}{x\ln x}\,dx &= \lim_{b\to\infty}\int_2^b \frac{1}{x\ln x}\,dx \\ &\qquad \text{(set } u=\ln x \text{ so that } du = \frac{1}{x}\,dx\text{)} \\ &= \lim_{b\to\infty}\int_{\ln 2}^{\ln b} \frac{1}{u}\,dx \\ &= \lim_{b\to\infty}[\ln u]_{\ln 2}^{\ln b} \\ &= \lim_{b\to\infty}[\ln(\ln b)-\ln(\ln 2)] = \infty.\end{aligned}$$

Since $\int_2^\infty f(x)\,dx$ *diverges, so does* $\sum_{n=2}^\infty \frac{1}{n\ln n}$ *by the Integral Test.*

(c) $\sum_{n=1}^\infty \frac{A}{Bn+C}$ *(here* A, B *and* C *are constants)*

Solution: *We will assume* A *and* B *are positive (otherwise factor out a negative sign; this does not change whether or not the series converges). Now let* $f(x) = \frac{A}{Bx+C}$*; note that* $f'(x) = \frac{-AB}{Bx+C} < 0$ *so* f *is decreasing. Now*

$$\begin{aligned}\int_1^\infty \frac{A}{Bx+C}\,dx &= \lim_{b\to\infty}\int_1^b \frac{A}{Bx+C}\,dx \\ &= \lim_{b\to\infty}[A\ln(Bx+C)]_1^b \\ &= \lim_{b\to\infty}[A\ln(Bb+C)-A\ln(B+C)] = \infty.\end{aligned}$$

Therefore $\int_1^\infty \frac{A}{Bx+C}\,dx$ *diverges, so by the Integral Test,* $\sum_{n=1}^\infty \frac{A}{Bn+C}$ *diverges as well.*

(d) $\sum \frac{1}{n^p}$ *where* $p > 1$ *is a constant.*

Solution: *Let* $f(x) = \frac{1}{x^p} = x^{-p}$*. Since* $f'(x) = -px^{-p-1} < 0$*,* f *is decreasing. Now*

$$\begin{aligned}\int_1^\infty x^{-p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b \\ &= \lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p}-\frac{1^{1-p}}{1-p}\right].\end{aligned}$$

Since $p > 1$, $1 - p < 0$ so $b^{1-p} \to 0$ as $b \to \infty$. Thus the above limit evaluates to

$$-\frac{1^{1-p}}{1-p},$$

a finite number. Thus $\int_1^\infty x^{-p}\,dx$ converges so $\sum \frac{1}{n^p}$ converges as well by the Integral Test.

(e) $\sum \frac{1}{n^p}$ *where $0 < p < 1$ is a constant.*

Solution: *Let $f(x) = \frac{1}{x^p} = x^{-p}$. Since $f'(x) = -px^{-p-1} < 0$, f is decreasing. Now*

$$\begin{aligned}\int_1^\infty x^{-p}\,dx &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \\ &= \lim_{b\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^b \\ &= \lim_{b\to\infty}\left[\frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right].\end{aligned}$$

Since $p < 1$, $1 - p > 0$ so $b^{1-p} \to \infty$ as $b \to \infty$. Thus the above limit evaluates to ∞, and therefore $\int_1^\infty x^{-p}\,dx$ diverges so $\sum \frac{1}{n^p}$ diverges by the Integral Test.

Examples (c), (d) and (e) above illustrate some general facts which should be memorized for their own sake. First, some definitions which describe series like those in Example (c) above (the difference between these definitions is that the first defines "**the** harmonic series" and the second defines "**a** harmonic series"):

Definition 1.7. ***The*** **harmonic series** *is the infinite series*

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

Definition 1.8. ***A*** **harmonic series** *is any infinite series of the form*

$$\sum \frac{A}{Bn + C}$$

where A, B and C are constants (with $B \neq 0$).

Notice that the harmonic series is a harmonic series (by setting $A = B = 1$ and $C = 0$. Example (c) above proves the following important theorem:

Theorem 1.8 (Harmonic Series Test)**.** *If $\sum a_n$ is a harmonic series, then $\sum a_n$ diverges.*

In particular,

$$\textbf{the harmonic series } \sum \frac{1}{n} \textbf{ diverges}.$$

The series in Examples (d) and (e) also have a name:

Definition 1.9. *An infinite series is called a $p-$***series** *it is of the form*

$$\sum \frac{1}{n^p}$$

for a constant $p > 0$.

We also use the term $p-$series to describe any series which is a constant times any series of the type described in the preceding definition. Here are some examples of $p-$series, and the corresponding value of p:

p-series	value of p
$\sum \frac{4}{n^3}$	$p = 3$
$\sum \frac{-2}{\sqrt{n}}$	$p = \frac{1}{2}$
$\sum \frac{2}{n}$	$p = 1$
$\sum 2n^{-2}$	$p = 2$
$\sum \frac{6}{n^{3/2}}$	$p = \frac{3}{2}$
$\sum \frac{4}{n^{\pi}}$	$p = \pi$

Two remarks are in order:

1. Notice that $p-$series for which $p = 1$ are harmonic series.

2. It is easy to confuse geometric series with $p-$series. The difference between these classes of series is that if the variable of summation is n, then
   - the terms of a geometric series have n in the exponent, but a constant base;
   - the terms of a $p-$series have n as the base of the exponent, but a constant exponent.

The results of Examples (c) to (e) above prove the following useful theorem:

Theorem 1.9 ($p-$Series Test)**.** *If $\sum a_n$ is a $p-$series, then*

- *$\sum a_n$ converges if $p > 1$.*
- *$\sum a_n$ diverges if $p \leq 1$.*

Applying this test to the series listed above, we see:

p-series	value of p	converges or diverges?
$\sum \frac{4}{n^3}$	$p = 3$	converges by the $p-$series test since $p > 1$
$\sum \frac{-2}{\sqrt{n}}$	$p = \frac{1}{2}$	diverges by the $p-$series test since $p \leq 1$
$\sum \frac{2}{n}$	$p = 1$	diverges by the Harmonic Series Test
$\sum 2n^{-2}$	$p = 2$	converges by the $p-$series test since $p > 1$
$\sum \frac{6}{n^{3/2}}$	$p = \frac{3}{2}$	converges by the $p-$series test since $p > 1$
$\sum \frac{4}{n^\pi}$	$p = \pi$	converges by the $p-$series test since $p > 1$

This concludes the discussion of the Integral Test and $p-$series Test.

1.4.2 The $n^{th}-$Term Test

We now return to the concept of convergence. Recall that a series $\sum a_n$ converges to a number L if and only if

$$\lim_{N\to\infty} S_N = L$$

where S_N is the N^{th} partial sum of the series (i.e. $S_N = \sum_{n\leq N} a_n$). Now if $\lim_{N\to\infty} S_N = L$, it must be that as N gets large, the consecutive partial sums S_{N-1}

and S_N are very close together (since the numbers S_{N-1} and S_N must both be close to L). Therefore:

$$S_N - S_{N-1} \text{ is very small.} \tag{1.8}$$

But the difference $S_N - S_{N-1}$ can be rewritten as

$$\begin{aligned} S_N - S_{N-1} &= [a_1 + ... + a_{N-1} + a_N] - [a_1 + ... + a_{N-1}] \\ &= a_1 + a_2 + ... + a_{N-1} + a_N - a_1 - a_2 - ... - a_{N-1}; \end{aligned}$$

notice at this point that all the terms other than a_N are both added and subtracted, so all the terms cancel other than a_N.

$$\begin{aligned} S_{N+1} - S_N &= a_1 + a_2 + ... + a_{N-1} + a_N - a_1 - a_2 - ... - a_{N-1} \\ &= a_N. \end{aligned}$$

In light of equation (1.8) above, this means that whenever $\sum a_n$ converges, we can conclude that when N is big, a_N must be small.

More precisely, this means that if $\sum a_n$ converges, then $\lim_{N\to\infty} a_N = 0$ (this is the same thing as saying $\lim_{n\to\infty} a_n = 0$. An equivalent formulation of this fact is the following test, which tells us that whenever $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ must diverge:

Theorem 1.10 (n^{th} Term Test)**.** *Suppose $\sum a_n$ is an infinite series. If*

$$\lim_{n\to\infty} a_n \neq 0$$

, then $\sum a_n$ diverges.

Some important remarks are in order: first, the n^{th} Term Test is only a test for divergence: you can **never, ever, ever** conclude that a series **converges** using the n^{th} Term Test. This is the single most common conceptual error made by Calculus II students. Frequently such students argue like this:

"Notice that $\lim\limits_{n\to\infty} \dfrac{1}{\ln n} = 0$. Therefore $\sum \dfrac{1}{\ln n}$ converges by the n^{th} term test."

The above argument is **totally wrong**. It is true that $\lim_{n\to\infty} \frac{1}{\ln n} = 0$. But this has nothing to do with the hypothesis of the n^{th} Term Test–the n^{th} Term Test only applies if $\lim a_n \neq 0$, not when $\lim a_n = 0$. Furthermore, the n^{th} Term test never allows you to conclude that a series converges (only that it diverges).

Second, to say that $\lim_{n\to\infty} a_n \neq 0$ includes two types of situations: situations where $\lim_{n\to\infty} a_n$ exists but is equal to some number other than zero, and situations where $\lim_{n\to\infty} a_n$ does not exist. In either of these cases, the n^{th} Term Test allows you to conclude that $\sum a_n$ diverges.

Third, the statement $\lim_{n\to\infty} a_n = 0$ is equivalent to the statement $\lim_{n\to\infty} |a_n| = 0$, since the only way the terms a_n can approach zero is if their absolute values approach zero, and vice versa. Therefore, the n^{th} Term Test can be rephrased as follows:

Theorem 1.1 (n^{th} Term Test, Restated). *Suppose $\sum a_n$ is an infinite series.*

$$\text{If } \lim_{n\to\infty} |a_n| \neq 0$$

$$\textit{(this includes any situation where } \lim_{n\to\infty} |a_n| \text{ DNE}\textit{)},$$

then $\sum a_n$ diverges.

This formulation is sometimes more useful to work with (especially when dealing with alternating series, as we will see in Section 1.6). The reason is that $\lim_{n\to\infty} |a_n|$ is often easier to calculate than $\lim_{n\to\infty} a_n$.

Example 1.16. *Determine whether or not the following series converge or diverge:*

(a) $\sum_{n=1}^{\infty} \sin n$

Solution: *Let $a_n = \sin n$. Let's try the n^{th} Term Test. Here,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin n \text{ DNE}$$

because as $n \to \infty$, $\sin n$ oscillates between 1 and -1. Since the limit does not exist, it is not equal to 0, so by the n^{th} Term Test, $\sum a_n$ diverges.

(b) $\frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \ldots$

Solution: *First of all, we need to translate this series into $\Sigma-$notation and find a formula for a_n. Notice that $26 = 5^2 + 1$, $37 = 6^2 + 1$, $50 = 7^2 + 1$ so if* *we let* $a_n = \frac{1}{n^2+1}$ *our given series is* $\sum_{n=5}^{\infty} a_n = \sum_{n=5}^{\infty} \frac{1}{n^2+1}$. *Let's first try to analyze this series using the* n^{th} *Term Test:*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = \frac{1}{\infty} = 0$$

so the n^{th} *Term Test is useless (we can draw no conclusion from it since its hypothesis is not satisfied).*

Since the n^{th} *Term Test didn't work, we need to try a different test. Since this series is not a* $p-$*series, and not a geometric series, the only test we know at this point that remains is the Integral Test. To apply this, let* $f(x) = \frac{1}{x^2+1}$. *Notice that*

- f *is decreasing for* $x \geq 5$;
- $f(x) \geq 0$ *for* $x \geq 5$; *and*
- $f(n) = a_n$ *for all integers* $n \geq 5$,

so the Integral Test does apply here. We compute the appropriate integral:

$$\begin{aligned}
\int_5^{\infty} f(x)\,dx &= \int_5^{\infty} \frac{1}{x^2+1}\,dx \\
&= \lim_{b\to\infty} \int_5^{b} \frac{1}{x^2+1}\,dx \\
&= \lim_{b\to\infty} [\arctan x]_5^b \\
&= \lim_{b\to\infty} (\arctan b - \arctan 5) \\
&= \frac{\pi}{2} - \arctan 5 < \infty
\end{aligned}$$

so the integral $\int_5^{\infty} f(x)\,dx$ *converges. By the Integral Test,* $\sum_{n=5}^{\infty} a_n$ *converges as well.*

(c) $\sum_{n=1}^{\infty} e^{1/n}$

Solution: *Let* $a_n = e^{1/n}$. *Let's try the* n^{th} *Term Test. Here,*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{1/n} = e^{1/\infty} = e^0 = 1$$

so by the n^{th} *Term Test,* $\sum a_n$ *diverges.*

(d) $\sum_{n=2}^{\infty} \frac{3n^2}{\sqrt{5n^5+2^n}}$

Solution: *In this example,* $a_n = \frac{3n^2}{5n^5+2^n}$*. We might start by trying the* n^{th} *Term Test. But here,*

$$lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2}{5n^5 + 2^n} = 0$$

(because after three applications of L'Hopital's Rule the numerator would be a constant but the denominator would still have n in it) so the n^{th} *Term Test is useless. This series is not a* $p-$*series, so the* $p-$*series test is useless also. The only other test we know at this point is the Integral Test. To apply this, we would have to study the integral*

$$\int_2^\infty \frac{3x^2}{\sqrt{5x^5 + 2^x}}\, dx.$$

Conceivably, you might be able to figure out whether or not this integral converges, but this seems very hard. At this point, we are stuck; we just don't know yet whether this series converges.

Example (d) above illustrates the need for more tests which will help determine whether or not a given series converges. In the next section, we will discuss another test which is useful in situations like example (d).

1.4.3 The Comparison Test

Remember that for an infinite series, to say that the series converges means that the terms of the series add up to give a finite number. To say that the series diverges means that the terms of the series cannot be added to give a finite number. More precisely, this means $\lim_{N\to\infty} S_N$ does not exist, where S_N is the N^{th} partial sum of the series. Now, recall from Calculus I what it means for such a limit to not exist. To say that $\lim_{N\to\infty} S_N$ does not exist means one of these three things:

1. that $\lim_{N\to\infty} S_N = \infty$, i.e. the partial sums S_N increase without bound; or
2. that $\lim_{N\to\infty} S_N = -\infty$, i.e. the partial sums S_N decrease without bound; or
3. that S_N oscillates as $N \to \infty$.

Now suppose $\sum a_n$ is a positive series. This rules out possibilities (2) and (3) above, because since all the terms being added are positive, the partial sums of the series are getting larger as N increases (so S_N cannot decrease or oscillate). This justifies the following important principle:

Informal characterization of convergence and divergence for positive series: Let $\sum a_n$ be a positive series.

- To say that $\sum a_n$ converges means that the sum $a_1 + a_2 + a_3 + \ldots$ is equal to a finite number L.
- To say that $\sum a_n$ diverges means that the sum $a_1 + a_2 + a_3 + \ldots$ is equal to ∞.

Warning: This informal characterization is only valid for positive series. For more general series, to say that the series diverges could mean any of the three things described above.

Now, suppose you have two series which are both positive: $\sum a_n$ and $\sum b_n$. Suppose further that all the terms of the first series are less than or equal to the terms of the second series, i.e. $a_n \leq b_n$ for all n. This means that the a_n's must add up to something which is less than or equal to the sum of the b_n's. In particular:

- If the a_ns add up to ∞, then the b_ns must add up to something that is at least infinity. The only way this is possible is if the b_ns add up to ∞ as well.
- If the b_ns add up to a finite number L, then the a_ns must also add up to a finite number (which must be less than or equal to L).

Rephrasing this in the language of convergence and divergence, we have explained the following theorem, which is called the Comparison Test because it involves the comparison of two positive series, $\sum a_n$ and $\sum b_n$.

Theorem 1.11 (Comparison Test)**.** *Suppose $0 \leq a_n \leq b_n$ for all n. Then:*

1. *If the infinite series $\sum a_n$ diverges, then $\sum b_n$ diverges as well.*
2. *If the infinite series $\sum b_n$ converges, then $\sum a_n$ converges as well.*

Some remarks are in order:

1. This test is only useful for positive series: indeed, the hypothesis $0 \leq a_n \leq b_n$ implies that $\sum a_n$ and $\sum b_n$ must both be positive series.
2. This test can be used either to conclude that an infinite series converges, or that an infinite series diverges, but you have to be very careful about the logic. When applying this test, it is useful to think of the series $\sum a_n$ as the "small series" and to think of the series $\sum b_n$ as the "big series" (since $0 \leq a_n \leq b_n$).

   What you can do with the Comparison Test:

- You can conclude that the small series converges.
- You can conclude that the big series diverges.

What you cannot do with the Comparison Test:

- The Comparison Test never allows you to conclude that the big series converges.
- The Comparison Test never allows you to conclude that the small series diverges.

Suppose you are given a positive series and asked whether the series converges or diverges. In light of the preceding remarks, if you wanted to use the Comparison Test to study this series, you would need first to write an inequality which compares the series you are given to some other series. Ideally the other series is "simpler" than the one you started with, and you can tell whether or not the series converges by a quick test (like the $p-$series Test or Geometric Series Test). So based on:

- whether or not the "simpler" series converges or diverges, and
- whether or not the "simpler" series was larger or smaller than the given series,

you reason based on the following chart:

	the given series $\leq$ the "simpler" series	the "simpler" series $\leq$ the given series
the "simpler" series converges	**Conclusion:** By the Comparison Test, the given series **converges**.	No conclusion can be drawn from the Comparison Test
the "simpler" series diverges	No conclusion can be drawn from the Comparison Test	**Conclusion:** By the Comparison Test, the given series **diverges**.

The last thing to understand before we proceed with examples is how to come up with an appropriate inequality that starts an argument using the Comparison Test. Usually, series for which the Comparison Test is appropriate fall into these classes:

Classes of series which suggest the use of the Comparison Test to determine their convergence/divergence:

1. Series whose terms contain addition in the denominator of a fraction.

2. Series whose terms contain subtraction in the denominator of a fraction.

3. Series whose terms contain sines and cosines.

There are other situations where the Comparison Test is useful, but the other examples do not follow any kind of pattern. The upcoming examples include not only examples of each type listed above, but also some more unusual situations where the Comparison Test works. For now, we make some comments on the common classes of series which suggest the use of the Comparison Test:

1. **Addition in the denominator.** Suppose the terms of a series are of the form

$$\frac{\square}{\triangle + \star}$$

where $\square$, $\triangle$ and $\star$ **are all positive** quantities. Then, one can think of this series as the "small series" in the Comparison Test by starting with one of these two inequalities:

$$\frac{\square}{\triangle + \star} \leq \frac{\square}{\triangle} \quad \text{or} \quad \frac{\square}{\triangle + \star} \leq \frac{\square}{\star}$$

(The reason these inequalities are valid is that removing positive terms from the denominator makes the denominator smaller, which makes the entire fraction bigger.)

2. **Subtraction in the denominator.** Suppose the terms of a series are of the form

$$\frac{\square}{\triangle - \star}$$

where $\square$, $\triangle$ and $\star$ **are all positive** quantities. Then, one can think of this series as the "big series" in the Comparison Test by starting with this inequality:

$$\frac{\square}{\triangle - \star} \geq \frac{\square}{\triangle}$$

(The reason this inequality is valid is that removing negative terms from the denominator makes the denominator bigger, which makes the entire fraction smaller.)

3. **Series whose terms contain sines or cosines.** Suppose the terms of a series contain some expression of the form $\cos\Box$ or $\sin\Box$ where $\Box$ is some expression. Then, one can start with the inequality

$$-1 \leq \cos\Box \leq 1 \quad (\text{ or } -1 \leq \sin\Box \leq 1)$$

to obtain an inequality which makes the given series into either the "bigger" or "smaller" series as necessary.

This concludes the discussion of the Comparison Test. Now for some examples:

Example 1.17. *Determine whether each of the following series converge or diverge:*

(a) $\sum_{n=1}^{\infty} \frac{2}{n\sqrt{n}}$

Solution: *Don't forget tests we learned in previous sections! This series is just a constant times a p−series because it can be rewritten as*

$$2\sum \frac{1}{n^{3/2}}.$$

Thus this series converges because it is (a constant times a) a p−series with $p = \frac{3}{2} > 1$.

(b) $\sum \frac{2n}{n^8+1}$

The thought process behind the solution: *This is a positive series where the denominator contains addition. Thus, the Comparison Test is a good test to start with. Above, we said that inequality to start with in this situation is something like*

$$\frac{\Box}{\triangle + \star} \leq \frac{\Box}{\triangle}$$

In other words, in this example $\Box = 2n$, $\triangle = n^8$ *and* $\star = 1$. *So by translating the general form of the inequality into the language of this example, we get*

$$\frac{2n}{n^8+1} \leq \frac{2n}{n^8}.$$

Now we simplify the right-hand side to get

$$\frac{2n}{n^8+1} \leq \frac{2n}{n^8} = \frac{2}{n^7}.$$

Notice that we have constructed a "simpler" series (namely the p−series $\sum \frac{2}{n^7}$ *which is bigger than the series we started with. Since the larger series converges, so does the smaller one (i.e. the series we were given), by the Comparison Test.*

What needs to be written in the solution:

Since $\sum 2n^7$ *is a p−series with* $p = 7 > 1$*, it converges.*

Notice $0 \le \frac{2n}{n^8+1} \le \frac{2n}{n^8} = \frac{2}{n^7}$.

Therefore, by the Comparison Test, $\sum \frac{2n}{n^8+1}$ *converges.*

Remarks on what needed to be written in the solution: *To apply the Comparison Test, your argument must contain three things (1 and 2 can be done in either order, but 3 must come last):*

1. *An explanation of why the "simpler" converges or diverges.*
2. *An inequality that relates the terms of the given series to the terms of the "simpler" series and shows that both series are positive.*
3. *A concluding sentence which states the conclusion and the name of the test used (i.e. the Comparison Test).*

(c) $\sum_{n=40}^{\infty} \frac{3}{\sqrt[4]{5n-2}}$

Thought process: *The terms of this positive series contain subtraction in the denominator, so the Comparison Test is probably a good place to start. Here the motivating inequality is*

$$\frac{\square}{\triangle - \star} \ge \frac{\square}{\triangle}$$

and we are thinking of $\square = 3$, $\triangle = \sqrt[4]{5n}$, *and* $\star = 2$. *Thus we have*

$$\frac{3}{\sqrt[4]{5n}-2} \ge \frac{3}{\sqrt[4]{5n}} = \frac{3}{\sqrt[4]{5}}\frac{1}{\sqrt[4]{n}} = \frac{3}{\sqrt[4]{5}}\frac{1}{n^{1/4}}$$

Now $\frac{3}{\sqrt[4]{5}} \sum \frac{1}{n^{1/4}}$ *diverges (since it is a constant times a p−series with* $p = \frac{1}{4} < 1$*, so by the Comparison Test* $\sum_{n=40}^{\infty} \frac{3}{\sqrt[4]{5n-2}}$ *diverges as well.*

What you write in your solution:

$$\frac{3}{\sqrt[4]{5n}-2} \ge \frac{3}{\sqrt[4]{5n}} = \frac{3}{\sqrt[4]{5}}\frac{1}{n^{1/4}} \ge 0;$$

$\sum \frac{3}{\sqrt[4]{5}}\frac{1}{n^{1/4}}$ *diverges (p−series,* $p = \frac{1}{4} < 1$*);*

therefore $\sum \frac{3}{\sqrt[4]{5n-2}}$ *diverges by the Comparison Test.*

(d) $\sum_{n=2}^{\infty} \frac{2n+\cos(n^2-3\ln n)}{n^2}$

Thought process: *Since this series has a complicated cosine term in it, the Comparison Test is a suggested method. Here, we start with the inequality*

$$-1 \leq \cos(n^2 - 3\ln n) \leq 1$$

and try to recreate the terms of the series from this inequality. (In particular, the expression "$n^2 - 3\ln n$" has nothing to do with whether or not the series converges; it could be any expression and the method of solution would be the same.) In order to get back to the terms of the sequence, we take the inequality above, add $2n$ to both sides, and divide both sides by n^2:

$$-1 \leq \cos(n^2 - 3\ln n) \leq 1$$

$$2n-1 \leq 2n + \cos(n^2 - 3\ln n) \leq 2n+1$$

$$\frac{2n-1}{n^2} \leq \frac{2n + \cos(n^2 - 3\ln n)}{n^2} \leq \frac{2n+1}{n^2}$$

Notice that the interior term of this inequality is the general term of the series we are given. The question is which side of the inequality is useful. To figure this out, take the left- and right-hand terms of the above inequality, put them in a series and figure out whether or not those series converge or diverge. First, the left-hand side:

$$\sum \frac{2n-1}{n^2} = \sum \left(\frac{2n}{n^2} - \frac{1}{n^2}\right) = \sum \frac{2}{n} - \sum \frac{1}{n^2}.$$

Thus the series formed by adding terms like those on the left-hand side is the difference of a divergent series and a convergent series, hence the series diverges. Now for the right-hand side:

$$\sum \frac{2n+1}{n^2} = \sum \left(\frac{2n}{n^2} + \frac{1}{n^2}\right) = \sum \frac{2}{n} + \sum \frac{1}{n^2}.$$

Thus we have difference of a divergent series and a convergent series, hence the series diverges. Since both the series diverge, this suggests that the original series diverges as well; thus we want to think of the original given series as the "bigger" series (since the Comparison Test only allows us to conclude that the bigger series diverges). So the inequality

$$\frac{2n-1}{n^2} \leq \frac{2n + \cos(n^2 - 3\ln n)}{n^2}$$

is important, and the inequality

$$\frac{2n + \cos(n^2 - 3\ln n)}{n^2} \leq \frac{2n+1}{n^2}$$

is ultimately irrelevant to the argument. (Of course, we didn't know it would be irrelevant when we started out.) Now for what you would write:

What you write: *Observe that*

$$\frac{2n + \cos(n^2 - 3\ln n)}{n^2} \geq \frac{2n-1}{n^2} = \frac{2n}{n^2} - \frac{1}{n^2} = \frac{2}{n} - \frac{1}{n^2} \geq 0.$$

Notice $\sum \left(\frac{2}{n} - \frac{1}{n^2}\right)$ *is the difference of a divergent (harmonic) and convergent (p−series,* $p = 2 > 1$*) series, hence diverges.*

Therefore, by the Comparison Test, $\sum \frac{2n+\cos(n^2-3\ln n)}{n^2}$ *diverges.*

(e) $\sum_{n=1}^{\infty} \frac{-3}{\sqrt{n^4+3n}}$.

