1. (9 pts) Calculate the indefinite integral

$$\int \left\langle 4e^t, \frac{3}{t^2+1}, \sin\left(\frac{t}{3}\right) \right\rangle dt.$$

Solution: Integrate component-wise to get

$$\int \left\langle 4e^t, \frac{3}{t^2+1}, \sin\left(\frac{t}{3}\right) \right\rangle dt \quad = \quad \left\langle 4e^t, 3\tan^{-1}(t), -3\cos\left(\frac{t}{3}\right) \right\rangle + \overrightarrow{c}.$$

2. (9 pts) Determine whether the two lines given below are parallel, intersecting, or skew:

$$\begin{cases} x = -2t - 2 \\ y = t - 3 \\ z = 4 \end{cases} \qquad \begin{cases} x = 2t - 1 \\ y = -t + 1 \\ z = 3t + 1 \end{cases}$$

Solution: The direction vectors for the two lines are $< -2, 1, 4 >$ and $< 2, -1, 3 >$ respectively so the lines are clearly not parallel (these vectors are not multiples of one another). To see whether they intersect, replace t with s in the second equation. Then set the expressions for x, y and z equal to each other to obtain the system of equations

$$\begin{cases} -2t - 2 = 2s - 1 \\ t - 3 = -s + 1 \\ 4 = 3t + 1 \end{cases}$$

From the last equation, t must be equal to 1. Substituting $t = 1$ into the second equation and solving for s, we obtain $s = 3$. But in the first equation, $s = 3$ and $t = 1$ yields $-4 = 5$, a false statement. So this system has no solution for the variables s and t; therefore the lines do not intersect. Thus they are skew.

3. (12 pts) Write an equation of the plane containing the point $(3, -3, 1)$ and the line given by the symmetric equations

$$\frac{x-1}{3} = y + 1 = \frac{z}{2}.$$

Solution: The point $(1, -1, 0)$ lies on the given line (choose x and solve for y and z to get this point). So the vector $(3, -3, 1) - (1, -1, 0) =< 2, -2, 1 >$ lies in the plane and so does the direction vector for the line, namely $< 3, 1, 2 >$. Therefore a normal vector to the plane is given by

$$< 2, -2, 1 > \times < 3, 1, 2 >=< -5, -1, 8 > .$$

Finally using this normal vector and any point in the plane (I use $(1, -1, 0)$), we see the equation of the plane is

$$-5(x-1) - (y+1) + 8z = 0.$$

4. (15 pts) Write down a definite integral (you do not need to evaluate the integral) which gives the circumference of the ellipse given by the equation

$$\frac{x^2}{5} + \frac{y^2}{3} = 1.$$

Solution: The vector-valued function which gives this ellipse is $\overrightarrow{r}(t) =< \sqrt{5}\cos t, \sqrt{3}\sin t >$ where $0 \leq t \leq 2\pi$. So the circumference of the ellipse is the length of $\overrightarrow{r}(t)$ which is given by

$$s = \int_0^{2\pi} ||\overrightarrow{r}'(t)||\, dt = \int_0^{2\pi} \sqrt{\left(-\sqrt{5}\sin t\right)^2 + \left(\sqrt{3}\cos t\right)^2}\, dt = \int_0^{2\pi} \sqrt{5\sin^2 t + 3\cos^2 t}\, dt.$$

5. (9 pts) Sketch a rough graph in 3−dimensional space of each of the following:

   (a) $x^2 = y^2 + z^2$
   Solution: This is a cone which opens in the positive and negative directions of the $x-$axis.

   (b) $y^2 = x^2 + z^2 + 1$
   Solution: This is a hyperboloid of two sheets opening in the direction of the positive and negative $y-$axis.

   (c) $y^2 + z^2 = 1$
   Solution: This is a cylinder parallel to the $x-$axis, cross-sections to the cylinder are circles of radius 1.

