1. (a) Find a vector of length 12 parallel to $\overrightarrow{v} = \langle 3, -2, 1\rangle$.
   (b) How many different possible solutions are there to the previous question?

2. Find a vector of length 3 which is orthogonal to both $\langle 1, -3, -1\rangle$ and $\langle -2, 3, 1\rangle$.

3. Find the measure of the angle between $\overrightarrow{v} = \langle -1, -2, 4\rangle$ and $\overrightarrow{w} = \langle -2, 3, 1\rangle$.

4. Let θ be the angle between the vectors $\langle -2, 4\rangle$ and $\langle 3, -5\rangle$. Is the measure of θ greater than, equal to, or less than $\pi/2$? How do you know?

5. Find the projection of $\langle -1, -1, 4\rangle$ onto $\langle -2, 1, -2\rangle$.

6. Classify the following statements as true or false.
   (a) TRUE FALSE $\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{w} \times \overrightarrow{v}$

   (b) TRUE FALSE $(\overrightarrow{v} \times \overrightarrow{w}) \cdot \overrightarrow{v} = (\overrightarrow{w} \times \overrightarrow{v}) \cdot \overrightarrow{v}$

   (c) TRUE FALSE $\mathbf{proj}_{\overrightarrow{b}}(\overrightarrow{v} + \overrightarrow{w}) = \mathbf{proj}_{\overrightarrow{b}}\overrightarrow{v} + \mathbf{proj}_{\overrightarrow{b}}\overrightarrow{w}$

   (d) TRUE FALSE $||c\overrightarrow{v}|| = c||\overrightarrow{v}||$ (for any scalar c)

   (e) TRUE FALSE $\overrightarrow{v} \times \overrightarrow{w} = ||\overrightarrow{v}||||\overrightarrow{w}|| \sin\theta$ (where θ is the angle between $\overrightarrow{v}$ and $\overrightarrow{w}$)

   (f) TRUE FALSE $\overrightarrow{v} \cdot \overrightarrow{v} \geq 0$

   (g) TRUE FALSE $(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w})$

7. Find the symmetric equations of the line passing through the point $(1, -2, 4)$ that is perpendicular to the lines

$$\begin{cases} x = t - 2 \\ y = -3t + 7 \\ z = t + 1 \end{cases} \qquad \text{and} \qquad \frac{x-5}{-2} = \frac{y+3}{8} = z - 3.$$

8. Find the equation of the line passing through the point $(-2, 5, 3)$ which lies in the plane $2x - y = -9$ and is perpendicular to the line

$$\frac{x+5}{1} = \frac{y+1}{2} = \frac{z-6}{-1}.$$

9. Do the following lines intersect? Justify your answer.

$$\frac{x+1}{3} = y - 2 = \frac{z+2}{-2} \qquad \text{and} \qquad \frac{x-8}{3} = \frac{y-8}{4} = \frac{z+4}{-2}$$

10. Find the equation of the plane containing the point $(2, -2, 3)$ and the line

$$\frac{x-4}{-3} = \frac{y+2}{-1} = \frac{z+1}{2}.$$

11. Describe the graph of each equation:

    (a) $z = x^2 - y^2$
    (b) $z^2 = x^2 - y^2$
    (c) $z^2 = x^2 + y^2$
    (d) $z = x^2 + y^2$
    (e) $z^2 - y^2 - x^2 = 1$
    (f) $z^2 - y^2 - x^2 = 0$
    (g) $z^2 - y^2 - x^2 = -1$
    (h) $x^2 + y^2 = 1$
    (i) $x^2 - y^2 = z$

12. Write down a definite integral which gives the length of the curve traced out by the endpoint of the vector-valued function $\overrightarrow{r}(t) = \left\langle \cosh(t^2), 4\sqrt{t}, 2\tan t \right\rangle$ from $t = 2$ to $t = 5$. You do not need (nor should you try) to evaluate the integral. Any derivatives, limits, etc. that appear inside the integrand should be evaluated.

13. Sketch the graph of the vector-valued function $\overrightarrow{r}(t) = \langle \cos t, -t^{-1}, \sin t \rangle$. Assume that the domain of the function is $(0, \infty)$.

14. Find the indicated quantities. If any of these do not exist, say so and why.

    (a) $\frac{d}{dt} \langle e^{\sin t}, \sin(e^t), e^t \sin t \rangle$
    (b) $\int \left\langle 5t^{-1}, e^{2t-3}, \frac{1}{\sqrt{1-t^2}} \right\rangle dt$
    (c) $\lim_{t \to 0} \left\langle \frac{e^t - 1}{t}, t^2 + 5, \cos 6t \right\rangle$

15. Consider the curve C which is the graph of the vector-valued function $\overrightarrow{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$.

    (a) Sketch the graph of C.

(b) Find the length of C between the points $(1,0,0)$ and $(1,0,\pi)$.