Solution: *First of all, this is a negative series since the denominator of each term is positive, and the numerator is negative. To treat this series as a positive series, first factor out a negative constant:*

$$\sum_{n=1}^{\infty} \frac{-3}{\sqrt{n^4+3n}} = -\sum_{n=1}^{\infty} \frac{3}{\sqrt{n^4+3n}}$$

We will show the series on the right converges using the Comparison Test. Notice that

$$0 \leq \frac{3}{\sqrt{n^4+3n}} \leq \frac{3}{\sqrt{n^4}} = \frac{3}{n^2}$$

because the terms on the right have smaller denominator (hence are larger fractions). Now $\sum \frac{3}{n^2}$ *converges by the p−series test (*$p = 2 > 1$*), so by the Comparison Test, so does* $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n^4+3n}}$ *and by linearity, so does* $\sum_{n=1}^{\infty} \frac{-3}{\sqrt{n^4+3n}}$.

(f) $\sum_{n=2}^{\infty} e^{-n^2}$

Solution: *Although this series does not fit into one of the typical classes of series analyzed by the Comparison Test, we can use the Comparison Test to study this series. First, notice that for all* $n \geq 2$, $n^2 \geq n$ *so* $-n^2 \leq -n$ *and therefore* $0 \leq e^{-n^2} \leq e^{-n}$. *Now the series* $\sum e^{-n}$ *converges (it is geometric with* $r = \frac{1}{e} \in (-1, 1)$*), so by the Comparison Test,* $\sum_{n=2}^{\infty} e^{-n^2}$ *converges.*

(g) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Thought process: *This series is neither positive nor negative, so the Integral Test and Comparison Test are useless. This series is also not a $p-$series so the $p-$series Test is also useless. So far, the only technique we have to study such a series (which is neither positive nor negative) is the n^{th} Term Test. But*

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \lim_{n\to\infty} \frac{|(-1)^{n+1}|}{|n|} = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the n^{th} term Test is also useless. At this point, we are stuck. We do not yet know if this series converges.

Example (g) above illustrates the need for more tests; in particular, we need a test (or tests) that apply to series which are not postive or negative.

1.5 Alternating series

Recall that our main question in the study of series is the following: given a series $\sum a_n$, does the series converge or diverge? In the previous sections, we have developed several tests that tell us when certain kinds of series converge or diverge. Other than the n^{th} term test, however, all the tests have the same drawback: *they only apply to positive series.* (Technically speaking, this means they also apply to negative series since if you are given a negative series, you can factor out (-1) from the series; what is left is a positive series.)

In this section we will study a class of series which arises naturally; this class of series consists of terms whose signs alternate between being positive and negative.

Definition 1.10. *A series $\sum a_n$ is called* **alternating** *if the terms being added alternate in sign, i.e. if*

(a) ($a_n \geq 0$ whenever n is even and $a_n \leq 0$ whenever n is odd), or

(b) ($a_n \geq 0$ whenever n is odd and $a_n \leq 0$ whenever n is even).

Usually the terms of an alternating series contain an expression like $(-1)^n$, $(-1)^{n+1}$ or $\cos(\pi n)$, because as n counts up by 1, these terms alternate between 1 and -1.

Example 1.18. *Are the following series alternating or not?*

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$

Solution: *The terms of this series alternate* $+, -, +, -, ...$ *so this series is alternating.*

(b) $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$

Solution: *If you write the terms of this series out, you will get*

$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n = \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right)^3 + \left(\frac{-1}{2}\right)^4 + \left(\frac{-1}{2}\right)^5 + ...$$
$$= -\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - ...$$

Now it is clear that the terms of the series alternate, so this series is alternating.

(c) $\sum_{n=1}^{\infty} \cos(\pi n) n^{-2}$

Solution: *Notice that* $\cos(\pi n) = 1$ *whenever* n *is even and* $\cos(\pi n) = -1$ *whenever* n *is odd. Therefore, when written out, this series is*

$$-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + ...$$

and this series is alternating.

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \sin n$

Solution: *It would be easy to say that since the terms of the series contain* $(-1)^{n+1}$*, that the series alternates. However,* $\sin n$ *is not always positive. So if you write the terms of this series out, you get*

$$\sin 1 - \sin 2 + \sin 3 - \sin 4 + \sin 5 - \sin 6 + ...$$

The signs of these terms are as follows:

n	n^{th} *term*	*sign of* n^{th} *term*
1	$\sin 1$	$+$
2	$-\sin 2$	$-$
3	$\sin 3$	$+$
4	$-\sin 4$	$+$ *(because* $\sin 4 < 0$*)*
5	$\sin 5$	$-$

Notice in particular that the signs do not alternate (the third and fourth terms have the same sign), so this series is not alternating.

Example (d) above illustrates a general principle: the series $\sum(-1)^n b_n$ (also $\sum(-1)^{n+1}b_n$, $\sum(-1)^{n-1}b_n$ and $\sum\cos(\pi n)\cdot b_n$) is only alternating if b_n is either always positive or always negative.

1.5.1 The Alternating Series Test

Here is a test which allows you to conclude (if you verify the hypotheses) that an alternating series converges:

Theorem 1.12 (Alternating Series Test)**.** *Suppose $\sum a_n$ is an infinite series. If*

1. *$\sum a_n$ is alternating,*
2. *$\lim_{n\to\infty}|a_n| = 0$, and*
3. *$|a_n| \geq |a_{n+1}|$ for all n,*

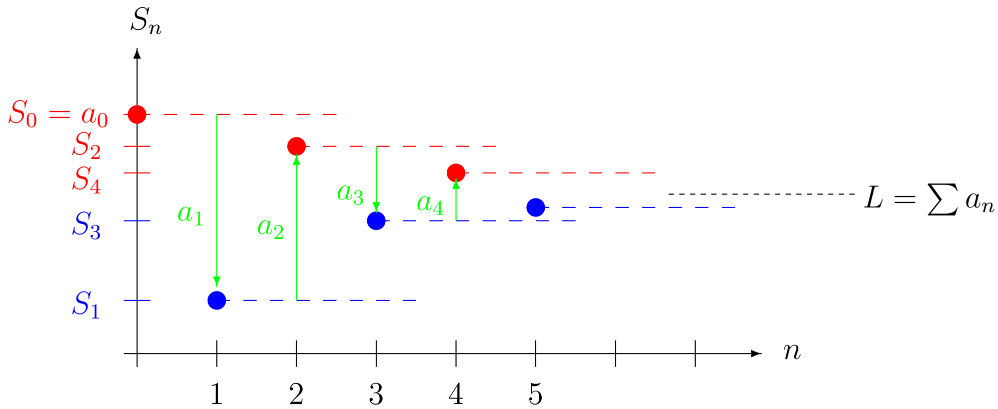
then $\sum a_n$ converges.

Notice two things: first, to draw a conclusion using the Alternating Series Test, there are three hypotheses that need to be verified (and you need to write that you have verified these hypotheses). Second, the Alternating Series Test can **never** be used to conclude that a series diverges.

An explanation of why the Alternating Series Test works: We are going for simplicity:

1. that the starting index of the series is $n = 0$, and
2. that the terms $a_0, a_2, a_4, ...$ are all positive and the terms $a_1, a_3, a_5, ...$ are all negative.

Now for each $n \geq 0$, let $S_n = a_0 + a_1 + ... + a_n$ be the n^{th} partial sum. Consider the following picture which plots S_n vertically and n horizontally (an explanation of the picture is below):



The partial sums are plotted with the red and blue points (the S_n where n is even are the red points and the S_n where n is odd are the blue points). Notice that to get from one partial sum to the next, i.e. to get from S_n to S_{n+1}, you have to add a_{n+1}. This is indicated by the green arrows. Now we will use the hypotheses of the Alternating Series Test:

- By hypothesis (1), the a_n alternate in sign. In our case, all the even a_n are negative and all the odd a_n are positive. Therefore, whenever n is odd, S_n is below the previous partial sum, and whenever n is even, S_n is above the previous partial sum. This makes each blue dot lower than each preceding red dot, and each red dot above each preceding blue dot.

- By hypothesis (3), $|a_n| \geq |a_{n+1}|$. Since $|a_n|$ is the length of the n^{th} green arrow, we are assured by hypothesis (3) that the green arrows are getting shorter as n increases. Thus the red and blue dots are getting closer and closer together.

- By hypothesis (2), $\lim_{n\to\infty} |a_n| = 0$. This means that since $|a_n|$ is the length of the green arrows, the length of these green arrows is going to zero as n increases. Thus the red and blue dots are both approaching the same height, so they have the same limit. This limit L is the limit of the partial sums, so by definition the infinite series converges to L.

This concludes the explanation of why the Alternating Series Test works. □

The Alternating Series Test, together with the n^{th} Term Test, determines in almost all cases whether or not an alternating series converges. If you are asked to determine whether an alternating series converges or diverges, follow the following procedure:

To determine whether an alternating series converges or diverges:

1. Make sure that the series is in fact alternating.
2. Compute the following limit:

$$\lim_{n\to\infty} |a_n|$$

   (a) If this limit is nonzero, then the series diverges by the n^{th} Term Test.

   (b) If this limit is zero, next try to show that $|a_n| \geq |a_{n+1}|$ for all n. If you can show this, then the series converges by the Alternating Series Test.

If the limit you found above is zero but you can't show that $|a_n| \geq |a_{n+1}|$, then you will need to try a different test. However, most of the time in Calculus II you will not encounter this situation.

Here are some examples which illustrate this procedure:

Example 1.19. *Do the following series converge or diverge? Explain your reasoning:*

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$

Solution: *First, we write the series in* $\Sigma-$*notation so that we know an explicit formula for* a_n*. Here, our series is*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

and therefore $a_n = \frac{(-1)^{n+1}}{n}$.

- *Since the terms alternate* $+, -, +, -, \ldots$*, the series is alternating.*
- *Next, compute the limit in step 2 of the above procedure:*

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

- *Last, verify that* $|a_n| \geq |a_{n+1}|$*:*

$$|a_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \geq \frac{1}{n+1} = \left|\frac{(-1)^{n+2}}{n+1}\right| = |a_{n+1}| \quad \checkmark$$

We have verified all hypotheses of the Alternating Series Test, so by the Alternating Series Test, this series converges.

(b) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

Solution: *Let* $a_n = (-1)^n \frac{n}{n+1}$*. Our given series* $\sum a_n$ *is clearly an alternating series. Now*

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left|(-1)^n \frac{n}{n+1}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Since this limit is not zero, $\sum a_n$ *diverges by the* n^{th} *Term Test.*

(c) $\sum_{n=1}^{\infty} \frac{n}{(-e)^{n-1}}$

Solution: *First, we rewrite this series to verify that it is alternating:*

$$\sum_{n=1}^{\infty} \frac{n}{(-e)^{n-1}} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{e^n}$$

so since each term is $(-1)^{n-1}$ *times something positive, the series is alternating. Next, we compute the appropriate limit (using L'Hopital's Rule):*

$$\lim_{n\to\infty} |a_n| = \lim \left|(-1)^{n-1}\frac{n}{e^n}\right| = \lim_{n\to\infty} \frac{n}{e^n} = \text{“}\frac{\infty}{\infty}\text{''} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

Last, we verify that $|a_n| \geq |a_{n+1}|$ *by rewriting this inequality into one we know is true:*

$$\begin{aligned}
|a_n| &\geq |a_{n+1}| \\
\Leftrightarrow \left|(-1)^{n-1}\frac{n}{e^n}\right| &\geq \left|(-1)^{n+1-1}\frac{n+1}{e^{n+1}}\right| \\
\Leftrightarrow \frac{n}{e^n} &\geq \frac{n+1}{e^{n+1}} \\
&\qquad \textit{(multiply through by } e^{n+1}\textit{)} \\
\Leftrightarrow en &\geq n+1 \\
\Leftrightarrow en - n &\geq 1 \\
\Leftrightarrow n(e-1) &\geq 1 \\
\Leftrightarrow n &\geq \frac{1}{e-1}
\end{aligned}$$

Since $e > 2$, $1/(e-1) < 1$ *so* n, *since it starts with index* 1, *is always bigger than* $1/(e-1)$. *Therefore, by tracing the previous steps backwards,* $|a_n| \geq |a_{n+1}|$. *We have verified all three hypotheses of the Alternating Series Test, so this series converges.*

1.6 Absolute and conditional convergence

So far, most of our tests for convergence (Comparison, Integral, etc.) apply only to positive series (or series which are eventually positive). They also apply to negative series because one can factor out -1 from a negative series; what remains can be treated as a positive series. We also have one test which applies to alternating series (the Alternating Series Test). But what about a series like

$$\sum_{n=1}^{\infty} \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n}\,?$$

Notice that for any number z,

$$\frac{z}{|z|} = \begin{cases} 1 & \text{if } z > 0 \\ 1 & \text{if } z < 0 \end{cases},$$

so the expression $\frac{\sin(e^n)}{|\sin(e^n)|}$ in the above sum is either 1 or -1 (depending on the sign of $\sin e^n$). The problem is that this sign doesn't alternate, nor is it always negative, nor is it always positive. When written out, this series ends up being

$$\sum_{n=1}^{\infty} \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n} = \pm e^{-1} \pm e^{-2} \pm e^{-3} \pm e^{-4} \pm e^{-5} \pm ...$$

where the $\pm$ signs are sometimes $+$ and sometimes $-$, but there is no pattern to which is which. Thus this series is neither positive nor negative nor alternating. At this point, we have only one test which can be used to study this series: the n^{th} Term Test. Let's try this test:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n} \right| = \lim_{n\to\infty} \left|(\pm 1)e^{-n}\right| = \lim_{n\to\infty} e^{-n} = 0$$

Unfortunately this means that we can draw no conclusion from the n^{th} Term Test. At this point we are stuck, and we need to develop some new machinery.

A big idea in the study of series which are neither positive, negative nor alternating is what is called the Triangle Inequality. This statement takes many diferent forms, which we now discuss. Here is the first version, which says that if you take two numbers, adding them and then taking the absolute value always produces a smaller result than taking their absolute values, then adding:

Theorem 1.13 (Triangle Inequality for $\mathbb{R}$)**.** *For all real numbers a and b,*

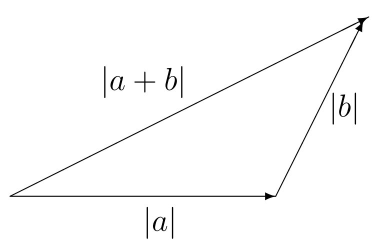
$$|a+b| \le |a| + |b|.$$

Proof of Triangle Inequality for $\mathbb{R}$: First, notice that for any two real numbers a and b, $ab \le |ab|$. We manipulate this inequality to produce the statement we want to prove:

$$\begin{aligned}
ab &\le |ab| \\
2ab &\le 2|ab| \\
&\qquad \text{(add } a^2 \text{ and } b^2 \text{ to both sides)} \\
a^2 + 2ab + b^2 &\le |a|^2 + 2|ab| + |b|^2 \\
&\qquad \text{(factor both sides)} \\
(a+b)^2 &\le (|a| + |b|)^2 \\
&\qquad \text{(take } \surd \text{ of both sides; note } \sqrt{z^2} = |z|) \\
|a+b| &\le ||a| + |b|| = |a| + |b|.
\end{aligned}$$

This finishes the proof. □

The reason this is called the Triangle Inequality is that if you consider the quantities a and b to be vectors rather than numbers, the quantities $|a|$, $|b|$ and $|a+b|$ are the lengths of the sides of a triangle:



You may remember a result from high-school geometry which says that the sum of two side lengths of any triangle is at least the length of the third side. In the language of this picture, this is exactly the Triangle Inequality: $|a+b| \le |a| + |b|$.

The same type of result holds if you are adding more than two numbers:

Theorem 1.14 (Generalized Triangle Inequality for $\mathbb{R}$)**.** *For all real numbers* $a_1, a_2, ..., a_n$,

$$\left|\sum_{j=1}^{n} a_j\right| \leq \sum_{j=1}^{n} |a_j|.$$

A similar statement holds for infinite series, but has to be phrased differently (because adding an infinite list of numbers is so fundamentally different from adding a finite list of numbers):

Theorem 1.15 (Triangle Inequality for Infinite Series)**.** *Let* $\sum a_n$ *be an infinite series. If* $\sum |a_n|$ *converges, then* $\sum a_n$ *also converges. In this case, we have*

$$\left|\sum a_n\right| \leq \sum |a_n|.$$

Proof of the Triangle Inequality for Infinite Series: Let $b_n = a_n + |a_n|$. Notice that if $a_n < 0$, $a_n + |a_n| = a_n - a_n = 0$ and if $a_n \geq 0$, then $a_n + |a_n| = |a_n| + |a_n| = 2|a_n|$. Thus b_n is always either 0 or $2|a_n|$. In particular, we have

$$0 \leq b_n \leq 2|a_n|.$$

Now we are given as a hypothesis of this theorem that $\sum |a_n|$ converges, so by linearity $\sum 2|a_n|$ converges, and by the Comparison Test, $\sum b_n$ converges. Last, we see

$$\sum a_n = \sum (b_n - |a_n|) = \sum b_n - \sum |a_n|$$

is the difference of two convergent series, hence $\sum a_n$ converges. This proves the first statement.

To verify the inequality in this theorem, first observe that for any number a_j, we have $a_j \leq |a_j|$. Therefore

$$a_1 + a_2 + ... + a_N \leq |a_1| + |a_2| + ... + |a_N| \tag{1.9}$$

Take the limit as $N \to \infty$ on both sides to get

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \tag{1.10}$$

Since each number a_j satisfies $a_j \geq -|a_j|$, we have

$$a_1 + a_2 + ... + a_N \geq -|a_1| - |a_2| - ... - |a_N|.$$

Take the limit as $N \to \infty$ on both sides of this equation to get

$$\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} -|a_n|$$

and factor out the -1 to get

$$\sum_{n=1}^{\infty} a_n \geq -\sum_{n=1}^{\infty} |a_n|. \tag{1.11}$$

Equations (1.10) and (1.11) together imply

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|,$$

i.e. $|\sum a_n| \leq \sum |a_n|$. This proves the theorem. $\square$

The Triangle Inequality for Infinite Series tells us that we can determine that a series $\sum a_n$ converges by proving that the series $\sum |a_n|$ converges. This is useful for series like the initial example of this section, which was the series

$$\sum_{n=1}^{\infty} \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n}.$$

Let $a_n = \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n}$. Rather than considering the original series $\sum a_n$, we will consider the series $\sum |a_n|$:

$$\sum |a_n| = \sum \left| \frac{\sin(e^n)}{|\sin(e^n)|} e^{-n} \right| = \sum e^{-n} = \sum \left(\frac{1}{e} \right)^n.$$

This series is a geometric series with $|r| = \frac{1}{e} < 1$, so $\sum |a_n|$ converges by the Geometric Series Test. Since $\sum |a_n|$ converges, $\sum a_n$ converges as well by the Triangle Inequality for Infinite Series.

Since it is important to know, given a series $\sum a_n$, whether or not $\sum |a_n|$ converges, we invent notation to describe this situation:

Definition 1.11. *Let $\sum a_n$ be an infinite series. We say the series is* **absolutely convergent** *(or* **converges absolutely***) if $\sum |a_n|$ converges.*

Some remarks on this definition:

- By the Triangle Inequality for Infinite Series, we know that if a series is absolutely convergent, then it converges.

- If a series diverges, then it cannot converge absolutely (this is the contrapositive of the immediate preceding statement).

- If a series $\sum a_n$ is positive, then there is no difference between $\sum |a_n|$ and $\sum a_n$, so saying that a positive series converges is the same as saying that it absolutely converges.

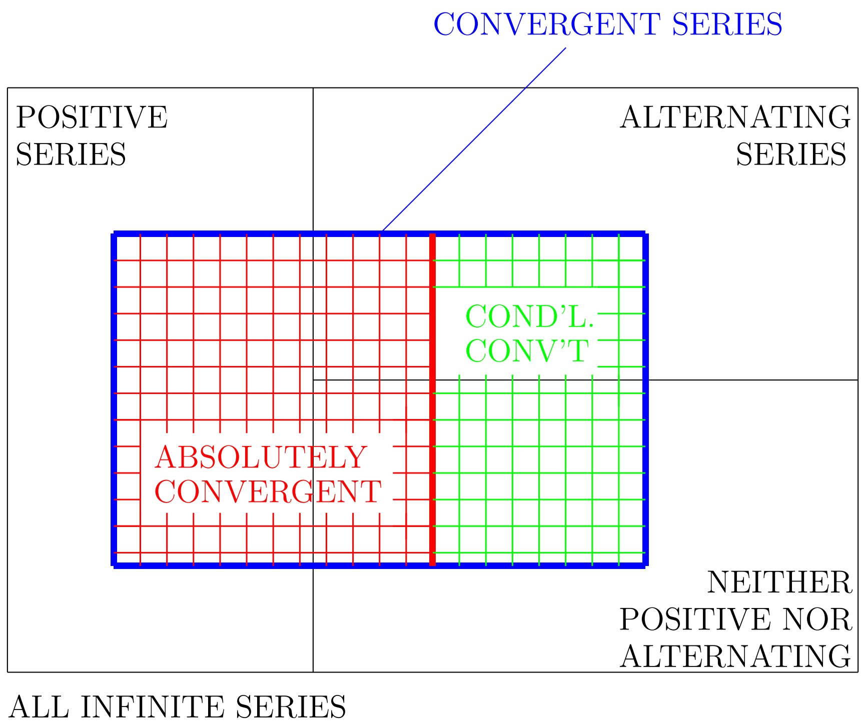
- If a series is negative, then $\sum |a_n| = -\sum a_n$ so to say a negative series converges is the same as saying that it absolutely converges.

- Based on these observations, there are three possibilities for an infinite series $\sum a_n$:
  1. The series $\sum a_n$ converges absolutely (i.e. $\sum |a_n|$ converges.
  2. The series $\sum a_n$ diverges.
  3. Something else (which given the remarks above must be that $\sum a_n$ converges but $\sum |a_n|$ diverges).

Based on this third possibility, we make the following definition:

Definition 1.12. *Let $\sum a_n$ be an infinite series. If $\sum a_n$ converges but $\sum |a_n|$ diverges, then we say $\sum a_n$ is* **conditionally convergent** *(or* **onverges conditionally***).*

Notice that there are now three disjoint classes of infinite series: those which converge absolutely, those which converge conditionally, and those which diverge. All series fall into exactly one of these three categories. Putting all this together, we have the following Venn diagram illustrating the various possibilities for infinite series. **In particular, the convergent series lie inside the blue box and the divergent series lie outside the blue box.** This Venn diagram is extremely important to understand:



Now, it is time to revise our "big picture" questions for our study of series to reflect this additional knowledge.

