We suppose throughout this handout that $\overrightarrow{r}(t) : \mathbb{R} \to \mathbb{R}^3$ is a 3-dimensional vector-valued function with graph C (although anything that doesn't use a cross product works for any dimension).

- Velocity vector $\overrightarrow{v}(t) = \overrightarrow{r}'(t)$. This gives the direction of the graph and (by taking the magnitude), gives the speed at time t. Velocity vectors are tangent to the graph of C.

- Arc length parameterization $s(t)$:

$$s(t) = \int_a^t ||\overrightarrow{r}'(t)||\, dt$$

$$\text{Note that } \frac{ds}{dt} = s'(t) = ||r'(t)||.$$

- Unit tangent vector $\overrightarrow{T}(t)$:

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{||\overrightarrow{r}'(t)||}$$

$\overrightarrow{T}(t)$ is tangent to C, has length 1.

- Curvature $\kappa(t)$. This is a non-negative scalar (not necessarily constant with respect to t) that tells you how sharply C curves; the greater $\kappa(t)$ is, the sharper the curve turns at time t.

$$\begin{aligned} \kappa(t) &= \left|\left|\frac{d\overrightarrow{T}}{ds}\right|\right| \\ &= \frac{||\overrightarrow{T}'(t)||}{||\overrightarrow{r}'(t)||} \\ &= \frac{||\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)||}{||\overrightarrow{r}'(t)||^3} \end{aligned}$$

The curvature of a 2-dim graph $y = f(x)$ is $\kappa(x) = \dfrac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}$.

- Principal unit normal vector $\overrightarrow{N}(t)$:

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{||\overrightarrow{T}'(t)||} = \frac{1}{\kappa(t)} \frac{d\overrightarrow{T}}{ds}$$

$$\overrightarrow{N}(t) \perp \overrightarrow{T}(t)$$

$\overrightarrow{N}(t)$ points to the concave side of C.

- Binormal vector $\overrightarrow{B}(t)$:

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$$

$\overrightarrow{B}(t)$ is orthogonal to both $\overrightarrow{T}(t)$ and $\overrightarrow{N}(t)$.

$\overrightarrow{T}(t), \overrightarrow{N}(t), \overrightarrow{B}(t)$ form the TNB frame at time t.

- Acceleration vector $\overrightarrow{a}(t)$:

$$\begin{aligned} \overrightarrow{a}(t) &= \overrightarrow{v}'(t) = \overrightarrow{r}''(t) \\ &= a_T \overrightarrow{T}(t) + a_N \overrightarrow{N}(t) \text{ where} \end{aligned}$$

  - a_T, the tangential component of acceleration, is

$$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}\left(||\overrightarrow{r}'(t)||\right) = \overrightarrow{T}(t) \cdot \overrightarrow{a}(t)$$

  - a_N, the normal component of acceleration, is

$$a_N = \kappa(t)\left(\frac{ds}{dt}\right)^2 = \kappa(t)\left(||\overrightarrow{r}'(t)||\right)^2 = \frac{||\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)||}{||\overrightarrow{r}'(t)||} = \left|\left|\overrightarrow{T}(t) \times \overrightarrow{a}(t)\right|\right|.$$

  The acceleration vector points in the direction the graph of C is turning towards, and is always orthogonal to the binormal vector.