6. (10 pts) Prove (using techniques from Math 230) that if the sides of a parallelogram all have the same length, then the diagonals of that parallelogram are perpendicular.
Solution: Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be vectors which give adjacent sides of the parallelogram. Then the two diagonals are $\overrightarrow{v} + \overrightarrow{w}$ and $\overrightarrow{v} - \overrightarrow{w}$. Now

$$(\overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{v} - \overrightarrow{w}) = (\overrightarrow{v} \cdot \overrightarrow{v}) - (\overrightarrow{v} \cdot \overrightarrow{w}) + (\overrightarrow{v} \cdot \overrightarrow{w}) - (\overrightarrow{w} \cdot \overrightarrow{w}) = ||\overrightarrow{v}||^2 - ||\overrightarrow{w}||^2 = 0 \text{ since by hypothesis, } ||\overrightarrow{v}|| = ||\overrightarrow{w}||.$$

Since their dot product is zero, $\overrightarrow{v} + \overrightarrow{w}$ and $\overrightarrow{v} - \overrightarrow{w}$ are perpendicular.

7. (20 pts) Consider the vector-valued function $\overrightarrow{r}(t) =< \cos t, t^3, 4\sin t >$, where $-\infty < t < \infty$.

(a) Sketch the graph of $\overrightarrow{r}(t)$ as seen from the positive $y-$axis.
Solution: This is an ellipse centered at the origin; the ellipse extends to 1 and -1 in the $x-$direction and extends to 4 and -4 in the $z-$direction. The orientation of the ellipse goes from the positive x-axis to the positive $z-$axis.

(b) Suppose $\overrightarrow{r}(t)$ gives the position vector for a particle moving in $3-$dimensional space. Find the particle's velocity, speed, and acceleration when $t = 0$.
Solution:

$$\begin{aligned}
\overrightarrow{v}(t) = \overrightarrow{r}'(t) &= <-\sin t, 3t^2, 4\cos t> \\
\overrightarrow{v}(0) &= <0, 0, 4> \\
\overrightarrow{a}(t) = \overrightarrow{r}''(t) &= <-\cos t, 6t, -4\sin t> \\
\overrightarrow{a}(0) &= <-1, 0, 0> \\
\text{speed at } t = 0 &= ||\overrightarrow{v}(0)|| = 4.
\end{aligned}$$

(c) Sketch the graph of $\overrightarrow{r}(t)$ as seen from the usual perspective. Indicate the velocity and acceleration vectors at $t = 0$ on your graph.
Solution: This is an elliptical helix; the graph moves from left to right in the $y-$axis as t increases; the size of the ellipse is always the same (see part (a)). The velocity vector at $t = 0$ points from the point $(1, 0, 0)$ upward to $(1, 0, 4)$ and the acceleration vector at $t = 0$ points from the point $(1, 0, 0)$ to the origin.

8. (16 pts) To say that two planes in $\mathbb{R}^3$ are perpendicular means heuristically that the two planes meet at right angles. For example, the $xy-$plane is perpendicular to the $xz-$plane.

(a) Here is an incorrect definition of what it means for two planes to be perpendicular:
Wrong definition: Two planes $\mathcal{P}$ and $\mathcal{Q}$ are said to be *perpendicular* if every vector in $\mathcal{P}$ is perpendicular to every vector in $\mathcal{Q}$.
Explain why this definition is wrong.
Solution: If two planes are perpendicular, then they intersect in a line. Take a nonzero vector that lies in both planes; it cannot be perpendicular to itself.

(b) Give a precise and correct mathematical definition of what it means for two planes to be perpendicular (there are many possible answers). Your answer should be a complete sentence that starts "Two planes $\mathcal{P}$ and $\mathcal{Q}$ are said to be perpendicular if ...".
Solution: Two planes are perpendicular if they have normal vectors which are perpendicular.

(c) Verify using the definition you gave in part (b) above that the $xy-$ and $xz-$planes are perpendicular.
Solution: A normal vector to the $xy-$plane is $< 0, 0, 1 >$; a normal vector to the $xz-$plane is $< 0, 1, 0 >$. These vectors are perpendicular as $< 0, 0, 1 > \cdot < 0, 1, 0 >= 0$.