Big picture questions with infinite series:

1. **Classification problem:** Given an infinite series $\sum a_n$, where on the above Venn diagram does the series belong? In particular, does the series converge absolutely, converge conditionally, or diverge?

2. **Calculation problem:** Given a convergent series $\sum a_n$, what is the sum of that series?

3. **Rearrangement problem:** When, if ever, can the terms of an infinite series be rearranged or regrouped without affecting the value of the sum?

In Section 1.7.2 there will be several examples where we determine whether some given series converge absolutely, converge conditionally, or diverge. For now, we turn our attention to the third problem.

1.6.1 Rearrangement issues

In this section our goal is to determine under what circumstances the terms of an infinite series can be rearranged without affecting the sum (or whether the series converges). We will start with the following motivating example, called the **alternating harmonic series**:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This series converges by the Alternating Series Test (this was Example 1.3 (a) earlier in this text). However,

$$\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} \text{ diverges}$$

since it is the harmonic series. Therefore $\sum(-1)^{n+1}\frac{1}{n}$ **converges conditionally**. In fact, this series is the prototypical example of a conditionally convergent series.

Now, let L be the sum of this series. We will figure out what L is in Section 2.5, but for now what we know is that since the series alternates and its terms of even index are negative, $L \geq S_n$ for all n even; in particular $L \geq S_2 = a_1 + a_2 = 1 - \frac{1}{2} = \frac{1}{2}$. Then

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Let's suppose it was legal to rearrange and regroup the terms of this series. Then we could write

$$\begin{aligned}
L &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \dots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \dots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\right) \\
&= \frac{1}{2}L.
\end{aligned}$$

We have obtained the equation $L = \frac{1}{2}L$, which is only possible if $L = 0$. But from the discussion at the start of this paragraph, we know $L \geq \frac{1}{2}$. This means that it is **definitely not possible to rearrange/regroup the terms of this series legally**.

Let's investigate the alternating harmonic series further. Suppose we only looked at the terms of the alternating harmonic series which are positive, and added those terms up. We would get

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ... = \sum_{n=1}^{\infty} \frac{1}{2n+1},$$

a series which diverges (since it is essentially harmonic). Since the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ is a positive series which diverges, the terms of this series must add up to give ∞ (otherwise, they would add up to a positive number and the series would converge).

Now let's look at the terms of the alternating harmonic series which are negative and add these up. We get

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - ... = \sum_{n=1}^{\infty} \frac{-1}{2n} = \frac{-1}{2} \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges (since it is a constant times the harmonic series). Since this is a negative series, the terms of this series must add up to $-\infty$ (otherwise the terms would add up to a negative number and the series would converge).

Notice, therefore, that the alternating series is made up of positive terms which add up to ∞ and negative terms which add up to $-\infty$. The series only converges because the positive and negative terms are added in a very special order, allowing the $+\infty$ and $-\infty$ to add to the number L (remember that $L \neq 0$!).

Rearranging an infinite series like this screws up the way in which the $+\infty$ and $-\infty$ combine. The way we rearranged terms at the top of this page essentially moves negative terms "closer to the front" of the series, meaning that the negative terms get added more quickly. This makes the sum come out to something smaller than what it was (because the negative terms "contribute" more). (In our rearrangement we made the sum that originally added to L add to $\frac{1}{2}L$.) If we did something different to make the positive terms "closer to the front" of the series, the sum of the series would become larger. In fact, if you move enough positive terms to the front of the series, you can actually make the series diverge.

Loosely speaking, the problem is that this series converges conditionally (as opposed to absolutely). Any conditionally convergent series is comprised of an infinite amount of "positive stuff" and an infinite amount of "negative stuff", and the series only converges because the terms cancel each other out in a very delicate way. On the other hand, an absolutely convergent series has only a finite amount of "positive stuff" and a finite amount of "negative stuff", so no matter how you rearrange the series you always get the same thing. To summarize, here is a very important theorem:

Theorem 1.16 (Rearrangement Theorem)**.** *Suppose $\sum a_n$ is an infinite series.*

1. *If $\sum a_n$ converges conditionally, then the terms of that series can be rearranged so that the rearranged series converges to any number you like! (The series can also be rearranged so that the rearranged series diverges.)*

2. *If $\sum a_n$ converges absolutely to L, then no matter how the terms of the series are regrouped or rearranged, the rearranged series still converges absolutely to L.*

This theorem solves one of our important "big picture" questions:

Solution of the Rearrangement Problem: The terms of an infinite series can be legally regrouped and/or rearranged if and only if the series converges absolutely.

Warning: Suppose $\sum a_n$ converges absolutely. Then you can rearrange or group terms without a problem. However, you can **not** insert infinitely many terms into the series legally. For example, the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

converges to $\frac{1}{1-\frac{1}{2}} = 2$, but the series

$$1 + 1 - 1 + \frac{1}{2} + 1 - 1 + \frac{1}{4} + 1 - 1 + \frac{1}{8} + 1 - 1...$$

diverges because it is the sum of the convergent series $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ and the divergent series $\sum_{n=0}^{\infty}(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + ...$

1.6.2 Examples

Now we return to the problem of classifying infinite series as absolutely convergent, conditionally convergent, or divergent. Here are some examples illustrating some of the various tests for convergence:

Example 1.20. *Classify the following infinite series as absolutely convergent, conditionally convergent, or divergent.*

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{e^{2n}}$

Solution: *This series is alternating, so we begin by calculating the limit of the absolute values of the terms as $n \to \infty$. In particular,*

$$\lim_{n\to\infty} \left| \left(\frac{-n}{e^2} \right)^n \right| = \lim_{n\to\infty} \left| \left(\frac{-n}{e^2} \right)^n \right| = \lim_{n\to\infty} |(-1)^n| \left(\frac{n}{e^2} \right)^n = \lim_{n\to\infty} \left(\frac{n}{e^2} \right)^n = (\infty)^n = \infty.$$

Since $\lim_{n\to\infty} \left| \left(\frac{-n}{e^2} \right)^n \right| \neq 0$*, the series* $\sum \left(\frac{-n}{e^2} \right)^n$ *diverges by the* n^{th} *Term Test.*

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution: *Let* $a_n = \frac{(-1)^n}{\sqrt{n}}$*. This series* $\sum a_n$ *is alternating, so we start by calculating the limit of the absolute values of the terms as $n \to \infty$. In particular,*

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

We now verify the last condition of the Alternating Series Test (namely, that $|a_n| \geq |a_{n+1}|$*):*

$$|a_n| = \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} = |a_{n+1}| \quad \checkmark$$

(The middle inequality follows from the fact that the term on the right has a larger denominator, hence is a smaller fraction.) Finally, by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ *converges.*

But we are not done yet! We have to determine whether or not the series converges absolutely or converges conditionally. To do this, we make a new series whose terms are the absolute values of the terms of the series in the problem. In particular, we consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a $p-$series with $p = \frac{1}{2} < 1$ *so it diverges. Now we can conclude that the original series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges conditionally (by definition), because the series $\sum a_n$ *converges but the series* $\sum_{n=1}^{\infty} |a_n|$ *diverges.*

(c) $\frac{1}{3} + \frac{1}{9} - \frac{1}{27} - \frac{1}{81} + \frac{1}{3^5} + \frac{1}{3^6} - \frac{1}{3^7} - \frac{1}{3^8} + ...$

Solution: *This series is neither positive nor negative nor alternating (since the signs have the pattern* $+, +, -, -, +, +, -, -, ...,$ *not* $+, -, +, -, +, -, ...$*). So at this point, none of our tests can be directly applied to our series. However, we can use some theory to help. Let* a_n *be the* n^{th} *term of this series. Then*

$$\sum |a_n| = \sum \left| \frac{\pm 1}{3^n} \right| = \sum \left(\frac{1}{3} \right)^n$$

is a geometric series with $r = \frac{1}{3}$*; this series converges by the Geometric Series Test. Since* $\sum |a_n|$ *converges,* $\sum a_n$ *converges absolutely by definition (hence converges by the Triangle Inequality).*

(d) $\sum_{n=1}^{\infty} \frac{2n}{n^3+5n+2}$

Solution: *This series is positive, hence it cannot converge conditionally–if it converges, it converges absolutely. To check whether or not it converges, since its terms have addition in the denominator, use the Comparison Test:*

$$0 \leq \frac{2n}{n^3 + 5n + 2} \leq \frac{2n}{n^3} = \frac{2}{n^2}$$

Now $\sum \frac{2}{n^2}$ *converges by the* $p-$*series Test* $(p = 2)$ *so by the Comparison Test,* $\sum_{n=1}^{\infty} \frac{2n}{n^3+5n+2}$ *converges (and converges absolutely since it is a positive series).*

(e) $\sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + \frac{2}{2} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + ...$

Solution: *This is a positive series, so it cannot converge conditionally. After that, however, there is not a lot we can say.*

- *One cannot write a function* $f(x) = \frac{2^x}{x!}$ *because* $x!$ *is not defined for real numbers* x *which are not whole numbers, so the Integral Test won't work here.*
- *This is not a* $p-$*series or a geometric series.*
- *The Comparison Test doesn't seem to work here because there is no addition or subtraction in the denominator, nor is there a sine or cosine term.*
- *The Alternating Series Test is useless because this series doesn't alternate.*
- *The* n^{th} *Term Test might work, but what is*

$$\lim_{n\to\infty} \frac{2^n}{n!}?$$

We don't have the means to figure this out. At this point, we are stuck.

We need one last test to help us with series like Example (e) above. This test is discussed in the next section and is one of the most useful tests for analyzing series.

1.7 The Ratio Test

To motivate our last major test, we recall that a series $\sum a_n$ is called a geometric series if there exists a real number r such that

$$\frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = ... = r.$$

If this number r satisfies $|r| < 1$, then the series converges (absolutely) and if $|r| \geq 1$ the series diverges.

Now for a general (not necessarily geometric) series $\sum a_n$, the ratios

$$\frac{a_1}{a_0}, \ \frac{a_2}{a_1}, \ \frac{a_3}{a_2}, \ \frac{a_4}{a_3}, \quad \text{etc.}$$

are not likely to be the same number. However, it may be the case that these ratios get closer and closer to some number ρ (this is the Greek letter "rho"), i.e.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho.$$

In this case, we would expect the series $\sum a_n$ to behave like a geometric series with common ratio ρ. Restated, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$, then for large n,

$$\frac{a_{n+1}}{a_n} \approx \rho.$$

Let's suppose this approximation holds for all $n \geq N$. Then taking the previous line and multiplying through by a_n, we see

$$a_{n+1} \approx \rho a_n$$

and arguing recursively, this means

$$a_{n+1} \approx \rho a_n \approx \rho(\rho a_{n-1}) = \rho^2 a_{n-1}$$

and

$$a_{n+1} \approx \rho^2 a_{n-1} \approx \rho^3 a_{n-1} \approx ... \approx \rho^{n+1-N} a_N.$$

Therefore

$$\begin{aligned}\sum_{n=N}^{\infty} a_n &= a_N + a_{N+1} + a_{N+2} + a_{N+3} + a_{N+4} + ... \\ &\approx a_N + \rho a_N + \rho^2 a_N + \rho^3 a_N + \rho^4 a_N + ... \\ &= a_N(1 + \rho + \rho^2 + \rho^3 + ...) \\ &= a_N \sum_{n=0}^{\infty} \rho^n.\end{aligned}$$

Therefore:

- The last series $a_N \sum_{n=0}^{\infty} \rho^n$ converges if $|\rho| < 1$, so the original series $\sum a_n$ should also converge if $|\rho| < 1$ (and in fact, it does in this case).
- The series $a_N \sum_{n=0}^{\infty} \rho^n$ diverges if $|\rho| > 1$, so the original series should also diverge if $|\rho| > 1$ (and in fact, it does in this case).
- The series $a_N \sum_{n=0}^{\infty} \rho^n$ diverges if $|\rho| = 1$, so the original series should also diverge if $|\rho| = 1$ (here, unfortunately, the inutition is wrong; in performing the **approximation** above the value of ρ might get "fudged" a tiny bit for technical mathematical reasons; if $|\rho| = 1$ then fudging it a tiny bit makes a huge difference as far as whether or not the series converges).

The conclusion of all this logic is what is called the Ratio Test:

Theorem 1.17 (Ratio Test)**.** *Suppose $\sum a_n$ is an infinite series, and let*

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

1. *If $\rho < 1$, then $\sum a_n$ converges absolutely.*
2. *If $\rho > 1$, then $\sum a_n$ diverges.*
3. *If $\rho = 1$ or if the limit DNE, then this test is inconclusive.*

Remarks:

1. Since you are taking the limit of ratios which are positive, the value of ρ must be nonnnegative. If you get $\rho < 0$, you have done something wrong (you forgot the absolute values inside the limit).

2. Notice that the Ratio Test never allows you to conclude that a series converges conditionally. This is because for any conditionally convergent series, it turns out that the value of ρ will work out to be 1. Unfortunately, there are also absolutely convergent series and divergent series for which ρ is 1 as well.

It is a good idea to know which kinds of series for which the Ratio Test works well. In a nutshell, if you are going to use the Ratio Test on a series, you will need to be able to calculate

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

This means you will have to somehow simplify the quotient

$$\frac{|a_{n+1}|}{|a_n|}$$

before taking the limit. This means that in order to use the Ratio Test, the terms should contain only multiplication and division, and not addition and subtraction (as any addition or subtraction in the terms will create a mess when considering the above fraction $\frac{|a_{n+1}|}{|a_n|}$).

Here are some examples which apply the Ratio Test:

Example 1.21. *Determine whether or not the following series converge absolutely, converge conditionally or diverge:*

(a) $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

Solution: *In this setting,* $a_n = \frac{5^n}{n!}$*. We try the Ratio Test. The first thing to do when using the Ratio Test is to compute* ρ*:*

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$$

$$= \lim_{n\to\infty} \frac{|\frac{5^{n+1}}{(n+1)!}|}{|\frac{5^n}{n!}|}$$

Since all terms are positive, the absolute value signs are unnecessary, and to divide one fraction by another, flip the second fraction over and multiply to get

$$= \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n}$$

Observe that $\frac{5^{n+1}}{5^n} = 5$*, so we can cancel the exponents to get*

$$= \lim_{n\to\infty} 5 \cdot \frac{n!}{(n+1)!}$$

From the last page, we have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} 5 \cdot \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} 5 \cdot \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \\
&= \lim_{n\to\infty} \frac{5}{n+1} = 0.
\end{aligned}$$

Since $\rho < 1$, the series $\sum a_n$ converges absolutely by the Ratio Test.

(b) $\sum_{n=0}^{\infty} \frac{n^2 2^{4n+5}}{3^{2n}}$

Solution: *Apply the Ratio Test. First, compute ρ:*

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{\left|\frac{(n+1)^2 2^{4(n+1)+5}}{3^{2(n+1)}}\right|}{\left|\frac{n^2 2^{4n+5}}{3^{2n}}\right|} \\
&= \lim_{n\to\infty} \frac{(n+1)^2 2^{4(n+1)+5}}{3^{2(n+1)}} \cdot \frac{3^{2n}}{n^2 2^{4n+5}} \\
&= \lim_{n\to\infty} \frac{(n+1)^2 2^{4n+9} 3^{2n}}{3^{2n+2} n^2 2^{4n+5}} \\
&= \lim_{n\to\infty} \frac{(n+1)^2 2^5}{3^2 n^2} \\
&= \frac{2^5}{3^2} \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \\
&= \frac{32}{9} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \\
&= \frac{32}{9} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \\
&= \frac{32}{9}(1+0)^2 = \frac{32}{9}.
\end{aligned}$$

Since $\rho > 1$, the series diverges by the Ratio Test.

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n^n}$

Solution: *Apply the Ratio Test:*

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}(n+1)!}{(n+1)^{n+1}}\right|}{\left|\frac{(-1)^n n!}{n^n}\right|} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} (n+1) \frac{n^n}{(n+1)^{n+1}} \\
&= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n
\end{aligned}$$

If you let $n \to \infty$, this expression goes to "1^∞", an indeterminate form. To evaluate this expression, we need to use L'Hopital's Rule:

Side calculation *of* $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$*:*

First, let $y = \left(\frac{n}{n+1}\right)^n$*. Then* $\ln y = n \ln\left(\frac{n}{n+1}\right)$*. Now take the limit as* $n \to \infty$ *of* $\ln y$*:*

$$\begin{aligned}
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) \\
&= \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = \frac{\text{“ } 0 \text{ “}}{0} \\
&= \lim_{n\to\infty} \frac{\left(\frac{n}{n+1}\right)^{-1}\left(\frac{1(n+1)-1(n)}{(n+1)^2}\right)}{\frac{-1}{n^2}} \quad \textit{(by L'Hopital's Rule)}
\end{aligned}$$

From the previous page,

$$
\begin{aligned}
\lim_{n\to\infty} \ln y &= \lim_{n\to\infty} \frac{\left(\frac{n+1}{n}\right)\left(\frac{1}{(n+1)^2}\right)}{\frac{-1}{n^2}} \\
&= \lim_{n\to\infty} \left[\frac{1}{(n)(n+1)}\right] \cdot \frac{n^2}{-1} \\
&= \lim_{n\to\infty} \left[\frac{-n}{n+1}\right] = \frac{\text{“}\,\infty\,\text{“}}{\infty} \\
&= \lim_{n\to\infty} \left[\frac{-1}{1}\right] \quad \textit{(by L'Hopital's Rule)} \\
&= -1.
\end{aligned}
$$

Now, since $\lim_{n\to\infty} \ln y = -1$*,* $\rho = \lim_{n\to\infty} y = e^{-1} = \frac{1}{e} < 1$*. Thus by the Ratio Test, the series converges absolutely.*

Remark: Notice that the evaluation of ρ in these examples often contains either a simplification of exponents of the form

$$
\frac{c^{n+1}}{c^n} = c \quad \text{or} \quad \frac{c^n}{c^{n+1}} = \frac{1}{c} \text{ etc.}
$$

or a simplification of factorials of the form

$$
\frac{n!}{(n+1)!} = \frac{1}{n+1} \quad \text{or} \quad \frac{(n+1)!}{n!} = n \quad \text{or} \quad \frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)} \text{ etc.}
$$

When simplifying a factorial expression, it is often useful to write out the terms being multiplied to see how they will be cancelled. For example, the last equality above comes from the algebra

$$
\frac{n!}{(n+2)!} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = \frac{1}{(n+1)(n+2)}.
$$

Because these kind of simplifications are usually needed to apply the Ratio Test, we observe the following general principle:

> ***When it is a good idea to use the Ratio Test:*** The Ratio Test is likely to work well for a series whose terms contain only things that are multiplied and divided, and for series whose terms contain expressions like $2^n, 3^n, c^n, n^n, n!$, etc.

Here is an example where the Ratio Test does not work:

Example 1.22. *Determine whether the following series converges absolutely, converges conditionally, or diverges:*

$$\sum_{n=2}^{\infty} (-1)^n \frac{2n}{n^2-1}$$

(Incorrect) Solution: *Let's start with the Ratio Test (first find ρ):*

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= \lim_{n\to\infty} \frac{|(-1)^{n+1} \frac{2(n+1)}{(n+1)^2-1}|}{|(-1)^n \frac{2n}{n^2-1}|} \\
&= \lim_{n\to\infty} \frac{2(n+1)}{(n+1)^2-1} \cdot \frac{n^2-1}{2n} \\
&= \lim_{n\to\infty} \frac{(2n+2)(n^2-1)}{(n^2+2n)2n} \\
&= \lim_{n\to\infty} \frac{2n^3+2n^2-2n-2}{2n^3+4n^2} \\
&= \lim_{n\to\infty} \frac{2n^3}{2n^3} \cdot \frac{(1+\frac{1}{n}-\frac{2}{n^2}-\frac{2}{n^3})}{(1+\frac{4}{n})} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}-\frac{2}{n^2}-\frac{2}{n^3}}{1+\frac{4}{n}} = \frac{1+0-0-0}{1+0} = 1.
\end{aligned}$$

Unfortunately, this tells us nothing as the Ratio Test is inconclusive when $\rho = 1$.

We need to use a different test to study the series in the previous example (in fact, this series can be shown to converge by the Alternating Series Test and by using the Comparison Test, the convergence can be shown to be conditional; we omit this work here).

The most recent example illustrates a more general principle:

> ***When it is a bad idea to use the Ratio Test:*** When the terms of the series under consideration contains only powers of n (and not exponential terms and/or factorials), the Ratio Test will not work because ρ will turn out to be 1.

You should be aware of this principle and avoid using the Ratio Test in these situations.

1.8 Summary of convergence/divergence tests

Having completed our discussion of methods to classify infinite series as absolutely convergent, conditionally convergent, or divergent, we now summarize the procedure one should use when trying to solve such a classification problem:

Procedure to classify an infinite series:
To classify an infinite series $\sum a_n$ as absolutely convergent, conditionally convergent or divergent, follow these steps:

1. First, check to see whether the series is a $p-$series or a geometric series (or is a sum or difference of series of this type). If it is, use the $p-$series Test and/or Geometric Series Test (together with linearity properties) to classify the series.

2. If the terms of the series contain only multiplication and division and contain exponentials or factorial terms, use the Ratio Test. (If the terms of the series are all polynomials in n, avoid the Ratio Test.)

3. Otherwise, classify the series as positive, negative, alternating, or none of these. If the series is negative, factor out -1 from the series and treat what remains as a positive series.

4. If the series is positive:

    (a) If the terms of the series contain addition/subtraction in the denominator, or if they contain sines or cosines, try the Comparison Test.

    (b) If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the $n^{th}-$Term Test.

    (c) If the terms of the series look like a function you can integrate, try the Integral Test (use this only as a last resort).

(continued on next page)

Procedure to classify an infinite series (continued):

5. If the series is alternating, compute $\lim_{n\to\infty} |a_n|$.

   (a) If this limit is not zero, then the series diverges by the $n^{th}-$Term Test.

   (b) If this limit is zero, you can usually verify that $|a_n| \geq |a_{n+1}|$; then the series converges by the Alternating Series Test. In this case, you then have to examine the series $\sum |a_n|$:

      i. If $\sum |a_n|$ converges, then $\sum a_n$ converges absolutely.

      ii. If $\sum |a_n|$ diverges, then $\sum a_n$ converges conditionally.

6. If the series is neither positive, negative nor alternating:

   (a) If you can show that $\lim_{n\to\infty} |a_n| \neq 0$, then the series diverges by the $n^{th}-$Term Test.

   (b) Forget the original series and try to show that the positive series $\sum |a_n|$ converges; in this case the original series converges absolutely by definition.

At this point, we have largely solved two of our three original "big picture" problems:

- **Classification problem:** we (mostly) know how to classify infinite series as absolutely convergent, conditionally convergent or divergent, using the procedure outlined above.

- **Rearrangement problem:** we know exactly when series can be rearranged and/or regrouped legally (when the series converges absolutely).

What we don't know how to do is approach the **computation problem**, i.e. find the sum of a convergent non-geometric series. While we will develop another method to find the sum of some series in the next chapter, it is in general a very hard problem to find the sum of an arbitrary infinite series, even if you know the series converges. It's so hard, that even expert mathematicians don't know how to sum most series.

1.9 Exercises for Chapter 1

(From Section 1.2) In problems 1-12, you are given an infinite series "written out". Write the given series in $\Sigma-$notation (keep in mind that there are multiple correct answers).

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...$
2. $\frac{3}{25} + \frac{4}{125} + \frac{5}{625} + \frac{6}{5^5} + \frac{7}{5^6} + ...$
3. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{3^5} + ...$
4. $-2 + 2 - 2 + 2 - 2 + 2 - 2 + 2...$
5. $\frac{4}{8} + \frac{7}{15} + \frac{10}{22} + \frac{13}{29} + \frac{16}{36} + \frac{19}{43} + ...$
6. $-\frac{2}{9} - \frac{2}{25} - \frac{2}{49} - \frac{2}{81} - \frac{2}{121} - ...$
7. $\frac{1}{16} - \frac{1}{64} + \frac{1}{4^4} - \frac{1}{4^5} + ...$
8. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + ...$
9. $\frac{1}{14} + \frac{1}{17} + \frac{1}{20} + \frac{1}{23} + ...$
10. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + ...$
11. $\frac{1}{3} - \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 3^3} - \frac{1}{4 \cdot 3^4} + ...$
12. $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + ...$

(From Section 1.2) In problems 13-20, you are given an infinite series in $\Sigma-$notation. For each series, find the third term of the series (if it exists), find the ninth term of the series, and find the fourth partial sum of the series.

13. $\sum_{n=1}^{\infty} \frac{2n-1}{n}$
14. $\sum_{n=0}^{\infty} [1 + (-1)^n]$
15. $\sum_{n=4}^{\infty} \frac{1}{n}$
16. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n+1}$
17. $\sum_{n=0}^{\infty} n!$
18. $\sum_{n=2}^{\infty} (-1)^n \frac{3}{2n-1}$
19. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(From Section 1.2.1) In problems 21-28, you are given that some infinite series converges to some number. Then you are given a second infinite series which relates to the first series in some way. Find the sum of the second series.

21. Given $\sum_{n=0}^{\infty} \frac{1}{n!} = e$, compute $\sum_{n=2}^{\infty} \frac{1}{n!}$.

22. Given $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, compute $\sum_{n=1}^{\infty} \frac{3}{n^2}$.

23. Given $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = -1$, compute $\sum_{n=2}^{\infty} \frac{(-1)^n 3\pi^{2n}}{(2n)!}$.

24. Given $\sum_{n=0}^{\infty} \frac{e^{-2}2^n}{n!} = 1$, compute $\sum_{n=3}^{\infty} \frac{2^{n+3}}{n!}$.

25. Given $\sum_{n=0}^{\infty} \frac{5^n}{n!} = e^5$, compute $\sum_{n=2}^{\infty} \frac{5^n}{(n+2)!}$.

26. Given $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$, compute $\sum_{n=-3}^{\infty} \frac{3}{2^n}$.

27. Given $\sum_{n=0}^{\infty} n \left(\frac{1}{2}\right)^n = 2$, compute $\sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^{n+2}$.

28. Given $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ... = \frac{\pi}{4}$, compute $\frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \frac{2}{13} + \frac{2}{15} - ....$

(From Section 1.2.2) In problems 29-36, you are given an infinite series. Rewrite each series in $\Sigma-$notation such that the initial term of each rewritten series corresponds to the given index.

29. $\sum_{n=3}^{\infty} \frac{4}{(n+1)(n+2)}$; starting index 0

30. $\sum_{n=2}^{\infty} \frac{3^{2n-1}}{n!}$; starting index 4

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$; starting index 2

32. $\sum_{n=5}^{\infty} \frac{2n-1}{(n-2)^3-n}$; starting index 1

33. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{e^{-3n}}$; starting index 0

34. $18 - 6 + 2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + ...$; starting index 3

35. $2 + \frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9} + ...$; starting index 4

36. $\frac{3}{8} - \frac{4}{11} + \frac{5}{14} - \frac{6}{17} + \frac{7}{20} - \frac{8}{23} + ...$; starting index 0

(From Section 1.3) In problems 37-52, find the sum of each finite or infinite series (if it converges; if the series diverges, say so):

37. $\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$

38. $\sum_{n=0}^{\infty}\frac{-3}{6^n}$

39. $9-\frac{9}{2}+\frac{9}{4}-\frac{9}{8}+\frac{9}{16}-\frac{9}{32}+...$

40. $\sum_{n=3}^{17}\frac{2}{5^n}$

41. $\sum_{n=2}^{11}\left(\frac{3^n}{4^{2n}}\right)$

42. $2+4+8+16+...+2^{100}$

43. $\sum_{n=3}^{\infty}\left(\frac{-4}{3}\right)^n$

44. $\sum_{n=2}^{\infty}\frac{6^{n-1}}{7^{n+1}}$

45. $\sum_{n=0}^{\infty}\left[\frac{3}{2^n}+\left(\frac{2}{3}\right)^n\right]$

46. $\sum_{n=1}^{\infty}\left[\frac{5}{(-1)^n3^n}-\left(\frac{3}{5}\right)^{n+2}\right]$

47. $80+40+20+10+5+\frac{5}{2}+\frac{5}{4}+...$

48. $\sum_{n=0}^{\infty}\frac{3\cdot 8^n}{5^{2n-3}}$

49. $\sum_{n=2}^{\infty}\frac{2\cdot 3^{2n-1}}{5\cdot 2^{4n+3}}$

50. $\sum_{n=0}^{\infty}3^{-n}$

51. $\sum_{n=1}^{\infty}\frac{3^n-5}{6^n}$

52. $\sum_{n=0}^{\infty}4^n5^{-n}$

53. Suppose the initial term of a geometric series is 3 and the series converges to 2. What is the common ratio of the series?

54. A geometric series with common ratio $\frac{2}{3}$ converges to 8. What is the initial term of the series?

55. Write the repeating decimal .71717171717... as a fraction in lowest terms.

56. Write the repeating decimal 1.314314314314... as a fraction in lowest terms.

57. Write the repeating decimal .256161616161... as a fraction in lowest terms.

58. Write the repeating decimal .132032032032032... as a fraction in lowest terms.

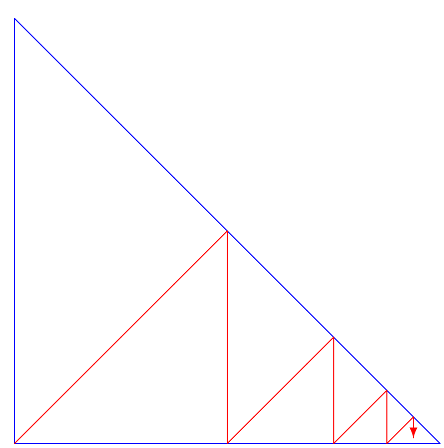
59. A ball is dropped from a height of 15 ft onto a concrete slab. Each time the ball bounces, it rebounds directly to $\frac{2}{3}$ of its previous height. Find the total distance the ball travels before it comes to rest.

60. Suppose a patient takes 25mg of a certain drug each day. If 80% of the drug is excreted by bodily functions each day, how much of the drug will be in the patient's body immediately before she takes her 22nd dose of the medicine?

61. Suppose your drinking water contains poison, and as such you injest 2.3mg of the poison each day. Although your body gets rid of 10% of the poison in your body each day, when you accumulate 20mg of the poison in your system you will be dead. How long do you have before you need to stop drinking your water?

62. Suppose your drinking water contains poison, and as such you injest 2.3mg of the poison each day. Suppose further that when you accumulate 250mg of the poison in your system you will be dead. What percent of the poison in your system does your body need to excrete daily in order to never die from the poison? (Assume that you will live forever if the poison doesn't kill you.)

63. In the figure below, the triangle indicated by the blue lines is an isoceles right triangle whose height is 1 unit. All the red line segments drawn are perpendicular to either the base of the triangle or the hypotenuse. If the red line segments continue indefinitely, find the total length of the red line segments.



(From Section 1.4) In problems 64-80, determine whether the following series converge or diverge. You should state the name of the test(s) you use and completely justify your reasoning, giving arguments like those in the examples of this text.

64. $\sum_{n=1}^{\infty} \frac{3}{n^4}$

65. $\sum_{n=1}^{\infty} \frac{3}{5n-3}$

66. $\sum_{n=1}^{\infty} \frac{-3}{2n^{2/3}}$

67. $\sum_{n=2}^{\infty} \frac{2n^2+3}{5n^2-4}$

68. $\sum_{n=1}^{\infty} \left(\frac{4}{n^5} + \frac{2}{n}\right)$

69. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

70. $\sum_{n=2}^{\infty} (n-1)^{-1/2}$

71. $\sum_{n=0}^{\infty} \frac{1}{e^n+e^{-n}}$

72. $\sum_{n=1}^{\infty} 3ne^{-n^2}$

73. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4n^2}$

74. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4\sqrt[3]{n}}$

75. $\sum_{k=2}^{\infty} \frac{3+2^n}{3^n+4}$

76. $\sum_{n=1}^{\infty} \frac{4+\sin(n^2+2n)}{\sqrt[3]{n^5+1}}$

77. $\sum_{n=2}^{\infty} 5^{-n^2-3n}$

78. $\sum_{k=1}^{\infty} \frac{3}{4+\sin^4(2k)}$

79. $\sum_{n=1}^{\infty} \frac{6^n}{6^{2n}+3}$

80. $\sum_{n=3}^{\infty} \frac{\ln n}{n}$

(From Sections 1.5 to 1.8) In problems 81-102, determine whether the following series converge absolutely, converge conditionally, or diverge. You should state the name of the test(s) you use and completely justify your reasoning, giving arguments like those in the examples of this text.

81. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

82. $\sum_{j=4}^{\infty} \frac{1}{\ln(2^j)}$

83. $\sum_{n=1}^{\infty} \frac{2+\cos(3n)}{n}$

84. $\sum_{n=2}^{\infty}(-1)^n \sin n$

85. $\sum_{n=2}^{\infty} \frac{n+1}{\ln(2n-5)}$

86. $\sum_{k=1}^{\infty} \frac{2\cos(\pi k)}{k^2}$

87. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{n^2+1}$

88. $\sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n^3+5}$

89. $\sum_{j=1}^{\infty}(-2)^j 2^{-j}$

90. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

91. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{7^n}$

92. $\sum_{n=2}^{\infty} \cos(n\pi) n^{-2/3}$

93. $\sum_{n=0}^{\infty} \frac{n^n}{n^3 n!}$

94. $\sum_{k=0}^{\infty} \frac{k^8}{1.01^k}$

95. $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

96. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

97. $\sum_{n=0}^{\infty} \frac{n! 2n!}{(3n)!}$

98. $\sum_{n=0}^{\infty} \frac{n!(3n)!}{[(2n)!]^2}$

99. $\sum_{n=0}^{\infty} \left[\frac{3}{n^5} + \frac{(-1)^n}{n}\right]$

100. $\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{1}{2}n(n+1)]}}{4^n+n^2}$

101. (Challenge) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

102. (Challenge) $1 + \frac{1}{1.1} + \frac{1}{1.11} + \frac{1}{1.111} + \frac{1}{1.1111} + ...$

1.9.1 Answers (not solutions) to the exercises

1. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ (there are multiple correct answers to numbers 1-12)
2. $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$ (there are multiple correct answers to numbers 1-12)
3. $\sum_{n=0}^{\infty} \frac{1}{3^n}$ (there are multiple correct answers to numbers 1-12)
4. $\sum_{n=1}^{\infty} 2 \cdot (-1)^n$ (there are multiple correct answers to numbers 1-12)
5. $\sum_{n=1}^{\infty} \frac{3n+1}{7n+1}$ (there are multiple correct answers to numbers 1-12)
6. $\sum_{n=1}^{\infty} \frac{-2}{(2n+1)^2}$ (there are multiple correct answers to numbers 1-12)
7. $\sum_{n=2}^{\infty} \left(\frac{-1}{4}\right)^n$ (there are multiple correct answers to numbers 1-12)
8. $\sum_{n=0}^{\infty} \frac{1}{n!}$ (there are multiple correct answers to numbers 1-12)
9. $\sum_{n=1}^{\infty} \frac{1}{3n+11}$ (there are multiple correct answers to numbers 1-12)
10. $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (there are multiple correct answers to numbers 1-12)
11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n3^n}$ (there are multiple correct answers to numbers 1-12)
12. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (there are multiple correct answers to numbers 1-12)
13. third term is $\frac{5}{3}$; ninth term is $\frac{17}{9}$; fourth partial sum is $\frac{71}{12}$.
14. third term is 0; ninth term is 0; fourth partial sum is 6.
15. third term is 0; ninth term is $\frac{1}{9}$; fourth partial sum is $\frac{1}{4}$.
16. third term is $\frac{-1}{4}$; ninth term is $\frac{-1}{10}$; fourth partial sum is $\frac{-13}{60}$.
17. third term is 6; ninth term is 362880; fourth partial sum is 34.
18. third term is $\frac{-3}{5}$; ninth term is $\frac{-3}{17}$; fourth partial sum is $\frac{29}{35}$
19. third term is $\frac{1}{12}$; ninth term is $\frac{1}{90}$; fourth partial sum is $\frac{4}{5}$
20. third term is $\frac{-1}{3}$; ninth term is $\frac{-1}{9}$; fourth partial sum is $\frac{-7}{12}$
21. $e - 2$
22. $\frac{\pi^2}{2}$
23. $-6 + \frac{3}{2}\pi^2$

24. $8e^2 - 40$

25. $\frac{1}{25}e^2 - \frac{118}{75}$

26. 48

27. $\frac{3}{8}$

28. $\frac{\pi}{2} - \frac{26}{15}$

29. $\sum_{n=0}^{\infty} \frac{4}{(n+4)(n+5)}$

30. $\sum_{n=4}^{\infty} \frac{3^{2n-5}}{(n-2)!}$

31. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

32. $\sum_{n=1}^{\infty} \frac{2n+7}{(n+2)^3-n-4}$

33. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{e^{-3n-3}}$

34. $\sum_{n=3}^{\infty}(-1)^{n+1}486\left(\frac{1}{3}\right)^n$

35. $\sum_{n=4}^{\infty} \frac{2}{2n-7}$

36. $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n+3}{3n+8}$

37. $\frac{1}{4}$

38. $\frac{-18}{5}$

39. 6

40. $\frac{1}{50}\left[1 - \left(\frac{1}{5}\right)^{15}\right]$

41. $\frac{9}{208}\left[1 - \left(\frac{3}{16}\right)^{10}\right]$

42. $2(2^{100} - 1)$

43. diverges

44. $\frac{6}{49}$

45. 9

46. $\frac{-179}{100}$

47. 160

48. $\frac{9375}{17}$

49. $\frac{27}{2240}$

50. $\frac{3}{2}$

51. 0

52. 5

53. $r = \frac{-1}{2}$

54. $\frac{8}{3}$

55. $\frac{71}{99}$

56. $\frac{1313}{999}$

57. $\frac{634}{2475}$

58. $\frac{1319}{9990}$

59. 75 ft

60. 6.25 mg

61. On the 19th day you will die.

62. .92%

63. $1 + \sqrt{2}$

64. $\sum_{n=1}^{\infty} \frac{3}{n^4}$ converges

65. $\sum_{n=1}^{\infty} \frac{3}{5n-3}$ diverges

66. $\sum_{n=1}^{\infty} \frac{-3}{2n^{2/3}}$ diverges

67. $\sum_{n=2}^{\infty} \frac{2n^2+3}{5n^2-4}$ diverges

68. $\sum_{n=1}^{\infty} \left(\frac{4}{n^5} + \frac{2}{n}\right)$ diverges

69. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ converges

70. $\sum_{n=2}^{\infty} (n-1)^{-1/2}$ diverges

71. $\sum_{n=0}^{\infty} \frac{1}{e^n+e^{-n}}$ converges

72. $\sum_{n=1}^{\infty} 3ne^{-n^2}$ converges

73. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4n^2}$ converges

74. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4\sqrt[3]{n}}$ diverges

75. $\sum_{k=2}^{\infty} \frac{3+2^n}{3^n+4}$ converges

76. $\sum_{n=1}^{\infty} \frac{4+\sin(n^2+2n)}{\sqrt[3]{n^5+1}}$ converges

77. $\sum_{n=2}^{\infty} 5^{-n^2-3n}$ converges

78. $\sum_{k=1}^{\infty} \frac{3}{4+\sin^4(2k)}$ diverges

79. $\sum_{n=1}^{\infty} \frac{6^n}{6^{2n}+3}$ converges

80. $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges

81. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges absolutely

82. $\sum_{j=4}^{\infty} \frac{1}{\ln(2^j)}$ diverges

83. $\sum_{n=1}^{\infty} \frac{2+\cos(3n)}{n}$ diverges

84. $\sum_{n=2}^{\infty}(-1)^n \sin n$ diverges

85. $\sum_{n=2}^{\infty} \frac{n+1}{\ln(2n-5)}$ diverges

86. $\sum_{k=1}^{\infty} \frac{2\cos(\pi k)}{k^2}$ converges absolutely

87. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$ converges conditionally

88. $\sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n^3+5}$ converges absolutely

89. $\sum_{j=1}^{\infty}(-2)^j 2^{-j}$ diverges

90. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$ converges absolutely

91. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{7^n}$ diverges

92. $\sum_{n=2}^{\infty} \cos(n\pi)n^{-2/3}$ converges conditionally

93. $\sum_{n=0}^{\infty} \frac{n^n}{n^3 n!}$ diverges

94. $\sum_{k=0}^{\infty} \frac{k^8}{1.01^k}$ converges absolutely

95. $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ diverges

96. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ converges absolutely

97. $\sum_{n=0}^{\infty} \frac{n!2n!}{(3n)!}$ diverges

98. $\sum_{n=0}^{\infty} \frac{n!(3n)!}{[(2n)!]^2}$ diverges

99. $\sum_{n=0}^{\infty} \left[\frac{3}{n^5} + \frac{(-1)^n}{n}\right]$ converges conditionally

100. $\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{1}{2}n(n+1)]}}{4^n+n^2}$ converges absolutely

101. answer not given here

102. answer not given here

2 Taylor and Power Series

2.1 Power series

In this chapter, we will use the concepts developed in Chapter 1 to define and study functions which are written as infinite series. What we will find is that several functions we know (like the exponential function, sine, cosine, arctangent, etc.) can be written as an infinite series which is relatively easy to work with. Furthermore, the representation of these and other functions by a class of infinite series called "power series" has many applications.

To begin, we start with some definitions:

Definition 2.1. *A* **power series (in x) centered at** $x = a$ *is a function of the form*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \ldots \end{aligned}$$

where the $a_0, a_1, \ldots$ are real numbers.

A **power series (in x)** *is a power series centered at $x = 0$, i.e. a power series is a function of the form*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \end{aligned}$$

The numbers $a_0, a_1, a_2, \ldots$ are called the **coefficients** *of the power series; the* **terms** *of the series are a_0, $a_1 x$, $a_2 x^2$, ...*

In particular, power series are **functions** where the variable of the function is x.

Remark: Suppose you have a power series centered at $x = a$. Then the power series can be written

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n;$$

plugging in a for x in this function produces

$$f(a) = \sum_{n=0}^{\infty} a_n (a-a)^n = a_0 0^0 + a_1 0^1 + a_2 0^2 + a_3 0^3 + ...$$

All the terms other than the initial one are zero. The initial term contains an expression of the form 0^0, which formally speaking is indeterminate. However, in the context of power series, the expression 0^0 is always taken to be 1; this is because if one writes out the sum for $f(x)$ before plugging in $x = a$ we get

$$f(a) = a_0 + a_1(a-a) + a_2(a-a)^2 + a_3(a-a)^3 + a_4(x-a)^4 + ... = a_0.$$

Another important thing to take note of is that with power series, the n^{th} term and n^{th} coefficient of the series correspond to the n^{th} power of $(x-a)$ or x, not the n^{th} thing that is written when one writes the series out:

Example 2.1. *For each power series, state where the power series is centered, identify its second coefficient, its first term, its sixth term, and its ninth coefficient:*

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

Solution: *Writing this series out, we see the power series is*

$$f(x) = 1 + \frac{1}{2}(x-2) + \frac{1}{5}(x-2)^2 + \frac{1}{10}(x-2)^3 + ...$$

Since the series contains powers of $x-2$, the power series is centered at 2. The second coefficient of the power series is the constant on the second power term $(x-2)^2$ which is $a_2 = \frac{1}{5}$. The first term of the power series is the term containing $(x-2)^1 = x-2$ which is $a_1(x-2) = \frac{1}{2}(x-2)$. The sixth term of the power series is the term containing $(x-2)^6$, which corresponds to $n = 6$ in this case. This term is $\frac{(x-2)^6}{6^2+1} = \frac{1}{37}(x-2)^6$. Finally, the ninth coefficient is the number multiplied by $(x-2)^9$. This comes from the coefficient when $n = 9$, which is $\frac{1}{9^2+1} = \frac{1}{82}$.

(b) $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{3x+1}$

Solution: *Writing this out, we see the power series is*

$$f(x) = 1 + \frac{1}{4}x^2 + \frac{1}{7}x^4 + \frac{1}{10}x^6 + \frac{1}{13}x^8 + ...$$

Since this series contains powers of x (i.e. powers of $x-0$), this power series is centered at 0. The second coefficient of the power series is the constant on the x^2 which is $a_2 = \frac{1}{4}$. The first term of the power series is the term containing $x^1 = x$ which in this case is 0, because the power series could really be written out as

$$f(x) = 1 + 0x + \frac{1}{4}x^2 + 0x^3 + \frac{1}{7}x^4 + 0x^5 + \frac{1}{10}x^6 + 0x^7 + \frac{1}{13}x^8 + ...$$

The sixth term of the power series is the term containing x^6, which corresponds to $n = 3$ in this case. This term is $a_6x^6 = \frac{1}{3\cdot 3+1}x^6 = \frac{1}{10}x^6$. Finally, the ninth coefficient is the number multiplied by x^9. This is zero since 9 is odd and the only powers of x that appear in the power series are even.

2.1.1 The Cauchy-Hadamard Theorem

Given any function f, the first thing you typically compute to study f is its domain. Since power series are functions, we will now discuss issues related to the domain of a power series. Recall that the domain of a function is the set of inputs (usually $x-$values) which "produce a valid output" (i.e. the domain of $f(x) = \sqrt{x}$ is $[0, \infty)$ because those are the numbers which are allowed as inputs to the square root function). Given a power series, to say that the series "produces a valid output" means that the series converges (since the entire concept of convergence means that a series adds up to give a finite number). In other words, **the domain of a power series is the set of x for which the power series converges**. Let's look at what this set of x can look like by considering some examples:

Example 2.2. *For what x does the given series converge?*

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n}$

Solution: *We can apply the Ratio Test to the terms of this series. First, we* *compute ρ (in general, we would expect ρ to depend on x):*

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th}\ term|}{|n^{th}\ term|} \\
&= \lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{(n+1)2^{n+1}}\right|}{\left|\frac{(x-3)^n}{n2^n}\right|} \\
&= \lim_{n\to\infty} \frac{|x-3|^{n+1}n2^n}{|x-3|^n(n+1)2^{n+1}} \\
&= \lim_{n\to\infty} |x-3|\frac{1}{2}\cdot\frac{n}{n+1} \\
&= \frac{1}{2}|x-3|\lim_{n\to\infty}\frac{n}{n+1} \\
&= \frac{1}{2}|x-3|\cdot 1 \\
&= \frac{1}{2}|x-3|.
\end{aligned}$$

By the Ratio Test, this series converges absolutely when $\rho < 1$. Since $\rho = \frac{1}{2}|x-3|$, this corresponds to the inequality

$$\begin{aligned}
\frac{1}{2}|x-3| < 1 &\Leftrightarrow |x-3| < 2 \\
&\Leftrightarrow -2 < x-3 < 2 \\
&\Leftrightarrow 1 < x < 5.
\end{aligned}$$

So the power series converges absolutely when $1 < x < 5$. Similarly, the series diverges when $\rho > 1$. This corresponds to the inequality

$$\begin{aligned}
\frac{1}{2}|x-3| > 1 &\Leftrightarrow |x-3| > 2 \\
&\Leftrightarrow -2 > x-3 \ \text{or}\ x-3 > 2 \\
&\Leftrightarrow x < 1 \ \text{or}\ x > 5.
\end{aligned}$$

So the power series diverges when $x < 1$ or $x > 5$. At this point, we know the behavior of the power series for all values of x other than $x = 1$ and $x = 5$. We analyze these cases individually:

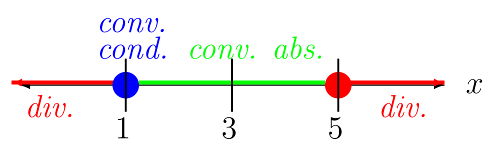
$$x = 1 \Rightarrow f(1) = \sum_{n=1}^{\infty}\frac{(1-3)^n}{n2^n} = \sum\frac{(-2)^n}{n2^n} = \sum\frac{(-1)^n}{n}.$$

This series converges conditionally (it is alternating harmonic).

$$x = 5 \Rightarrow f(5) = \sum_{n=1}^{\infty} \frac{(5-3)^n}{n2^n} = \sum \frac{2^n}{n2^n} = \sum \frac{1}{n}.$$

This series diverges since it is harmonic.

In conclusion, we have determined that the series converges absolutely when $x \in (1,5)$*; the series converges conditionally when* $x = 1$*; and the series diverges for all other* x*. We summarize this with the following picture, where we take a number line which represents values of* x *and color the portion of the number line green/blue/red for* $x-$ *values where the series converges absolutely/converges conditionally/diverges (respectively):*



(b) $\sum_{n=0}^{\infty} n!(x+4)^n$

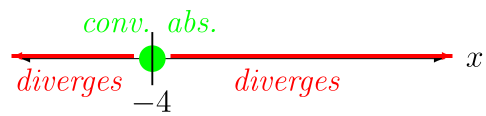
Solution: *We can apply the Ratio Test to the terms of this series. First, we compute* ρ*. Here, the computation is trickier because the process eventually splits into two cases:*

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th} \ \textit{term}|}{|n^{th} \ \textit{term}|} \\ &= \lim_{n\to\infty} \frac{|(n+1)!(x+4)^{n+1}|}{|n!(x+4)^n|} \\ &= \lim_{n\to\infty} (n+1)|x+4|.\end{aligned}$$

At this point, the value of this limit depends on x*. If* $x = -4$*, then the expression inside the limit is* $(n+1)0 = 0$ *so the limit is zero. However, if* $x \neq -4$*, then this limit is* ∞ *since the limit is* $n+1$ *times a nonzero constant. So altogether we see*

$$\rho = \begin{cases} 0 & \textit{if } x = -4 \\ \infty & \textit{else} \end{cases}$$

Since any series converges absolutely when $\rho < 1$ *and diverges when* $\rho > 1$*, we see that this power series converges absolutely when* $x = -4$ *and diverges for all other* x*. We show this conclusion with the following picture:*

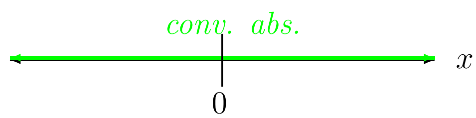


(c) $\sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$

Solution: *Again, we use the Ratio Test; first, calculate ρ:*

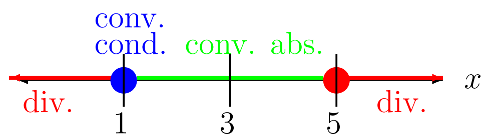
$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th}\ term|}{|n^{th}\ term|} \\
&= \lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{((n+1)!)^2}\right|}{\left|\frac{x^n}{(n!)^2}\right|} \\
&= \lim_{n\to\infty} \frac{|x|}{(n+1)^2} = 0
\end{aligned}$$

(and in particular, $\rho = 0$ no matter what x is). Since $\rho < 1$ for all x, by the Ratio Test this power series converges absolutely for all x. The picture is



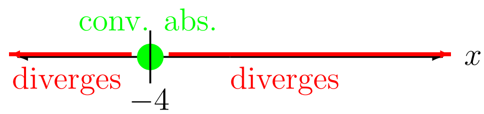
There is a pattern in these three examples. In each example, there is an open interval I whose midpoint is the place where the power series is centered. If x belongs to that interval I, the power series converges absolutely. If x is on the edge (a.k.a. boundary) of I, then anything can happen (the power series might converge conditionally, converge absolutely or diverge). If x is outside the edge of I, then the power series diverges. Here are the pictures from the previous three examples repeated with a description of I:

Example (a):



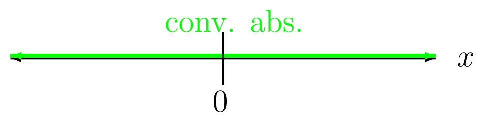
$I = (1, 5)$
Midpoint of $I = 3$
Edge of I is $1, 5$

Example (b):



$I = (-4, -4) = \emptyset$
Midpoint of $I = -4$
Edge of I is -4

Example (c):



$I = (-\infty, \infty) = \mathbb{R}$
Midpoint of $I = 0$
Edge of I does not exist

Note: It is technically incorrect to say that I has a midpoint in the case where $I = (-\infty, \infty)$. In this situation, we will say that the "midpoint" of $(-\infty, \infty)$ is the place where the power series is centered.

These three examples in Example 2.2 above essentially describe all possible behavior for power series. More specifically, the set of x for which a given power series converges is described by the following theorem, which is long to state but extremely important:

Theorem 2.1 (Cauchy-Hadamard (C-H) Theorem)**.** *Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series centered at a. There exists a quantity R (which is either a nonnegative real number or ∞) called the* **radius of convergence** *of the series such that:*

1. *If $0 < R < \infty$, then:*
   (a) *the power series converges absolutely for $x \in (a - R, a + R)$;*
   (b) *the power series diverges for $x \in (-\infty, a - R)$ or $x \in (a + R, \infty)$;*
   (c) *anything can happen when $x = a - R$ or $x = a + R$.*

2. *If $R = 0$, then the power series converges absolutely when $x = a$ but diverges for all other x.*

3. *If $R = \infty$, the power series converges absolutely for all x.*

The cases 1, 2 and 3 of this theorem correspond, respectively, to situations like (a), (b) and (c) of Example 2.2 above. In Example 2.2 (a), the radius of convergence is $R = 2$; in Example 2.2 (b), the radius of convergence is $R = 0$; in Example 2.2 (c) the radius of convergence is $R = \infty$.

To find the radius of convergence of a power series, one can use the following formula:

Theorem 2.2 (Abel's Formula)**.** *Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series centered at a. Then, the radius of convergence of this power series is given by*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

(assuming this limit exists).

Remarks:

1. Notice that Abel's Rormula uses only the *coefficients* of the power series and *not the powers of* $(x-a)$.

2. Abel's Formula looks similar to the computation done in the Ratio Test, but if you look carefully you will see that the fraction used to compute R is "upside-down" compared to what was used to compute ρ in the Ratio Test.

3. Abel's Formula has nothing to do with numerical series (those studied in Chapter 1); it is only useful for power series.

Proof of the Cauchy-Hadamard Theorem and Abel's Formula: Given power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, let $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$. We'll begin by proving statement (1) of the C-H Theorem, so we assume for now that $0 < R < \infty$. Now try to determine the convergence of the power series using the Ratio Test; first compute ρ:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th} \text{ term}|}{|n^{th} \text{ term}|} \\
&= \lim_{n\to\infty} \frac{|a_{n+1}(x-a)^{n+1}|}{|a_n(x-a)^n|} \\
&= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x-a| \\
&= |x-a| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \\
&= |x-a| \lim_{n\to\infty} \left(\frac{|a_n|}{|a_{n+1}|}\right)^{-1} \\
&= |x-a|R^{-1}.
\end{aligned}$$

Now by the Ratio Test, if $\rho < 1$ the power series converges absolutely. This corresponds to

$$\begin{aligned}
|x-a|R^{-1} < 1 &\Leftrightarrow |x-a| < R \\
&\Leftrightarrow -R < x-a < R \\
&\Leftrightarrow a-R < x < a+R.
\end{aligned}$$

Thus the power series converges absolutely for $x \in (a-R, a+R)$ as desired. Now the Ratio Test also tells us that if $\rho > 1$ the power series diverges. This corresponds to

$$\begin{aligned}
|x-a|R^{-1} > 1 &\Leftrightarrow |x-a| > R \\
&\Leftrightarrow -R > x-a \text{ or } x-a > R \\
&\Leftrightarrow x < a-R \text{ or } x > a+R.
\end{aligned}$$

Thus the power series diverges when $x \in (-\infty, a-R)$ or $x \in (a+R, \infty)$ as desired. This proves statement (1) of the C-H Theorem.

Now we prove statement (2). Assume that $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = 0$ so that (by taking reciprocals) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$. Again apply the Ratio Test to the power series. Repeating the argument above to compute ρ, we see

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th} \text{ term}|}{|n^{th} \text{ term}|} \\ &= \lim_{n\to\infty} \frac{|a_{n+1}(x-a)^{n+1}|}{|a_n(x-a)^n|} \\ &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x-a| \\ &= |x-a| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\end{aligned}$$

and this evaluates to 0 if $x = a$ (in which case the power series converges absolutely) and evaluates to ∞ if $x \neq a$ (in which case the power series diverges). This proves statement (2) of the C-H Theorem.

For the last statement, assume that $R = \infty$ so that (again by taking reciprocals) $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$. Again apply the Ratio Test to the power series. Repeating the argument above to compute ρ, we see

$$\begin{aligned}\rho &= \lim_{n\to\infty} \frac{|(n+1)^{th} \text{ term}|}{|n^{th} \text{ term}|} \\ &= \lim_{n\to\infty} \frac{|a_{n+1}(x-a)^{n+1}|}{|a_n(x-a)^n|} \\ &= \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|x-a| \\ &= |x-a| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0.\end{aligned}$$

Since $\rho < 1$ for all x, the power series converges absolutely for all x. This completes the proof. $\square$

The Cauchy-Hadamard Theorem ensures that **the set of x for which a power series converges is always an interval centered at the same place the power series is**, so the set of x for which a power series converges is also called the **interval of convergence** of the power series.

Abel's Formula and the Cauchy-Hadamard Theorem can (in most cases) be used to find the interval of convergence of a power series using the following mechanism:

Finding the interval of convergence of a power series: To find the values of x for which a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges (i.e. to find the interval of convergence), follow the following steps:

1. First, read off the value of a (where the power series is centered).

2. Next, compute the radius of convergence R using Abel's Formula:
$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

3. If $R = 0$, then the series converges absolutely for $x = a$ and diverges for all other x. The interval of convergence is $[a, a]$ or just $\{a\}$.

4. If $R = \infty$, then the series converges absolutely for all x. The interval of convergence is $(-\infty, \infty)$.

5. If $0 < R < \infty$, then by the C-H Theorem we can immediately conclude:
   - the series converges absolutely on $(a-R, a+R)$;
   - the series diverges on $(-\infty, a-R)$ and $(a+R, \infty)$.

   The interval of convergence in this case always runs from $a-R$ to $a+R$; what you have to subsequently determine is whether the endpoints $a-R$ and $a+R$ should be included in the interval. To figure this out, plug each of these endpoints in for x (do them one-by-one) and check whether the numerical series you obtain converge or diverge.

Here are some examples illustrating this procedure:

Example 2.3. *For each power series, find the radius of convergence and the set of x for which the series converges.*

(a) $\sum_{n=0}^{\infty} 2^n(x+1)^n$

Solution: *This power series is centered at $a = -1$. Apply Abel's Formula to* *calculate the radius of convergence:*

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \\ &= \lim_{n\to\infty} \frac{|2^n|}{|2^{n+1}|} \\ &= \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Since $R = \frac{1}{2}$, we know that the interval of convergence will run from $a - R$ to $a + R$, i.e. from $-1 - \frac{1}{2} = \frac{-3}{2}$ to $-1 + \frac{1}{2} = \frac{-1}{2}$. The next thing to do is to determine whether or not the power series converges at the endpoints $\frac{-3}{2}$ and $\frac{-1}{2}$. We test these endpoints individually:

$$x = \frac{-3}{2}: \quad \textit{series is } \sum 2^n \left(\frac{-3}{2} + 1\right)^n = \sum 2^n \left(\frac{-1}{2}\right)^n = \sum (-1)^n$$

This series diverges since it is geometric with $r = -1$.

$$x = \frac{-1}{2}: \quad \textit{series is } \sum 2^n \left(\frac{-1}{2} + 1\right)^n = \sum 2^n \left(\frac{1}{2}\right)^n = \sum 1^n$$

This series diverges since it is geometric with $r = 1$. Since the power series diverges at both its endpoints, the interval of convergence is open on both ends and is therefore $\left(\frac{-3}{2}, \frac{-1}{2}\right)$.

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{n 6^n}$

Solution: *The power series is centered at $a = 2$. Apply Abel's Formula:*

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \\ &= \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{n6^n}\right|}{\left|\frac{(-1)^{n+1}}{(n+1)6^{n+1}}\right|} \\ &= \lim_{n\to\infty} \frac{6(n+1)}{n} = 6. \end{aligned}$$

Since $R = 6$, we know that the interval of convergence will run from $a - R$ to $a + R$, i.e. from $2 - 6 = -4$ to $2 + 6 = 8$. The next thing to do is to determine whether or not the power series converges at the endpoints -4 and 8. We test these endpoints individually:

$$x = -4: \quad \textit{series is } \sum \frac{(-1)^n(-4-2)^n}{n6^n} = \sum \frac{(-1)^n(-6)^n}{n6^n} = \sum \frac{1}{n}$$

This series diverges (it is harmonic).

$$x = 8: \quad \textit{series is } \sum \frac{(-1)^n(8-2)^n}{n6^n} = \sum \frac{(-1)^n 6^n}{n6^n} = \sum \frac{(-1)^n}{n}$$

This series converges conditionally (it is alternating harmonic). Since the power series converges at $x = -4$ but not at $x = 8$, the interval of convergence of the power series is $[-4, 8)$.

(c) $\sum_{n=0}^{\infty} (2n)! x^n$

Solution: *This power series is centered at $a = 0$. Apply Abel's Formula:*

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \\ &= \lim_{n\to\infty} \frac{|(2n)!|}{|(2(n+1))!|} \\ &= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \\ &= \lim_{n\to\infty} \frac{1}{(2n+1)(2n+2)} = 0. \end{aligned}$$

Since $R = 0$, the power series converges only at the place where the power series is centered, i.e. at $x = 0$ (and diverges for all other x). The interval of convergence is therefore $[0, 0]$ or just $\{0\}$.

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n (x+3)^n}{n^n}$

Solution: *This power series is centered at $a = -3$. Apply Abel's Formula:*

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \\ &= \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{n^n}\right|}{\left|\frac{(-1)^{n+1}}{(n+1)^{n+1}}\right|} \\ &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \\ &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n (n+1) \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n (n+1) \\ &= \lim_{n\to\infty} e(n+1) = \infty. \end{aligned}$$

Since $R = \infty$, this series converges for all x. The interval of convergence is $(-\infty, \infty)$.

2.1.2 Calculus with power series

Recall that a power series is a function. In calculus, we learn two major operations on functions: differentiation and integration. In this subsection, we discuss the differentiation and integration of power series; one of the reasons power series are so useful is that they are "easy" to differentiate and integrate. The reason they are easy to differentiate and integrate is that they are comprised of (infinite) sums and differences of constants times nonnegative integer powers of x. We know from Calculus I that

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ and } \int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

so by applying these rules we can differentiate and integrate power series. Let's start with differentiation. Suppose we wanted to differentiate the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

First, write the series out to obtain

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ...$$

and differentiate each term to get

$$f'(x) = 0 + a_1 + a_2 \cdot 2(x-a) + a_3 \cdot 3(x-a)^2 + a_4 \cdot 4(x-a)^3 + ...$$

Writing this in $\Sigma-$notation, we obtain

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-a)^{n-1}.$$

Notice that the starting index of the series of $f'(x)$ is different from the starting index of $f(x)$ (since the constant term of $f(x)$ disappears under differentiation). Apart from that, the formula for $f'(x)$ is what one would "expect".

Now let's try integration. Suppose we wanted to integrate the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n.$$

First, write the series out to obtain

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ...$$

and integrate to get

$$\int f(x)\,dx = C + a_0(x-a) + a_1 \cdot \frac{(x-a)^2}{2} + a_2 \cdot \frac{(x-a)^3}{3} + a_3 \cdot \frac{(x-a)^4}{4} + a_4 \cdot \frac{(x-a)^5}{5} + ...$$

Writing this in $\Sigma-$notation, we obtain

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}.$$

Again the formula is what one would "expect".

In a nutshell, the idea behind the work in this section is that power series can be differentiated and integrated "term-by-term". (Interestingly, for series with variables in them that are not power series, i.e. things like $\sum e^{-nx}$, term-by-term differentiation or integration may not work. But we won't discuss series like this in Calculus II.) What's also true is that when one differentiates or integrates a power series, the radius of convergence is preserved. To summarize, we have the following theorem (stated on the next page):

Theorem 2.3 (Calculus of Power Series)**.** *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series centered at a with radius of convergence R. Then:*

1. *$f(x)$ is continuous on its interval of convergence;*
2. *$f'(x)$ is differentiable on $(a-R, a+R)$ and*

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-a)^{n-1}.$$

In particular, the power series for $f'(x)$ has the same radius of convergence as the original power series $f(x)$. However, the behavior of $f'(x)$ at endpoints may be different.

3. *$f(x)$ is integrable on $(a-R, a+R)$ and*

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-a)^{n+1}}{n+1}.$$

In particular, the power series for $\int f(x)\,dx$ has the same radius of convergence as the original power series $f(x)$. However, the behavior of $\int f(x)\,dx$ at endpoints may be different.

The exposition at the beginning of this section makes it seem as though this theorem is"obviously true". But in fact, this theorem is very deep. Rather than give a proof, we discuss why this theorem is deep here. As motivation, consider these two examples:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x}{x+y}\right) \quad \text{and} \quad \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x}{x+y}\right).$$

These may look the same, but if you look carefully, these are the same two limits on the same expression *done in different orders*. In the left-hand example, we do the $y-$limit first, then the $x-$limit to obtain

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x}{x+y}\right) = \lim_{x\to 0}\frac{x}{x} = \lim_{x\to 0} 1 = 1$$

but in the right-hand example, we do the $x-$limit first, then the $y-$limit to get

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x}{x+y}\right) = \lim_{y\to 0}\frac{0}{y} = 0.$$

Notice that the answers do not coincide! This illustrates a general point which you should understand, especially if you take more math courses: **in an expression with one limit inside another, you cannot (in general) interchange the order in which you take the limits**.

Now suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$. Thinking all the way back to the way infinite series are defined, this means formally that $f(x)$ is the limit of some partial sums, i.e.

$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-a)^n.$$

Let's now take a look at the statement

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-a)^{n-1}. \tag{2.1}$$

The left-hand side of this statement is (by definition)

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x+h-a)^n - \lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-a)^n}{h} \\
&= \lim_{h\to 0} \lim_{N\to\infty} \frac{1}{h} \left[\sum_{n=0}^{N} a_n[(x+h-a)^n - (x-a)^n]\right]
\end{aligned}$$

and the right-hand side of (2.1) is

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n n(x-a)^{n-1} &= \lim_{N\to\infty} \sum_{n=0}^{N} a_n \frac{d}{dx}(x-a)^n \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \lim_{h\to 0} \frac{1}{h} a_n \left[(x+h-a)^n - (x-a)^n\right] \\
&= \lim_{N\to\infty} \lim_{h\to 0} \frac{1}{h} \left[\sum_{n=0}^{N} a_n[(x+h-a)^n - (x-a)^n]\right]
\end{aligned}$$

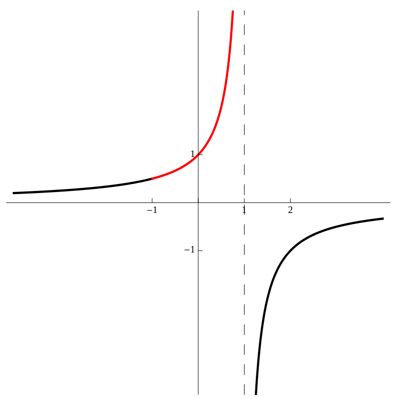
Notice that the left- and right-hand sides of (2.1) are the same limits of the same expression, but the limits are taken in different order. In general this doesn't work, but when working with power series there is some "magic" that allows you to change the order in which the limits are taken and show that they are equal.

The key application of the theorem of this section is that by taking derivatives and/or integrals (together with algebra and substitutions), we can write power series which represent many different functions, if we start with a power series for one function which we memorize.

At this point, there is one function whose power series we "know". From the geometric series summation formula, we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n$$

This summation is of course not valid for all x; it works only when $|x| < 1$ (i.e. $(-1,1)$). In other words, if we were to graph the two functions $\frac{1}{1-x}$ and $\sum_{n=0}^{\infty} x^n$, these graphs would coincide on the interval $(-1,1)$ but the graph of $\sum_{n=0}^{\infty} x^n$ does not extend past -1 or 1 (since such x values would produce a divergent series when plugged into $\sum_{n=0}^{\infty} x^n$):



In the above picture, the red graph is the graph of the function $\sum_{n=0}^{\infty} x^n$ and the red and black curves together comprise the function $\frac{1}{1-x}$.

Now, from the formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n \tag{2.2}$$

we can derive lots of other power series. For example:

- by taking derivatives of each side of (2.2) we obtain

  $$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + ...\right),$$

  i.e.

  $$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + ... = \sum_{n=1}^{\infty} nx^{n-1}.$$

- By integrating both sides of (2.2) we obtain

$$\int \left(\frac{1}{1-x}\right) dx = \int \left(1 + x + x^2 + x^3 + x^4 + ...\right) dx,$$

  i.e.

$$\ln(1-x) = C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...$$

  To find C, plug in $x = 0$ to both sides. On the left, we get $\ln(1-0) = \ln 1 = 0$ and on the right, we get $C + 0 + 0 + 0 + 0 + ... = C$ so $C = 0$. Thus

$$\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ... = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

- By substituting in $(-x)$ in for x on both sides of (2.2) we obtain

$$\frac{1}{1-(-x)} = 1 - x + (-x)^2 + (-x)^3 + (-x)^4 + ...$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + ... = \sum_{n=0}^{\infty} (-1)^n x^n.$$

- By substituting in x^2 for x on both sides of (2.2) we obtain

$$\frac{1}{1-x^2} = 1 + x^2 + (x^2)^2 + (x^2)^3 + ... = 1 + x^2 + x^4 + x^6 + ... = \sum_{n=0}^{\infty} x^{2n}.$$

- Suppose we wanted a power series representation of the function $g(x) = \frac{3}{4-7x}$. To find this, we will start with some algebra:

$$g(x) = \frac{3}{4-7x} = \frac{3}{4}\left(\frac{1}{1-\frac{7}{4}x}\right).$$

  Now since we know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n,$$

  we also know (by substituting in $\frac{7}{4}x$ for x,

$$\frac{1}{1-\frac{7}{4}x} = 1 + \frac{7}{4}x + \left(\frac{7}{4}x\right)^2 + \left(\frac{7}{4}x\right)^3 + ... = \sum_{n=0}^{\infty} \left(\frac{7}{4}x\right)^n$$

and by multiplying both sides by $\frac{3}{4}$ we get

$$g(x) = \frac{3}{4}\left(\frac{1}{1-\frac{7}{4}x}\right) = \frac{3}{4} + \frac{3}{4}\cdot\frac{7}{4}x + \frac{3}{4}\left(\frac{7}{4}x\right)^2 + \frac{3}{4}\left(\frac{7}{4}x\right)^3 + ... = \sum_{n=0}^{\infty}\frac{3}{4}\left(\frac{7}{4}\right)^n x^n.$$

In general, given a power series you "know" (the only one we know now is the geometric series given in equation (2.2) above), you can obtain a power series for a different function $g(x)$ by thinking of a sequence of substitutions/algebraic operations/differentiation/integration that produces $g(x)$ from the function you know. Then perform the same operations on the power series you know to produce the power series for $g(x)$.

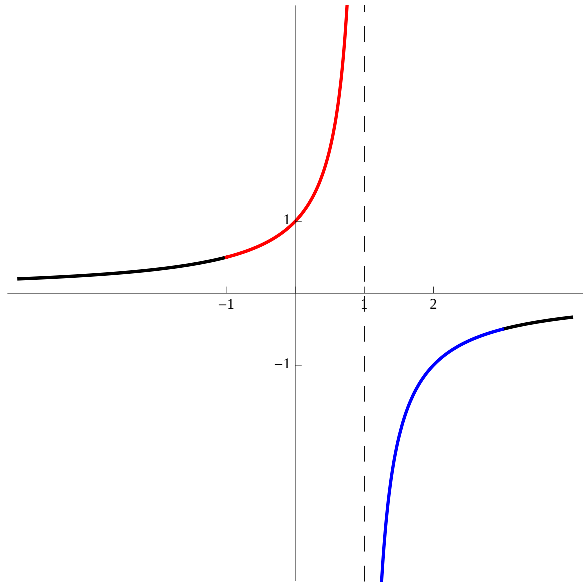
We finish this section by remarking that different power series can represent the same function on different intervals. For example, we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ on } (-1,1).$$

At the same time,

$$\begin{aligned}
\frac{1}{1-x} &= \frac{1}{1-(x-2)-2} \\
&= \frac{1}{-1-(x-2)} \\
&= -1\frac{1}{1-[-(x-2)]} \\
&= -1\left[1-(x-2)+(x-2)^2-(x-2)^3+(x-2)^4-(x-2)^5+...\right] \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n.
\end{aligned}$$

By Abel's Formula (the computation is omitted here), the radius of convergence of this series can be found to be $R = 1$ and this series has interval of convergence $(1,3)$. What we conclude is best described by the following graph: the red curve is $\sum_{n=0}^{\infty} x^n$, the blue curve is $\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$ and the red, blue and black curves together are the graph of $\frac{1}{1-x}$:



This example tells us that a function can have different power series which represent it on different intervals (notice that the power series in our example are centered at different values of x, namely 0 and 2). A natural question is the following: can a function have two different power series which represent it on the same interval? Put another way, can a function be represented on an open interval by two different power series centered at the same value of x? We consider this question in the next section.

2.2 Uniqueness of power series

We recall some notation related to derivatives: given a function f, we denote the first derivative of f by (among other things) f'; denote the second derivative of f as f''; etc. To represent the eighth derivative of a function using this kind of notation, we do not use primes. Instead, we write $f^{(8)}$. This means that $f^{(13)}(x)$ means the thirteenth derivative of f at x; $f^{(n)}(a)$ is the n^{th} derivative of f at a; $f^{(2)}$ is another way of writing f''; etc. (Note that $f^8(x)$ is not the eighth derivative of f; this means either $[f(x)]^8$ or f composed with itself eight times, depending on the context.) The **zeroth derivative** of any function is defined to be the function itself: $f^{(0)}(x) = f(x)$. Finally, a function is called **infinitely differentiable** on a set if $f^{(n)}$ exists for all n at all points in the set.

Recall that we asked in the last section if a function could be represented by two different power series centered at a. To address this question, suppose that function

f can be represented by some power series on an open interval containing a, i.e. that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-2)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ...$$

on $(a-R, a+R)$ where $R > 0$ is the radius of convergence of the series. By part (2) of Theorem 2.3, $f'(x)$ can be expressed as a power series centered at a with the same radius of convergence, so by repeated application of part (2) of Theorem 2.3, f is **infinitely differentiable** on $(a-R, a+R)$ (i.e. it is a function which can be repeatedly differentiated without anything becoming undefined). Furthermore, by the formula in part (2) of Theorem 2.3:

$$\begin{aligned}
f^{(0)}(x) = f(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + ... \\
f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + ... \\
f''(x) &= 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + ... \\
f^{(3)}(x) = f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + ... \\
\vdots \quad &= \quad \vdots \\
f^{(n)}(x) &= n(n-1)(n-2)(n-3)\cdots 3 \cdot 2 \cdot 1a_n + \\
&\qquad (n+1)n(n-1)(n-2)\cdots 3 \cdot 2a_{n+1}(x-a) + ... \\
&= n!a_n + (n+1)!a_{n+1}(x-a) + ...
\end{aligned}$$

Now, plug in a for x in each of the following formulas above. Notice that when we do this, every term of the form $(x-a)^j$ becomes $(a-a)^j = 0^j = 0$, so things are greatly simplified. In particular, we obtain

$$\begin{aligned}
f(a) &= a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + ... &&= a_0 = 0!a_0 \\
f'(a) &= a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0 + 4a_4 \cdot 0 + ... &&= a_1 = 1!a_1 \\
f''(a) &= 2a_2 + 3 \cdot 2a_3 \cdot 0 + 4 \cdot 3a_4 \cdot 0 + ... &&= 2a_2 = 2!a_2 \\
f^{(3)}(a) = f'''(x) &= 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4 \cdot 0 + ... &&= 3 \cdot 2a_3 = 3!a_3 \\
\vdots \quad &= \quad \vdots \\
f^{(n)}(a) &= n!a_n + (n+1)!a_{n+1} \cdot 0 + ... &&= n!a_n
\end{aligned}$$

The key formula that thas been derived is in the last line above:

$$f^{(n)}(a) = n!\, a_n.$$

By dividing both sides of the previous line by $n!$, we have proven the following theorem:

Theorem 2.4 (Formula for coefficients of a power series)**.** *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *where this series converges on an open interval containing* a *(equivalently, the series has positive radius of convergence). Then, for every* n *the coeffcients* a_n *of the power series must satisfy*

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

This extremely important result has several immediate consequences. Most importantly, it demonstrates that **the coefficients of a power series representation of f centered at $x = a$ are determined completely by the derivatives of f at a**. Since there is only one formula for the coefficients a_n, we can conclude:

Theorem 2.5 (Unique representation by power series)**.** *Given a constant* a*, a function* $f(x)$ *has at most one power series centered at* a *which represents it on an open interval containing* a*.*

Also, we have the following result, which says that if you have two "different" power series, both centered at a, which coincide on some interval containing a, then the series really aren't different: their coefficients must be identical.

Theorem 2.6 (Uniqueness of coefficients)**.** *Suppose*

$$\sum_{n=0}^{\infty} a_n(x-a)^n = \sum_{n=0}^{\infty} b_n(x-a)^n$$

on an open interval containing $x = a$*. Then* $a_n = b_n$ *for all* n*.*

Proof of Theorem 2.6: By Theorem 2.4, both a_n and b_n must be equal to $\frac{f^{(n)}(a)}{n!}$ for all n, thus they are equal to one another. $\square$

Collectively, Theorems 2.4, 2.5, and 2.6 are referred to as the **uniqueness of power series**. They tell us that given a function f and a number a, there is only one possible power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ which can represent f on an open interval containing a, and this power series has its coefficients $a_n = \frac{f^{(n)}(a)}{n!}$. Here is a problem which applies this principle:

Example 2.4. *Suppose* $f(x) = \sum_{n=0}^{\infty} \frac{3}{(n+1)^2} x^n$. *Find* $f^{(9)}(0)$.

Solution: *You could find the solution by differentiating this power series term-by-term nine times and then plugging in* $x = 0$. *But this would take forever. An easier way is to use the uniqueness of power series. By Theorem 2.4 with* $n = 9$, *we know that*

$$a_9 = \frac{f^{(9)}(0)}{9!}.$$

Now a_9 *can be found by the formula for* f *that is given; it is the coefficient on the* x^9 *term which is* $\frac{3}{(9+1)^2} = \frac{3}{100}$. *Thus we have*

$$\frac{3}{100} = \frac{f^{(9)}(0)}{9!}$$

and multiplying through by $9!$, *we have* $f^{(9)}(0) = \frac{3 \cdot 9!}{100}$.

Combining the results of the last two sections, at this point we know:

$$\begin{array}{c} f \text{ is infinitely differentiable} \\ \text{on } (a-R, a+R) \text{ and} \\ a_n = \frac{f^{(n)}(a)}{n!} \ \forall n. \end{array} \quad \Leftarrow \quad \begin{array}{c} f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \\ \text{for } x \in (a-R, a+R) \end{array}$$

The next question we ask is the converse: if you start with a function f which is infinitely differentiable on $(a-R, a+R)$, is it the case that f is representable by a power series? This leads to the discussion in the next section.

2.3 Taylor series and Taylor polynomials

Remember from the last section that our goal is to determine when an infinitely differentiable function can be represented by a power series. We begin with a definition:

Definition 2.2. *Given a function f which is infinitely differentiable on some open interval containing a, the* **Taylor series of f centered at a** *is the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+...$$

If $a = 0$, then the series in this definition, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}}{3!}x^3 + ...$$

is called the **Taylor series of f** *or the* **Maclaurin series of f**.

It is easy to confuse the terms "power series" and "Taylor series". A power series is any expression of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$. A Taylor series is a particular power series associated to some function f which is specified in advance. The reason for defining such a series is based on the reasoning in the previous section: from the uniqueness of power series, **the Taylor series of a function centered at a is the only possible power series which can represent f on an open interval containing a.**

There are two main theoretical questions with Taylor series:

Main questions related to Taylor series:

1. For what x does the Taylor series of a function f centered at a converge?

2. What function does the Taylor series of f converge to?

At this point, we know enough to answer the first question. The Taylor series of a function f centered at a is an example of a power series centered at a. Therefore, by the Cauchy-Hadamard Theorem, the Taylor series converges (absolutely) to $f(a)$ when $x = a$. This is because

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \Bigg|_{x=a} &= \left[f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} \right]_{x=a} \\ &= f(a) + 0 + 0 + 0 + ... \\ &= f(a). \end{aligned}$$

We also know that there is some interval $(a-R, a+R)$ centered at a on which the Taylor series of f converges to **something**. Ideally, the Taylor series of f should converge to f itself (since it is the only possible power series representation of f). But we don't know at this point whether or not this happens, or under what circumstances this happens.

Throughout the rest of this section, we will develop the terminology and theory associated to Taylor series. As we go along, we will work out this theory as it relates to two examples: the functions $f(x) = e^x$ and $g(x) = \sin x$. These two functions will be called our "prototype examples".

Prototype Example 1: $f(x) = e^x$; $a = 0$.

Here, we see that $f^{(n)}(x) = e^x$ for all n (since e^x is its own derivative). Therefore $f^{(n)}(a) = f^{(n)}(0) = 1$ for all n and therefore the Taylor series of $f(x) = e^x$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To find where this series converges, we use Abel's Formula:

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{\left|\frac{1}{n!}\right|}{\left|\frac{1}{(n+1)!}\right|} = \lim_{n\to\infty} (n+1) = \infty.$$

Since $R = \infty$, this series converges for all x by the Cauchy-Hadamard Theorem.

Prototype Example 2: $g(x) = \sin x$; $a = 0$.

Here, we see that the derivatives of g follow a pattern:

$$\begin{aligned}
g^{(0)}(x) = g(x) &= \sin x; \\
g'(x) &= \cos x; \\
g''(x) &= -\sin x; \\
g'''(x) &= -\cos x; \\
g^{(4)}(x) &= \sin x; \\
g^{(5)}(x) &= \cos x; \\
g^{(6)}(x) &= -\sin x; \\
g^{(7)}(x) &= -\cos x; \\
g^{(8)}(x) &= \sin x; \\
g^{(9)}(x) &= \cos x; \\
\vdots \quad & \quad \vdots
\end{aligned}$$

Plugging in $a = 0$ for x in each of these expressions gives

$$\begin{aligned}
g(0) = g^{(4)}(0) = g^{(8)}(0) = g^{(12)}(0) = \ldots &= \sin 0 = 0; \\
g'(0) = g^{(5)}(0) = g^{(9)}(0) = g^{(13)}(0) = \ldots &= \cos 0 = 1; \\
g''(0) = g^{(6)}(0) = g^{(10)}(0) = g^{(14)}(0) = \ldots &= -\sin 0 = 0; \\
g'''(0) = g^{(7)}(0) = g^{(11)}(0) = g^{(15)}(0) = \ldots &= -\cos 0 = -1.
\end{aligned}$$

So the Taylor series of $\sin x$ is

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n &= g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(0)}{3!}x^3 + \ldots \\
&= 0 + x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 + \frac{-1}{7!}x^7 + 0x^8 + \frac{1}{9!}x^9 + 0x^{10} + \ldots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\end{aligned}$$

By an argument similar to the previous example (Abel's Formula gives $R = \infty$), this series converges absolutely for all x.

To study the convergence of Taylor series for arbitrary functions, we return to the basics of infinite series. Recall from Chapter 1 that a series converges if the limit of its partial sums exists and is finite. Therefore, to understand the convergence of Taylor series, it makes sense to talk about the partial sums of a Taylor series. These partial sums are called Taylor polynomials:

Definition 2.3. *Let $n \geq 0$. Given a function f which can be differentiated n times on an open interval containing a, we can define the* **Taylor polynomial of order n centered at a**, *also called the n^{th}* **Taylor polynomial centered at a** *to be*

$$P_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+...+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

In other words, the n^{th} Taylor polynomial of f is what you get when you start writing out the terms of the Taylor series of f, stopping at the $(x-a)^n$ term. (If the phrase "centered at a" is omitted, we assume as with power series and Taylor series that the expression is centered at 0.)

Properties of Taylor polynomials: Given any function f, where P_n denotes the n^{th} Taylor polynomial centered at a, the following hold:

1. $P_n(x)$ is a polynomial of degree $\leq n$;
2. If $f^{(n)}(a) \neq 0$, then $P_n(x)$ is a polynomial whose degree is exactly n;
3. $P_0(x)$ is the constant function $f(a)$;
4. $P_1(x) = f(a) + f'(a)(x-a)$ is the tangent line to f when $x = a$;
5. $P_n(x)$ is the n^{th} partial sum of the Taylor series of f centered at a; therefore

$$\lim_{n\to\infty} P_n(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

if the limit exists.

Prototype Example 1: $f(x) = e^x$; $a = 0$; recall that the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Given this, we see that

$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= 1 + x\\
P_2(x) &= 1 + x + \frac{x^2}{2}\\
P_3(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}\\
\vdots \quad &\vdots\\
P_n(x) &= 1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n!}.
\end{aligned}$$

Prototype Example 2: $g(x) = \sin x$; $a = 0$; recall that the Taylor series of g is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ...$$

We rewrite this Taylor series to make sure we do not forget to consider the even powers of x (even though they are not present in this Taylor series). The Taylor series can also be written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = 0 + x - \frac{x^3}{3!} + 0x^4 + \frac{x^5}{5!} + 0x^6 - \frac{x^7}{7!} + 0x^8 + \frac{x^9}{9!} ...$$

Now, we can see that

$$\begin{aligned}
P_0(x) &= 0\\
P_1(x) &= 0 + x = x\\
P_2(x) &= 0 + x + 0x^2 = x\\
P_3(x) &= 0 + x + 0x^2 - \frac{x^3}{3!} = x - \frac{x^3}{3!}\\
P_4(x) &= 0 + x + 0x^2 - \frac{x^3}{3!} + 0x^4 = x - \frac{x^3}{3!}\\
\vdots \quad &\vdots
\end{aligned}$$

$$\begin{aligned}
\vdots \quad & \quad \vdots \\
P_7(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \\
\vdots \quad & \quad \vdots \\
P_{2n+1}(x) &= 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \pm \frac{x^{2n+1}}{(2n+1)!} \\
P_{2n+2}(x) &= 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \pm \frac{x^{2n+1}}{(2n+1)!} \\
\vdots \quad & \quad \vdots
\end{aligned}$$

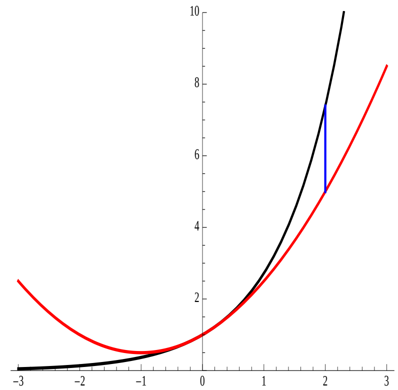
Notice that $P_1(x)$ and $P_2(x)$ are the same (as are $P_3(x)$ and $P_4(x)$, etc.); this is because the even power terms in the Taylor series of g are all zero.

Next, we turn to the problem of determining whether the Taylor series of a function f converges to f, or to something else. To do this, we introduce a new function, called the n^{th} remainder, which measures the difference between the original function f and its n^{th} Taylor polynomial:

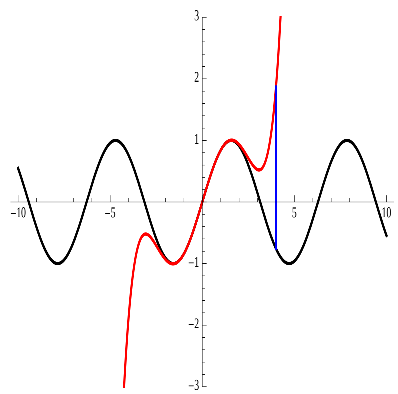
Definition 2.4. *Let f be infinitely differentiable on $(a-R, a+R)$ and let P_n be the n^{th} Taylor polynomial of f, centered at $x = a$. Define the n^{th}* **remainder (of f centered at a)** *to be the function*

$$R_n(x) = f(x) - P_n(x).$$

Prototype Example 1: $f(x) = e^x$; $a = 0$; recall that $P_2(x) = 1 + x + \frac{x^2}{2}$. In the picture below, f is graphed in black, P_2 is graphed in red, and $R_2(2)$ is the length of the blue line segment:



Prototype Example 2: $g(x) = \sin x$; $a = 0$; recall that $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. In the picture below, f is graphed in black, P_5 is graphed in red, and $R_5(4)$ is the length of the blue line segment:



It turns out that remainders have a lot to do with whether or not a function is equal to its Taylor series. In particular, we have the following theorem:

Theorem 2.7 (Remainder Theorem)**.** *Let f be infinitely differentiable on $(a - R, a + R)$ and let P_n and R_n be the n^{th} Taylor polynomial and n^{th} remainder of f, centered at $x = a$. Then if $\lim_{n\to\infty} R_n(x) = 0$, we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n,$$

i.e. f is equal to its Taylor series on $(a - R, a + R)$.

Proof of the Remainder Theorem: Recall that $P_n(x)$ is the n^{th} partial sum of the Taylor series of f. Therefore, since any infinite series is defined to be the limit of its partial sums, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n &= \lim_{n\to\infty} P_n(x) \\
&= \lim_{n\to\infty} (f(x) - R_n(x)) \quad \text{(by definition of } R_n) \\
&= f(x) - \lim_{n\to\infty} R_n(x) \\
&= f(x) - 0 \quad \text{(by hypothesis)} \\
&= f(x). \ \square
\end{aligned}$$

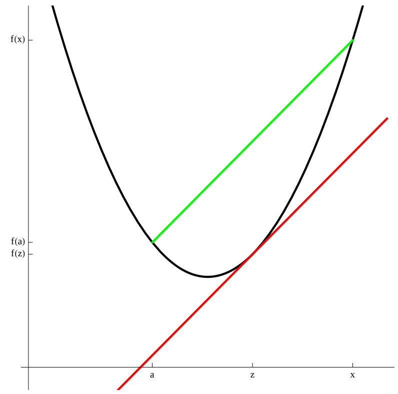
The Remainder Theorem sufficiently (for our purposes) answers (at least theoretically) the second main question related to Taylor series, because it gives a condition under which the Taylor series of f converges to f itself. In particular, the Remainder Theorem tells us that to show an infinitely differentiable function is equal to its Taylor series, we need only to show that $\lim_{n\to\infty} R_n(x) = 0$.

However, the definition of $R_n(x)$ alone is insufficient to evaluate this limit. We need an alternate representation of the n^{th} remainders which will allow us to show that $\lim_{n\to\infty} R_n(x) = 0$. To get this alternate representation, we first recall a theorem from Calculus I:

Theorem 2.8 (Mean Value Theorem (MVT))**.** *Let f be differentiable on the interval $[a, x]$. Then, there exists a $z \in (a, x)$ such that*

$$\frac{f(x) - f(a)}{x - a} = f'(z).$$

The Mean Value Theorem says that if you take two points on a graph (say $(a, f(a))$ and $(x, f(x))$), then there is a point z in between a and x such that the secant line between the two points you started with is parallel to the tangent line to the graph at z. The following picture illustrates this; if you are given the two points on the graph connected by the green line segment, then there is a point z where red tangent line at z is parallel to the green line segment:



The equation of the Mean Value Theorem can be rewritten as follows: in the theorem, it said

$$\frac{f(x) - f(a)}{x - a} = f'(z) \text{ for some } z \in (a, x).$$

Multiplying through by $(x - a)$, we have

$$f(x) - f(a) = f'(z)(x - a) \text{ for some } z \in (a, x)$$

and by adding $f(a)$ to both sides we obtain the equation

$$f(x) = f(a) + f'(z)(x - a) \text{ for some } z \in (a, x).$$

Notice that $f(a) = P_0(x)$, the zeroth Taylor polynomial of f centered at a. Therefore the rest of the right-hand side of the above equation must be $R_0(x)$, the zeroth remainder, and the Mean Value Theorem therefore says that

$$R_0(x) = f'(z)(x - a) = \frac{f'(z)}{1!}(x - a) \text{ for some } z \in (a, x).$$

Notice that the right-hand side of this equation looks a lot like the 1^{st} degree term of the Taylor series of x, except that there is a z in the derivative of f instead of an x. This equation generalizes into a very important theorem called Taylor's Theorem, which says that the n^{th} remainder looks like the $(n + 1)^{st}$ term of the Taylor series, except with a z in the $(n + 1)^{st}$ derivative instead of x:

Theorem 2.9 (Taylor's Theorem)**.** *Suppose f can be differentiated $n + 1$ times in an open interval $(a - R, a + R)$ (where $R > 0$). Then, for all $x \in (a - R, a + R)$ and all $n \geq 0$, there exists a z between a and x such that*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - a)^{n+1}.$$

Remarks on Taylor's Theorem:

- The main usage of Taylor's Theorem is as a mechanism to rewrite $R_n(x)$, so that you can subsequently use the Remainder Theorem to conclude that f is equal to its Taylor series.
- Taylor's Theorem is also used in approximation problems to give bounds on the error when one approximates a function with its n^{th} Taylor polynomial, but we won't consider these kinds of problems in Calculus II.
- Taylor's Theorem guarantees the existence of some number z which makes the formula work, but it doesn't tell you what z is or how to find z. In general we aren't interested in finding z.
- The Mean Value Theorem is just Taylor's Theorem in the special case where $n = 0$.

Proof of Taylor's Theorem: (This is a hard proof and can be skipped if you like.) First, a remark: this proof will use the Mean Value Theorem. The proof of the Mean Value Theorem is deep; take an advanced calculus course if you want to see that.

Now, let's prove the theorem. Fix $x \in (a - R, a + R)$ and recall that $R_n(x) = f(x) - P_n(x)$. Define a new function g, whose input variable will be called t, by setting $g(t)$ equal to

$$f(x) - \left[f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x-t)^n \right] - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

Observe that $g(a) = 0$ and $g(x) = 0$; this is because

$$g(a) = f(x) - [P_n(x)] - R_n(x)\frac{(x-a)^{n+1}}{(x-a)^{n+1}} = f(x) - P_n(x) - R_n(x) = R_n(x) - R_n(x) = 0$$

and

$$g(x) = f(x) - [f(x) + 0 + 0 + ... + 0] - R_n(x) \cdot 0 = f(x) - f(x) = 0.$$

Now apply the Mean Value Theorem to g to find a point z between a and x such that

$$g'(z) = \frac{g(x) - g(a)}{x - a} = \frac{0 - 0}{x - a} = 0.$$

Last, evaluate the derivative of g. We have

$$\begin{aligned} g'(z) &= \left.\frac{d}{dt}g(t)\right|_{t=z} \\ &= \left(0 - \left[f'(t) + (f''(t)(x-t) - f'(t)) + \left(\frac{f'''(t)}{2!}(x-t)^2 - f''(t)(x-t)\right) + \ldots \right.\right. \\ &\qquad \left.\left. + \left(\frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n+1}\right)\right] + \frac{R_n(x)}{(x-a)^{n+1}}(n+1)(x-t)^n\right)\Bigg|_{t=z}; \end{aligned}$$

notice that the terms inside the brackets cancel out to leave

$$g'(z) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{R_n(x)}{(x-a)^{n+1}}(n+1)(x-t)^n.$$

Since $g'(z) = 0$, we have

$$0 = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + \frac{R_n(x)}{(x-a)^{n+1}}(n+1)(x-t)^n.$$

Multiply through by $(x-t)^n$ and $(x-a)^{n+1}$, and divide through by $(n+1)$ to get

$$0 = -\frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1} + R_n(x);$$

this gives the desired result: $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$. $\square$

Here's a summary of what we have developed so far in this section:

1. Given an infinitely differentiable function f, the only possible power series centered at a which can represent f is its Taylor series (centered at a).

2. The partial sums of this Taylor series of f are called the Taylor polynomials of f (centered at a).

3. The difference between f and its n^{th} Taylor polynomial is called the n^{th} remainder of f.

4. The n^{th} remainder of f can be rewritten in terms of some unknown number z using Taylor's Theorem.

5. If $\lim_{n\to\infty} R_n(x) = 0$, then f is equal to its Taylor series.

Let's see how this is applied in the context of our two prototype examples:
Prototype Example 1: $f(x) = e^x$; $a = 0$. Fix $x \in \mathbb{R}$ and recall that

$$P_n(x) = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}.$$

By Taylor's Theorem, we have

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = \frac{e^z}{(n+1)!}x^{n+1}$$

for some z between 0 and x. Now

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^z}{(n+1)!}x^{n+1} = e^z \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!}.$$

Now we can show (using the Ratio Test or Abel's formula) that the series

$$\sum \frac{x^{n+1}}{(n+1)!}$$

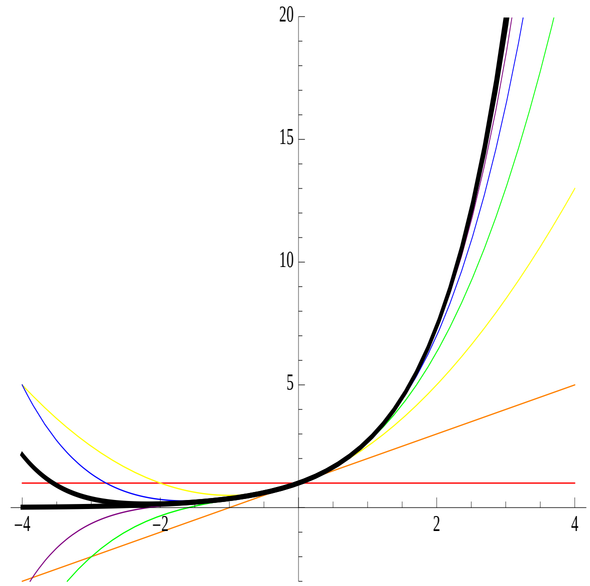
converges for all x. Thus, by the $n^{th}-$Term Test, the individual terms of this series must have limit 0, i.e. $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$. Therefore,

$$\lim_{n\to\infty} R_n(x) = 0$$

so by the Remainder Theorem, f is equal to its Taylor series, i.e.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This conclusion makes sense in light of the following picture. Here, e^x is graphed in black and the Taylor polynomials of e^x for various values of n are graphed in colors. $n = 0$ is in red, $n = 1$ is in orange, $n = 2$ is in yellow, $n = 3$ is green, $n = 4$ is blue, $n = 5$ is purple. Notice that as n increases, the graphs of $P_n(x)$ get closer and closer to the graph of e^x, so it makes sense that as $n \to \infty$, $P_n(x)$ goes to e^x. Since the Taylor series of f is $\lim_{n\to\infty} P_n(x)$, we see that f is equal to its Taylor series:



Prototype Example 2: $g(x) = \sin x$; $a = 0$. Fix $x \in \mathbb{R}$ and recall that

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \pm \frac{x^{2n+1}}{(2n+1)!}.$$

By Taylor's Theorem, we have

$$R_n(x) = \frac{g^{(n+1)}(z)}{(n+1)!}x^{n+1}.$$

No matter what n is, since $g(x) = \sin x$, the $(n+1)^{st}-$derivative of x is either $\sin x$, $\cos x$, $-\sin x$ or $-\cos x$. Since all these functions are bounded by 1 and -1, we can conclude that $|g^{(n+1)}(z)| \leq 1$ and therefore that

$$|R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}.$$

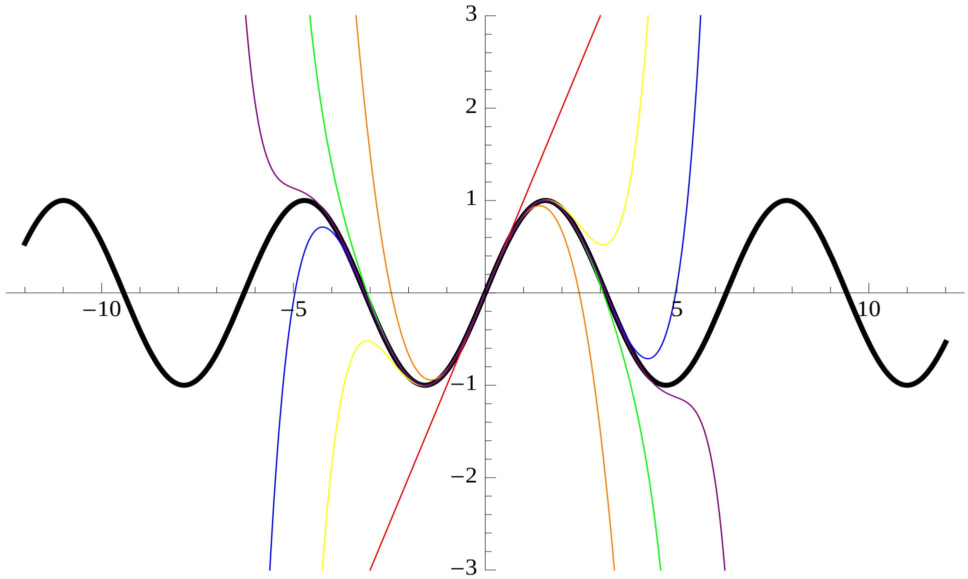
The right-hand side of this expression goes to 0 as $n \to \infty$ (see Prototype Example 1 for the details), so $\lim_{n\to\infty} R_n(x) = 0$ and by the Remainder Theorem, g is equal to its Taylor series, i.e.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

Writing this in $\Sigma-$notation, we have

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

Here are the pictures which explain this: in this graphic, $\sin x$ is graphed in black, and Taylor polynomials for various n are shown: P_1 is red, P_3 is orange, P_5 is yellow, P_7 is green, P_9 is blue and P_{11} is purple. Notice that as n increases, P_n gets closer and closer to more of the graph of $\sin x$. As before, if n were to go to infinity, we would get the entire sine graph, so g is equal to its Taylor series.



What we have discussed in this section can be summarized in the box on the next page. We started with an infinitely differentiable function f (i.e. we started in the upper left corner of the box) and wrote down its Taylor polynomials and remainders (in the upper right corner of the box). Taylor's Theorem tells us that we can rewrite the remainders as shown in the upper right block of the box on the next page. The Remainder Theorem tells us that if $\lim_{n\to\infty} R_n(x) = 0$, then f is equal to its Taylor series (described in the lower right corner of the box) and that f is therefore equal to a power series (the lower left corner).

Thinking more abstractly, if f itself was described as a power series, then that power series must be the Taylor series of f (since the Taylor series of a function is the only power series representation of a function centered at a particular value a).

Summary of the theory of Taylor polynomials and series:

$$\begin{array}{c} f \text{ infinitely} \\ \text{diff'ble on } (a-R, a+R) \\ (R>0) \end{array} \Longrightarrow \left\{ \begin{array}{l} P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ R_n(x) = f(x) - P_n(x) \\ \qquad\quad = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1} \\ \qquad\qquad \text{for some } z \in (a,x) \end{array} \right.$$

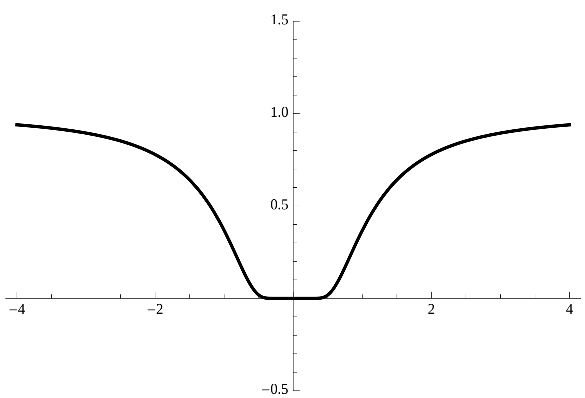
$$\Uparrow \qquad\qquad\qquad \Downarrow \leftarrow \text{ if and only if} \lim_{n\to\infty} R_n(x)=0$$

$$\left. \begin{array}{c} f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \\ \\ \text{(i.e. } f \text{ is equal to} \\ \text{some power series} \\ \text{on } (a-R, a+R)) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ \\ \text{(i.e. } f \text{ is equal to its Taylor} \\ \text{series on } (a-R, a+R)) \\ \text{(so } f(x) \approx P_n(x)) \end{array} \right.$$

There is one further example that requires consideration, however. Consider the function

$$h(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

The graph of this function is shown below:



It turns out that even though this function is piecewise-defined at zero, you can show that for every n, the n^{th} derivative $h^{(n)}(0)$ exists and is equal to zero. Therefore every Taylor polynomial of h of every order is just 0, since

$$P_n(x) = h(0) + h'(0)x + \frac{h''(0)}{2!}x^2 + ... + \frac{h^{(n)}(0)}{n!}x^n = 0 + 0x + ... + 0x^n = 0.$$

Similarly the Taylor series of h is just 0, which converges for all $x \in \mathbb{R}$. But then $R_n(x) = f(x) - P_n(x) = f(x) - 0 = f(x)$, and this does not go to zero as $n \to \infty$ (it goes to $f(x)$). Therefore the Remainder Theorem does not apply to this function, so this function h is not equal to its Taylor series centered at zero (even though the Taylor series converges everywhere). The importance of this example is that it illustrates the following fact: **an infinitely differentiable function is not necessarily equal to its Taylor series, even if the Taylor series converges everywhere.**

Having completed our discussion of the theory of Taylor polynomials and series, we now turn to applications.

2.4 Methods for determining the Taylor series of a function

In order to solve problems involving Taylor polynomials and Taylor series, we first need to learn how to quickly write the Taylor series for a given function. There are two general methods for doing this:

1. Memorize a few commonly used Taylor series, and manipulate one of these series that you memorize to produce the Taylor series for the function you want.

2. Work out the Taylor series of a function using the definition of Taylor series.

The first method is far superior to the second, and is much more commonly used. With regard to this method, there are two issues. First, what Taylor series are commonly used? Second, what do we mean by "manipulate" a series? The second question is easy to answer: "manipulate" means differentiating or integrating the series, and/or doing a substitution in the series; this is the kind of thing that was done in Section 2.1.2. That leaves the first question: what Taylor series need to be memorized?

The first three Taylor series are the ones we have already encountered. Our prototypical examples were

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$

and

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...;$$

both these series converge absolutely for all x. We have also seen a third Taylor series, even though it wasn't labeled as such. Recall from our study of geometric series that

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}$$

for all $r \in (-1, 1)$. Thinking of r as a variable x rather than a constant, we obtain a Taylor series for the function $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

This series only converges when $x \in (-1, 1)$, however, so it is not a valid representation of $\frac{1}{1-x}$ outside the interval $(-1, 1)$ (as we have seen in Section 2.1).

Having memorized these three Taylor series, we can quickly write the Taylor series (centered at 0) for some other functions:

Example 2.5. *Write the Taylor series centered at 0 for each function and give the values of x for which the Taylor series converges:*

(a) $f(x) = e^{2x}$.

Solution: *We know the Taylor series for e^x, and we can subsequently get the Taylor series for e^{2x} by replacing all the x's in the Taylor series for e^x with $2x$'s:*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$

Therefore

$$\begin{aligned} e^{2x} &= \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \\ &= 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!} + ... \\ &= 1 + 2x + \frac{2^2}{2!} x^2 + \frac{2^3}{3!} x^3 + ... \end{aligned}$$

This series converges for all $x \in \mathbb{R}$, since the series for e^x converges for all x.

(b) $f(x) = x \sin x^2$.

Solution: *We know the Taylor series for $\sin x$, so we can subsequently get the Taylor series for $x \sin x^2$ by first replacing all the x's with x^2's, then multiplying through by x:*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...;$$

Therefore

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2}$$

$$\begin{aligned} x \sin x^2 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+2} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3} \\ &= x^3 - \frac{1}{3!} x^7 + \frac{1}{5!} x^{11} - \frac{1}{7!} x^{15} + \frac{1}{9!} x^{19} - \ldots \end{aligned}$$

This series converges for all $x \in \mathbb{R}$, since the series for $\sin x$ converges for all x.

(c) $f(x) = \cos x$.

Solution: *We know the Taylor series for $\sin x$, so we can subsequently get the Taylor series for $\cos x$ by differentiating the series for $\sin x$ term-by-term:*

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots;$$

Therefore

$$\begin{aligned} \cos x &= \frac{d}{dx} \sin x \\ &= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dx} (x^{2n+1}) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots \end{aligned}$$

This series converges for all x, since term-by-term differentiation or integration of a series does not change its radius of convergence.

(d) $f(x) = \ln(1 + x)$.

Solution: *We know the Taylor series of* $\frac{1}{1-x}$*, so we can get the Taylor series of* $\ln(1+x)$ *by replacing* x *with* $(-x)$ *and then integrating term-by-term.*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

Therefore

$$\begin{aligned}
\frac{1}{1+x} = \frac{1}{1-x} &= \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + ... \\
\ln(1+x) &= \int \frac{1}{1+x}\, dx = \int \left(1 - x + x^2 - x^3 + ...\right)\, dx \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ... \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.
\end{aligned}$$

This series has the same radius of convergence as the Taylor series of $\frac{1}{1-x}$ *(namely* 1*), and if you check the endpoints of the interval of convergence you will find that this series converges for* $x \in (-1, 1]$*.*

The last two examples above are commonly used Taylor series, so they should be memorized. There is one other Taylor series to memorize: that of $\arctan x$ (which you will derive in the homework). Here is a complete list of Taylor series that a Calculus II student should know (you should know both the $\Sigma-$notation and written out forms, and you should know for which x the series converge):

Taylor series to memorize:
Here are the six commonly used Taylor series (all of these are centered at $a = 0$):

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$
(holds when $x \in (-1, 1)$)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$
(holds for all x)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$
(holds for all x)

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$
(holds for all x)

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$
(holds when $x \in (-1, 1]$)

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$
(holds when $x \in (-1, 1)$)

Now for the second method of writing a Taylor series. Here, you must use the definition of Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Here, you write down the derivatives of f at a for $n = 0, 1, 2, 3, ...$ and try to find a pattern that allows you to write a general formula for $f^{(n)}(a)$. Then you plug this into the definition to get the Taylor series.

Example 2.6. *Write the Taylor series for $f(x) = e^x$, centered at $a = 6$.*

Solution: *First, notice that this is not the Taylor series for e^x on the memorized list, since that Taylor series is centered at 0, not 6. In fact, whenever you are asked to find the Taylor series of a function centered somewhere other than zero, you must use this method to find the Taylor series.*

Now, let's compute $f^{(n)}(6)$ for various values of n. Since $f'(x) = f(x)$, $f^{(n)}(x) = f(x)$ for all x so $f^{(n)}(6) = e^6$ for all n. Thus, by definition the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} \frac{e^6}{n!}(x-6)^n.$$

2.5 Applications of Taylor polynomials and Taylor series

In this section we consider some of the many applications of the material in this chapter.

2.5.1 Evaluation of limits

Let's consider, as a motivating example, the following limit:

$$\lim_{x \to 0} \frac{12\cos x - 6x^2 - 12}{x^4}$$

Based on what we know from Calculus I, the first thing you might try is to plug in 0 for x and see what happens:

$$\frac{12\cos x - 6x^2 - 12}{x^4} = \frac{12 - 0 - 12}{0} = \frac{0}{0}$$

This is an indeterminate form which we have learned is usually handled using L'Hopital's Rule. Let's use L'Hopital's Rule (in this text, we will use $=^L$ to represent a step where L'Hopital's Rule is used). It turns out, as we see below, that we need to use

L'Hopital's Rule four times to get the answer:

$$\begin{aligned}
\lim_{x\to 0} \frac{12\cos x + 6x^2 - 12}{x^4} &=^L \lim_{x\to 0} \frac{-12\sin x + 12x}{4x^3} = \frac{0}{0} \\
&=^L \lim_{x\to 0} \frac{-12\cos x + 12}{12x^2} = \frac{0}{0} \\
&=^L \lim_{x\to 0} \frac{12\sin x}{24x} = \frac{0}{0} \\
&=^L \lim_{x\to 0} \frac{12\cos x}{24} = \frac{12}{24} = \frac{1}{2}.
\end{aligned}$$

This example illustrates a drawback with L'Hopital's Rule. When you start using L'Hopital's Rule, you don't know how many times you will have to use it to get an answer (and it is not a sure thing that you will ever get an answer at all). Plus, all it takes is one small mistake in any differentiation step to produce an incorrect answer.

Taylor series can be used to solve limits like this with a method that is far superior to L'Hopital's Rule. To do this, simply write the Taylor series of the top and/or bottom of the fraction, factor out any powers of x that remain and cancel the powers of x. Here is how this method works in the context of our motivating example: given

$$\lim_{x\to 0} \frac{12\cos x - 6x^2 - 12}{x^4}$$

we will first write the Taylor series of the expression we want the limit of (it is better to use the written-out form rather than the $\Sigma-$notation). Start with the Taylor series of $\cos x$, which we memorize:

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ... \\
12\cos x &= 12 - 12\frac{x^2}{2} + 12\frac{x^4}{4!} - 12\frac{x^6}{6!} + 12\frac{x^8}{8!} - ... \\
&= 12 - 6x^2 + \frac{12}{4!}x^4 - \frac{12}{6!}x^6 + \frac{12}{8!}x^8 - ... \\
12\cos x - 6x^2 - 12 &= \frac{12}{4!}x^4 - \frac{12}{6!}x^6 + \frac{12}{8!}x^8 - ... \\
\frac{12\cos x - 6x^2 - 12}{x^4} &= \frac{\frac{12}{4!}x^4 - \frac{12}{6!}x^6 + \frac{12}{8!}x^8 - ...}{x^4} \\
&= \frac{12}{4!} - \frac{12}{6!}x^2 + \frac{12}{8!}x^4 - ...
\end{aligned}$$

Now we evaluate the limit:

$$\begin{aligned}
\lim_{x\to 0}\frac{12\cos x-6x^2-12}{x^4} &= \lim_{x\to 0}\left[\frac{12}{4!}-\frac{12}{6!}x^2+\frac{12}{8!}x^4-...\right]\\
&= \frac{12}{4!}-0+0-0+0...\\
&= \frac{12}{4!}=\frac{12}{24}=\frac{1}{2}.
\end{aligned}$$

Here is one more example:

Example 2.7. *Evaluate* $\lim_{x\to 0}\frac{\arctan 3x-3x}{x^2}$ *without using L'Hopital's Rule.*

Solution: *First, write a power series for the expression whose limit we are to evaluate:*

$$\begin{aligned}
\arctan x &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...\\
\arctan 3x &= 3x-\frac{(3x)^3}{3}+\frac{(3x)^5}{5}-\frac{(3x)^7}{7}+...\\
&= 3x-\frac{3^3}{3}x^3+\frac{3^5}{5}x^5-\frac{3^7}{7}x^7+...\\
\arctan 3x-3x &= -\frac{3^3}{3}x^3+\frac{3^5}{5}x^5-\frac{3^7}{7}x^7+...\\
\frac{\arctan 3x-3x}{x^2} &= \frac{=-\frac{3^3}{3}x^3+\frac{3^5}{5}x^5-\frac{3^7}{7}x^7+...}{x^2}\\
&= -\frac{3^3}{3}x+\frac{3^5}{5}x^3-\frac{3^7}{7}x^5+...
\end{aligned}$$

Now, evaluate the limit:

$$\lim_{x\to 0}\frac{\arctan 3x-3x}{x^2}=\lim_{x\to 0}\left[-\frac{3^3}{3}x+\frac{3^5}{5}x^3-\frac{3^7}{7}x^5+...\right]=-0+0-0+0...=0.$$

2.5.2 Approximation of function values and definite integrals

The most important application of Taylor series and polynomials is in approximation problems. Suppose we have some function f which is known to be equal to its Taylor series (this includes exponential, trigonometric and logarithmic functions). Then, since any series is the limit of its partial sums, one can approximate a series by calculating one of its partial sums; in the context of Taylor series, this means one can approximate $f(x)$ by computing the n^{th} Taylor polynomial $P_n(x)$. The larger n is, the better the approximation is, but in general one does not need to consider a large value of n to obtain a very good approximation of f via P_n. Here are some examples which illustrate this:

Example 2.8. *(a) Approximate* $\ln 1.35$ *using a Taylor polynomial of order* 3 *for an appropriately chosen function.*

Solution: *First, we need to choose a function appropriately. Since we are asked to approximate a natural logarithm, we need to choose a function which has something to do with a natural logarithm. We also need a function whose Taylor series we know. In light of the list of memorized Taylor series from the previous section, we should choose* $f(x) = \ln(1 + x)$*. We want to know* $\ln(1.35) = \ln(1 + .35) = f(.35)$*. To do this, we first write the Taylor polynomial* P_3 *of order* 3 *for* f*. The Taylor series of* f *is*

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

so the Taylor polynomial of order 3 *is obtained by truncating this series at the* x^3 *term, i.e.*

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Now for the conclusion. Since $f \approx P_3$*, we have*

$$\ln(1.35) = f(.35) \approx P_3(.35) = (.35) - \frac{(.35)^2}{2} + \frac{(.35)^3}{3} = \frac{7273}{24000} = .303042.$$

(Remark: the actual value of $\ln(1.35)$ *is* $.300105$*, so this approximation is correct to within less than* 1% *error).*

(b) Approximate $e^{.1}$ *using a Taylor polynomial of order* 4 *for an appropriately chosen function.*

Solution: *First, we need to choose a function appropriately. Since we are asked to approximate an exponential, we need to choose a function which has something to do with an exponential. We also need a function whose Taylor series we know. In light of the list of memorized Taylor series from the previous section, we should choose* $f(x) = e^x$*. We want to know* $e^x = e^{.1} = f(.1)$*. To do this, we first write the Taylor polynomial* P_4 *of order* 4 *for* f*. The Taylor series of* f *is*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

so the Taylor polynomial of order 4 *is obtained by truncating this series at the* x^4 *term, i.e.*

$$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

Now for the conclusion. Since $f \approx P_4$, *we have*

$$e^{.1} = f(.1) \approx P_4(.1) = 1 + .1 + \frac{(.1)^2}{2} + \frac{(.1)^3}{3!} + \frac{(.1)^4}{4!} = \frac{265241}{240000} \approx 1.10517.$$

(Remark: this approximation is correct to seven decimal places.)

Observe in these examples that to obtain a decimal approximation for the number you are to approximate, you need only to add/subtract/multiply fractions. You don't need to know anything about e or ln. In fact, this is how your calculator computes exponentials and logarithms. For instance, when you tell your calculator to evaluate $e^{2.3}$, it computes $P_n(2.3)$ for a large value of n (which can be done just be adding, multiplying and subtracting fractions) and spits out the decimal answer.

Also, as shown in the remarks following each example, these approximations are extremely accurate, even for relatively small values of n.

Notice also that in these examples, the order of the Taylor polynomial is always given to you. In Calculus II, you will always be told which order Taylor polynomial to use, but in the real world, you might have to figure out the value of n which produces a "good enough" approximation. This is one topic that might be studied in a Numerical Analysis course.

One can also approximate the values of definite integrals using Taylor polynomials. Suppose you wanted to know the value of

$$\int_0^1 e^{-x^2}\, dx.$$

Usually one evaluates integrals like this using the Fundamental Theorem of Calculus. However, to use the Fundamental Theorem of Calculus one needs to know an antiderivative of the integrand, and for the function e^{-x^2} there is no way to write down an antiderivative, so we cannot find the exact value of this integral. However, we can approximate the integral by replacing the integrand with one of its Taylor polynomials (let's use the fourth Taylor polynomial for no particular reason). Since

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...,$$

by replacing the xs with $(-x^2)$s we get

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + ...$$

so to get the fourth Taylor polynomial of e^{-x^2}, we truncate the Taylor series at the x^4 term:

$$P_4(x) = 1 - x^2 + \frac{x^4}{2}.$$

Finally, since $e^{-x^2} \approx P_4(x)$, we see

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 P_4(x) = \int_0^1 \left[1 - x^2 + \frac{x^4}{2}\right] dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30}.$$

This fraction is roughly .7666..., which is close to the value of the integral which is .74682....

2.5.3 Higher-order derivatives

In Calculus I you learn that the first derivative of a function gives you the slope of that function, and the sign of the first derivative tells you where that function is increasing or decreasing. You also learn that the sign of the second derivative of a function tells you where that function is concave up or down. A natural question is to ask whether any information about a function can be obtained by looking at the sign or value of the third derivative, or fourth derivative, or 100^{th} derivatve of the function. It turns out that the third derivative of a function has something to do with a property of its graph called *curvature*, and higher-order derivatives (especially at $x = 0$) are relevant in certain problems in probability.

Taylor series can be useful in finding higher-order derivatives of functions, by using analysis first touched on in Section 2.2. Suppose we were interested in knowing the 1000^{th} derivative, at $x = 0$, of a function like $f(x) = xe^{x^3}$. To do this, we can write the Taylor series of f, starting with the Taylor series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Substitute x^3 for x to obtain

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + ...$$

Next, multiply through by x to obtain

$$xe^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!} = x + x^4 + \frac{x^7}{2!} + \frac{x^{10}}{3!} + ... \tag{2.3}$$

This is the actual formula for the Taylor series of $f(x) = xe^{x^3}$. At the same time, by definition, the Taylor series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n. \tag{2.4}$$

Although the n in Equation (2.3) is not the same n as in Equation (2.4), it must be that the series in Equations (2.3) and (2.4) match, i.e. have the same coefficients. What is important here is that the coefficients on the x^{1000} term match. From Equation (2.3), we see that the x^{1000} term occurs when $3n + 1 = 1000$, i.e. when $n = 333$, and is therefore

$$\frac{1}{333!} x^{1000}.$$

From Equation (2.4), we see that the x^{1000} term occurs when $n = 1000$, i.e. is

$$\frac{f^{(1000)}(0)}{1000!} x^{1000}.$$

By uniqueness of the power series representation of xe^{x^3}, these last two terms must coincide, so we get

$$\frac{1}{333!} x^{1000} = \frac{f^{(1000)}(0)}{1000!} x^{1000}.$$

We can therefore solve for $f^{(1000)}(0)$ by multiplying through by 1000! and cancelling the x^{1000} terms. This gives

$$f^{(1000)}(0) = \frac{1000!}{333!}.$$

This method generalizes:

General procedure to find $f^{(n)}(0)$ for large n:

1. Write the Taylor series for f (centered at 0). This usually involves a manipulation of a commonly used Taylor series.
2. Identify the term of this Taylor series which has x^n in it; call this term $a_n x^n$.
3. By uniqueness of power series, it must be that
$$a_n x^n = \frac{f^{(n)}(0)}{n!} x^n.$$
Use this equation to solve for $f^{(n)}(0)$.

Example 2.9. *Compute each of the following derivatives:*

(a) Find $f^{(30)}(0)$ if $f(x) = \arctan(3x^2)$.

Solution: *First, we write the Taylor series for f, starting with the Taylor series for* $\arctan x$*:*

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Substitute $3x^2$ in for x to obtain

$$\begin{aligned}\arctan(3x^2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (3x^2)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{2n+1} x^{4n+2}\end{aligned}$$

The x^{30} term occurs when $4n+2 = 30$, i.e. when $n = 7$, so this term is

$$\frac{(-1)^7 3^{2\cdot 7+1}}{2\cdot 7+1} x^{4\cdot 7+2} = \frac{-3^{15}}{15} x^{30}.$$

By uniqueness of power series, this term must equal $\frac{f^{(30)}(0)}{30!} x^{30}$, so we have

$$\frac{-3^{15}}{15} x^{30} = \frac{f^{(30)}(0)}{30!} x^{30},$$

and by solving for $f^{(30)}(0)$*, we get*

$$f^{(30)}(0) = \frac{-3^{15} \cdot 30!}{15}.$$

(b) *Find* $f^{(2000)}(0)$ *if* $f(x) = x^5 \cos 2x$.

Solution: *First, we write the Taylor series for* f*, starting with the Taylor series for* $\cos x$*:*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Substitute $2x$ *in for* x *to obtain*

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n}$$

Next, multiply through by x^5 *to obtain*

$$x^5 \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} x^{2n+5}$$

The x^{2000} *term occurs when* $2n + 5 = 2000$*. But there is no integer* n *which solves this equation, so in reality the* x^{2000} *term in the above series is zero. By uniqueness of power series, this term must equal* $\frac{f^{(2000)}(0)}{2000!} x^{2000}$*, so we have*

$$0 = \frac{f^{(2000)}(0)}{2000!} x^{2000}$$

and therefore $f^{(2000)}(0) = 0$.

2.5.4 Analysis of some numerical series

Taylor series can also be used to analyze series, because one can use the series expansion of an expression inside the summation to write an inequality that is useful in the context of the Comparison Test. Here is an example:

Example 2.10. *Determine whether the series* $\sum_{n=1}^{\infty} \left(e^{1/n} - 1\right)$ *converges or diverges.*

Solution: *None of the standard tests for convergence discussed in Chapter 1 work well with this series. As a last resort, let's write the Taylor series for the terms being added. We know*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

so by substituting $1/n$ in for x, we get

$$e^{1/n} = 1 + \frac{1}{n} + \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3!} + ...$$

and therefore

$$e^{1/n} - 1 = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3!n^3} + \frac{1}{4!n^4} + ... \tag{2.5}$$

Now observe that for $n \geq 1$, all the terms being added in the above expression are positive, so the right-hand side of (2.5) above must be greater than its first term, i.e.

$$e^{1/n} - 1 \geq \frac{1}{n}$$

Now we know that $\sum \frac{1}{n}$ diverges (since it is harmonic); therefore by the Comparison Test, $\sum \left(e^{1/n} - 1\right)$ diverges as well.

We can also find the sum of some series by recognizing them as a Taylor series of a commonly used function, with a value of x plugged in. For example, consider the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$$

which was shown to converge conditionally in Chapter 1. Now we can find the sum of this series; to do this, we want to think of a Taylor series where the denominators of the terms resemble the denominators of the given series. In this case, since the terms alternate and the denominators increase by 1 each term, we think of the Taylor series of $\ln(1 + x)$, since this Taylor series has the same phenomena–the Taylor series of $\ln(1 + x)$ is

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + ...$$

Suppose we were to plug in $x = 1$ to both sides of the preceding equation. On the left-hand side,we get $\ln(1 + 1) = \ln 2$, but on the right-hand side, all the powers x^n just become 1 so we get

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$$

which is exactly the series we wanted to sum. Therefore this series must converge to $\ln 2$.

This example generalizes using the following procedure:

General procedure for finding the sum of a series:

1. If the series is geometric, use the formulas developed in Chapter 1.

2. If the series is not geometric, write the series as something like
$$\sum a_n c^n$$
where c is some constant, and then think about which function f has a Taylor series that looks like $\sum a_n x^n$. The series will converge to $f(c)$.

Example 2.11. *Find the sum of each series:*

(a) $1 - \frac{1}{2} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - ...$

Solution: *Writing this in* $\Sigma-$*notation, we see that the given series is*
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}.$$
To think of this as $\sum a_n c^n$*, we can let* $c = 1$ *and treat the whole series as*
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 1^{2n};$$
this is the Taylor series for $\cos x$ *with* 1 *plugged in for* x*. Thus the series sums to* $\cos 1$*.*

(b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + ...$

Solution: *Writing this in* $\Sigma-$*notation, we see that the given series is*
$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}.$$
To think of this as $\sum a_n c^n$*, we can let* $c = \pi$ *and treat the whole series as*
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \pi^{2n+1};$$
this is the Taylor series for $\sin x$ *with* π *plugged in for* x*. Thus the series sums to* $\sin \pi = 0$*.*

(c) $\sum_{n=2}^{\infty} \frac{2^n}{n!}$

Solution: *We know from the Taylor series of e^x that for any x,*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

By plugging in $x = 2$ to both sides, we see

$$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!},$$

which is almost the series we are given (it has a different starting index). To account for the different starting index, subtract the $n = 0$ and $n = 1$ terms as in Chapter 1:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{2^n}{n!} &= \sum_{n=0}^{\infty} \frac{2^n}{n!} - \frac{2^1}{1!} - \frac{2^2}{2!} \\
&= e^2 - 2 - 2 \\
&= e^2 - 4.
\end{aligned}$$

2.6 Exercises for Chapter 2

(From Section 2.1) In problems 1-6, you are given a power series. For each power series, state where the power series is centered, identify is fourth coefficient, its first term, its sixth coefficient, and its eighth term.

1. $x - x^3 + x^5 - x^7 + x^9 - ...$
2. $1 + (x-1) + (x-1)^2 + (x-1)^3 + (x-1)^4 + ...$
3. $\sum_{n=1}^{\infty} \frac{2}{n^n} x^n$
4. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n^2} (x+8)^{2n}$
5. $\sum_{n=0}^{\infty} \frac{3x^{3n+1}}{n+1}$
6. $\sum_{n=2}^{\infty} \frac{(x+3)^n}{(3n)^n}$

For problems 7-11, write each power series in $\Sigma-$notation:

7. $\frac{1}{2}x + \frac{4}{2^2 \cdot 2!}x^2 + \frac{9}{2^3 \cdot 3!}x^3 + \frac{16}{2^4 \cdot 4!}x^3 + ...$
8. $\frac{x-2}{1} + \frac{(x-2)^2}{2 \cdot 2!} + \frac{(x-2)^3}{3 \cdot 3!} + \frac{(x-2)^4}{4 \cdot 4!} + ...$
9. $2(x-5)^5 + 2(x-5)^7 + 2(x-5)^9 + 2(x-5)^{11} + ...$
10. $3!x^2 + 6!x^4 + 9!x^6 + 12!x^8 + ...$
11. $1 - \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} - \frac{(x-1)^6}{6!} + ...$

(From Section 2.1.1) In problems 12-19, you are given a power series. Find the radius of convergence of each power series, and find the set of x for which the power series converges:

12. $\sum_{n=0}^{\infty} (5x)^n$
13. $\sum_{n=0}^{\infty} n^3 x^n$
14. $\sum_{n=0}^{\infty} \frac{x^n}{n^n}$
15. $\sum_{n=2}^{\infty} \frac{(x-3)^n}{n}$
16. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-5)^n}{2^n n^2}$
17. $\sum_{n=1}^{\infty} n! x^n$

18. $\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{(2n!)}$ (In this problem, you only need to find the radius of convergence, not the set of x for which the power series converges.)

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}(x+2)^n$

20. Suppose a power series of the form $\sum a_n x^n$ converges when $x = -5$ and diverges at $x = 7$. Given this information, classify the following statements as "true", "false", or "not possible to determine":

    (a) $\sum a_n x^n$ converges when $x = 3$.
    (b) $\sum a_n x^n$ converges when $x = 5$.
    (c) $\sum a_n x^n$ converges when $x = 6$.
    (d) $\sum a_n x^n$ converges when $x = -7$.
    (e) $\sum a_n x^n$ converges when $x = 8$.
    (f) $\sum a_n x^n$ converges when $x = -10$.

21. Suppose a power series of the form $\sum a_n (x-3)^n$ converges when $x = 7$ and diverges when $x = 10$. Given this information, classify the following statements as "true", "false", or "not possible to determine":

    (a) $\sum a_n (x-3)^n$ converges when $x = 0$.
    (b) $\sum a_n (x-3)^n$ converges when $x = 11$.
    (c) $\sum a_n (x-3)^n$ converges when $x = -2$.
    (d) $\sum a_n (x-3)^n$ converges when $x = -7$.
    (e) $\sum a_n (x-3)^n$ converges when $x = 8$.
    (f) $\sum a_n (x-3)^n$ converges when $x = 4$.

(From Section 2.1.2) In problems 22-29, find a power series representation of the given function, centered at 0:

22. $f(x) = \frac{3}{1-x}$

23. $f(x) = \frac{1}{1-x^2}$

24. $f(x) = \ln(1+x)$

25. $f(x) = \arctan x$ (*Hint:* First write the power series of $\frac{1}{1+x^2}$, then integrate.)

26. $f(x) = \frac{1}{(1-x)^3}$ (*Hint:* Differentiate $\frac{1}{1-x}$ twice.)

27. $f(x) = \frac{1}{1-x^3}$

28. $f(x) = \frac{2}{2+5x}$

29. $f(x) = \frac{-3}{-2-x}$

(From Section 2.4) In problems 30-45, write the Taylor series (centered at 0) of the given function f. Also, write the first and fourth Taylor polynomials of f, centered at 0.

30. $f(x) = e^{-x}$

31. $f(x) = \cos 4x$

32. $f(x) = \ln(1-2x)$

33. $f(x) = \frac{1}{1+2x}$

34. $f(x) = \cos x^2$

35. $f(x) = \ln(1-x^2)$

36. $f(x) = x \sin 3x$

37. $f(x) = \frac{x}{e^{x^2}}$

38. $f(x) = x \sin x^2 - x^3$

39. $f(x) = (x+1)^3$

40. $f(x) = e^x + e^{-x}$ (*Hint:* Find the Taylor series of e^x and e^{-x} independently, and then add them term-by-term.)

41. $f(x) = (x^2+2)\cos x$ (*Hint:* Find the Taylor series of $x^2 \cos x$ and $2\cos x$ independently, and then add them term-by-term.)

42. $f(x) = \arctan x$ (you have to derive the formula given in Section 2.4).

43. $f(x) = \ln(x^2+1)$

44. $f(x) = \frac{2}{x^2+1}$

45. $f(x) = \frac{1}{(1-x)^2}$

(From Section 2.4) In problems 46-48, write the Taylor series (centered at the given value of a) of the given function.

46. $f(x) = e^x$; $a = 5$

47. $f(x) = \sin x$; $a = \pi$

48. $f(x) = \cos 2x$; $a = \frac{\pi}{4}$

(From Section 2.5.1) For problems 49-59, evaluate the following limits without using L'Hopital's Rule:

49. $\lim_{x\to 0} \frac{\sin x}{x}$

50. $\lim_{x\to 0} \frac{\cos x - 1}{x}$

51. $\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9}$

52. $\lim_{x\to 0} \frac{\arctan x - x}{x^3}$

53. $\lim_{x\to 0} \frac{e^x - 1 - x}{2x^2}$

54. $\lim_{x\to 0} \frac{2e^x - 2 - 2x - x^2}{x^3}$

55. $\lim_{x\to 0} \frac{\arctan 6x^2 - 6x^2}{x^6}$

56. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{20}}$

57. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{24}}$

58. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{30}}$

59. $\lim_{x\to 0} \frac{\arctan x^9 - \ln(x^9+1)}{x^{18}}$

(From Section 2.5.2) In problems 60-64, approximate each of the following numbers using the second Taylor polynomial of an appropriately chosen function:

60. $\ln .8$

61. $\sin .3$

62. $e^{3/5}$

63. $\arctan \frac{1}{6}$

64. $\sqrt{2}$ (*Hint:* Here, the appropriate function is $f(x) = \sqrt{x+1}$. You will have to figure out the second Taylor polynomial of $f(x)$ by computing derivatives of f at zero and using the definition.)

(From Section 2.5.2) In problems 65-69, approximate each of the following numbers using the fourth Taylor polynomial of an appropriately chosen function:

65. $\sin \frac{1}{4}$

66. $\cos .2$

67. $\sqrt{e}$

68. $\arctan \frac{1}{2}$

69. $\sqrt{2}$ (*Hint:* As in problem 64, the appropriate function is $f(x) = \sqrt{x+1}$.)

70. Approximate $\int_0^{1/2} \cos(4x^2)\, dx$ by replacing the integrand with its fourth Taylor polynomial.

71. Approximate $\int_0^{1/2} \arctan x^2\, dx$ by replacing the integrand with its fourth Taylor polynomial.

72. Approximate $\int_{-1}^{1} e^{-x^3}\, dx$ by replacing the integrand with its fourth Taylor polynomial.

73. Approximate $\int_0^1 \ln(2x^2+1)\, dx$ by replacing the integrand with its fourth Taylor polynomial.

74. Approximate $\int_0^1 x^2 \sin(x^2)\, dx$ by replacing the integrand with its sixth Taylor polynomial.

75. Approximate $\int_0^1 x^8 \sin x\, dx$ by replacing the integrand with its twelfth Taylor polynomial. Describe the integration technique that one would use to find the exact value of this integral. (Isn't using a Taylor polynomial better?)

(From Section 2.5.3) In problems 76-82, find each higher-order derivative:

76. Find $f^{(6)}(0)$ if $f(x) = \sin x^2$.

77. Find $f^{(36)}(0)$ if $f(x) = \cos x^2$.

78. Find $f^{(100)}(0)$ if $f(x) = 4\ln(2x^2+1)$.

79. Find $f^{(40)}(0)$ if $f(x) = e^{2x^3}$.

80. Find $f^{(42)}(0)$ if $f(x) = e^{2x^3}$.

81. Find $f^{(30)}(0)$ if $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(4n)!}$.

82. Find $f^{(30)}(0)$ if $f(x) = \sum_{n=0}^{\infty} \frac{x^{4n+2}}{(4n)!}$.

83. Suppose the third-order Taylor polynomial (centered at 0) of some unknown function f is given by $P_3(x) = 2 - x - \frac{x^2}{3} + 2x^3$. Find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$.

84. (From Section 2.5.4) Determine whether the series $\sum_{n=1}^{\infty} \left(e^{1/\sqrt{n}} - 1\right)$ converges or diverges.

85. Determine whether the series $\sum_{n=1}^{\infty} \left(\cos(\frac{1}{n}) - 1\right)$ converges or diverges.

86. Determine whether the series $\sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{n}\right)$ converges or diverges.

(From Section 2.5.4) In problems 87-96, find the sum of each of the following series (you may assume without proof that the series converge):

87. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$

88. $2 + 2 + \frac{2}{2!} + \frac{2}{3!} + \frac{2}{4!} + \frac{2}{5!} + \ldots$

89. $\frac{\pi}{4} - \frac{\pi^2}{4^2 2!} + \frac{\pi^4}{4^4 4!} - \frac{\pi^6}{4^6 6!} + \ldots$

90. $1 - 3 + \frac{9}{2!} - \frac{27}{3!} + \frac{81}{4!} - \frac{3^5}{5!} + \ldots$

91. $1 + e + \frac{e^2}{2} + \frac{e^3}{3!} + \frac{e^4}{4!} + \ldots$

92. $2 - \frac{2\pi^2}{2!} + \frac{2\pi^4}{4!} - \frac{2\pi^6}{6!} + \ldots$

93. $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} - \ldots$

94. $\frac{100}{2!} - \frac{10000}{4!} + \frac{10^6}{6!} - \frac{10^8}{8!} + \ldots$

95. $1 - \frac{\pi^2}{2^2 3!} + \frac{\pi^4}{2^4 5!} - \frac{\pi^6}{2^6 7!} + \frac{\pi^8}{2^8 9!} - \ldots$

96. $\frac{1}{2} - \frac{1}{2^2 \cdot 2} + \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} + \ldots$

2.6.1 Answers (not solutions) to the exercises

1. centered at 0; fourth coefficient is 0; first term is x; sixth coefficient is 0; eighth term is 0.

2. centered at 1; fourth coefficient is 1; first term is $(x-1)$; sixth coefficient is 1; eighth term is $(x-1)^8$.

3. centered at 0; fourth coefficient is $\frac{2}{4^4}$; first term is $2x$; sixth coefficient is $\frac{2}{6^6}$; eighth term is $\frac{2}{8^8}x^8$.

4. centered at -8; fourth coefficient is $\frac{3}{4}$; first term is 0; sixth coefficient is $\frac{-4}{9}$; eighth term is $\frac{5}{16}x^8$.

5. centered at 0; fourth coefficient is $\frac{3}{2}$; first term is $3x$; sixth coefficient is 0; eighth term is 0.

6. centered at -3; fourth coefficient is $\frac{1}{12^4}$; first term is 0; sixth coefficient is $\frac{1}{18^6}$; eighth term is $\frac{1}{24^8}(x+3)^8$.

7. $\sum_{n=1}^{\infty} \frac{n^2}{2^n n!}x^n$ (multiple answers are possible)

8. $\sum_{n=1}^{\infty} \frac{1}{n \cdot n!}(x-2)^n$ (multiple answers are possible)

9. $\sum_{n=2}^{\infty} 2(x-5)^{2n+1}$ (multiple answers are possible)

10. $\sum_{n=1}^{\infty}(3n)!x^{2n}$ (multiple answers are possible)

11. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-1)^{2n}$ (multiple answers are possible)

12. $R = \frac{1}{5}$; $\left(\frac{-1}{5}, \frac{1}{5}\right)$

13. $R = 1$; $(-1, 1)$

14. $R = \infty$; $(-\infty, \infty)$

15. $R = 1$; $[2, 4)$

16. $R = 2$; $[3, 7]$

17. $R = 0$; $\{0\}$

18. $R = 4$

19. $R = 3$; $(-1, 5]$

20. (a) True
    (b) Not possible to determine
    (c) Not possible to determine
    (d) Not possible to determine
    (e) False
    (f) False

21. (a) True
    (b) False
    (c) Not possible to determine
    (d) False
    (e) Not possible to determine
    (f) True

22. $\sum_{n=0}^{\infty} 3x^n = 3 + 3x + 3x^2 + 3x^3 + 3x^4 + ...$

23. $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + ...$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ...$

25. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - ...$

26. $\sum_{n=0}^{\infty}(n+1)(n+2)x^n = 2 + 6x + 12x^2 + 20x^3 + 30x^4 + ...$

27. $\sum_{n=0}^{\infty} x^{3n} = 1 + x^3 + x^6 + x^9 + ...$

28. $\sum_{n=0}^{\infty} \left(\frac{-5}{2}\right)^n x^n = 1 - \frac{5}{2}x + \left(\frac{5}{2}\right)^2 x^2 - \left(\frac{5}{2}\right)^3 x^3 + \left(\frac{5}{2}\right)^4 x^4 - ...$

29. $\sum_{n=0}^{\infty} \frac{3}{2} \left(\frac{-1}{2}\right)^n x^n = \frac{3}{2} - \frac{3}{2} \cdot \frac{1}{2}x + \frac{3}{2} \left(\frac{1}{2}\right)^2 x^2 - \frac{3}{2} \left(\frac{1}{2}\right)^3 x^3 + \frac{3}{2} \left(\frac{1}{2}\right)^4 x^4 - ...$

30. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...$

31. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{(2n)!} x^{2n} = 1 - \frac{4^2}{2!}x^2 + \frac{4^4}{4!}x^4 - \frac{4^6}{6!}x^6 + ...$

32. $\sum_{n=1}^{\infty} \frac{-2^n}{n} x^n = -2x - \frac{4}{2}x^2 - \frac{8}{3}x^3 - \frac{16}{4}x^4 - \frac{32}{5}x^5 - ...$

33. $\sum_{n=0}^{\infty}(-2)^n x^n = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - ...$

34. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + ...$

35. $\sum_{n=1}^{\infty} \frac{-1}{n} x^{2n} = -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 \frac{1}{5}x^{10} - \frac{1}{6}x^{12} + ...$

36. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2} = \frac{3}{1!}x^2 - \frac{3^3}{3!}x^4 + \frac{3^5}{5!}x^6 - \frac{3^7}{7!}x^8 + \frac{3^9}{9!}x^{10} - ...$

37. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1} = x - x^3 + \frac{x^5}{2!} - \frac{x^7}{3!} + \frac{x^9}{4!} - ...$

38. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3} = -\frac{1}{3!}x^7 + \frac{1}{5!}x^{11} - \frac{1}{7!}x^{15} + \frac{1}{9!}x^{19} - ...$

39. $1 + 3x + 3x^2 + x^3$ (this series is finite)

40. $\sum_{n=0}^{\infty} \frac{2}{(2n)!} x^{2n} = 2 + x^2 + \frac{2}{4!}x^4 + \frac{2}{6!}x^6 + ...$

41. $2 + \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{1}{(2n)!} - \frac{2}{(2n+2)!} \right) x^{2n+2} = 2 + \left(1 - \frac{2}{2!}\right) x^2 - \left(\frac{1}{2!} - \frac{2}{4!}\right) x^4 + ...$

42. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + ...$

43. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + ...$

44. $\sum_{n=0}^{\infty} 2(-1)^n x^{2n} = 2 - 2x^2 + 2x^4 - 2x^6 + 2x^8 - ...$

45. $\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$

46. $\sum_{n=0}^{\infty} \frac{e^5}{n!} (x-5)^n = e^5 + e^5(x-5) + \frac{e^5}{2!}(x-5)^2 + \frac{e^5}{3!}(x-5)^3 + ...$

47. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x-\pi)^{2n+1} = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} - ...$

48. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1}}{(2n+1)!} \left(x - \frac{\pi}{4}\right)^{2n+1} = -2(x - \frac{\pi}{4}) + \frac{2^3}{3!}(x - \frac{\pi}{4})^3 - \frac{2^5}{5!}(x - \frac{\pi}{4})^5 + ...$

49. 1

50. 0

51. $\frac{-1}{6}$

52. $\frac{-1}{3}$

53. $\frac{1}{4}$

54. $\frac{1}{3}$

55. -72

56. 0

57. $\frac{-1}{6}$

58. $-\infty$

59. $\frac{1}{2}$

60. $\frac{-11}{50}$

61. .3

62. $\frac{89}{50}$

63. $\frac{1}{6}$

64. $\frac{11}{8}$

65. $\frac{95}{384}$

66. $\frac{14701}{15000}$

67. $\frac{211}{128}$

68. $\frac{11}{24}$

69. $\frac{179}{128}$

70. $\frac{9}{20}$

71. $\frac{1}{24}$

72. 2

73. $\frac{4}{15}$

74. $\frac{1}{5}$

75. $\frac{2557}{14040}$; you would perform integration by parts eight times to evaluate this integral.

76. -120

77. $\frac{-36!}{18!}$

78. $\frac{-2^{52}100!}{50}$

79. 0

80. $\frac{2^{14}42!}{14!}$

81. $\frac{30!}{120!}$

82. $30 \cdot 29$

83. $f(0) = 2$; $f'(0) = -1$; $f''(0) = \frac{-2}{3}$; $f'''(0) = 12$.

84. $\sum_{n=1}^{\infty} \left(e^{1/\sqrt{n}} - 1\right)$ diverges

85. $\sum_{n=1}^{\infty} \left(\cos(\frac{1}{n}) - 1\right)$ converges

86. $\sum_{n=1}^{\infty} \ln\left(1 - \frac{1}{n}\right)$ diverges

87. $\frac{\pi}{4}$

88. $2e$

89. $\frac{\sqrt{2}}{2}$

90. e^{-3}

91. e^{e}

92. -2

93. $\cos 1$

94. $\cos 10 - 1$

95. $\frac{2}{\pi}$

96. $\ln(\frac{1}{2})$