16. Find parametric equations that describe the intersection of the surfaces $z^2 = x^2+2y^2$ and $2x^2+y^2=1$.

17. Let $\overrightarrow{r}(t) = \langle t\sin t, 4\sec^2 t, e^{3t-1}\rangle$. Find the indicated quantities; if any of these do not exist, say so and why.

    (a) $\overrightarrow{r}'(0)$

    (b) $\int \overrightarrow{r}(t)\,dt$ (*Hint:* use integration by parts.)

Quizzes after Exam 1

18. Suppose a particle moves in three dimensional space such that it's velocity $\overrightarrow{r}'$ at time t is given by
$$\overrightarrow{r}'(t) = \left\langle \frac{\cos t}{\sin t}, \frac{-\sin t}{\cos t}, 1 \right\rangle.$$

    (a) Find the principal unit normal vector to the graph of the particle's motion at time $t=\pi/4$.

    (b) Find the tangential component of the particle's acceleration at $t=\pi/3$.

19. (a) Convert the point $(18,\pi/3,\pi/6)$ from spherical coordinates to rectangular coordinates.

    (b) Find a rectangular equation of $\rho = 2\sec\phi$.

    (c) Convert the point $(2\sqrt{3},-2,2)$ from rectangular coordinates to cylindrical coordinates.

    (d) Find the rectangular coordinates of the point whose spherical coordinates are $(12,\frac{2\pi}{3},\pi)$.

    (e) Write the equation $z=-x^2-y^2$ in either cylindrical or spherical coordinates (be sure to say whether you are using cylindrical or spherical coordinates).

20. Find a parametrization of the plane $x+y+z=6$.

21. Find a parametrization of the sphere of radius 3 centered at the origin.

22. Find the domain of $f(x,y) = y^2\sqrt[4]{y-x}$. Sketch a (2-dimensional) picture of the domain.

23. Sketch level curves of $f(x,y) = \frac{1}{x^2+y^2}$ corresponding to $z = \frac{1}{4}$, $z = 1$, and $z = 4$. Use these level curves (and some additional reasoning) to give a rough sketch of the (3-dimensional) graph of f.

24. Find the domain of $f(x,y) = \cos y \ln(y + x^2)$. Sketch a (2-dimensional) picture of the domain.

25. Find the values of the partial derivatives f_x, f_y and f_z for the function $f(x,y,z) = \sin(xy)\ln(z)$ at the point $(\pi, 1, e)$.

26. Find the following limits or state that they do not exist (answers must be appropriately justified):

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{\sqrt{x^2+y^2}}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^6}{x^4+y^2}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$ (e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

27. Let $f(x) = \ln(x \sin y)$; find the partial derivatives f_x and f_y.

28. Find the equation of the tangent plane to $f(x,y) = (x^4+1)e^{x+2y}$ at $(0, \frac{1}{2}\ln 2)$.

29. Let $f(x,y) = x^{40}y^{70}$. Evaluate the following:

$$\frac{\partial^{100} f}{\partial x^{20}\partial y^{40}\partial x^{10}\partial y^{10}\partial x^{20}}(2007, 2007)$$

30. Consider the function $f(x,y) = 2x^4y^2 - 3xy^3$.

(a) Find the total differential.

(b) Find the equation of the plane tangent to the graph of $f(x,y)$ at the point $(1,-1)$.

(c) Use the linear approximation from part (b) to estimate $f(.998, -.997)$.

31. Suppose $f(x,y) = \sin(x^2+y)$ and let r and θ be the usual polar coordinates. Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

32. Given the equation $x^2y^2z^2 = 16$, find $\frac{dy}{dx}$ at the point $(2,-1,-2)$.

33. Find the total differential of $z = x^2 \sin y$.

34. Consider the function $f(x,y) = 3x^2y + yx^3$.

(a) Find $\nabla f(1,2)$.

(b) Find $D_{\overrightarrow{u}}f(1,2)$ if $\overrightarrow{u} = \left\langle -\frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle$.

35. Suppose $z = x^3y^2$, $x = \cos(3st)$ and $y = \sin(s-t)$. Find $\frac{\partial z}{\partial t}$ in terms of s and t.

36. Find an equation of the tangent plane to the surface represented by the equation $\sin z + \cos y + x^4 = 2$ at the point $(1, 0, \pi)$.

37. Find a vector in the direction in which the value of the function $f(x,y,z) = x^2e^{y\sin z}$ is decreasing the fastest at the point $(-1, 2, \pi)$.

Answers

1. (a) First write

$$\overrightarrow{v} = ||\overrightarrow{v}|| \left(\frac{\overrightarrow{v}}{||\overrightarrow{v}||} \right) = \sqrt{14} \left\langle \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$$

and then take the unit vector from this decomposition and multiply by 12:

$$12 \left\langle \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle = \left\langle \frac{36}{\sqrt{14}}, \frac{-24}{\sqrt{14}}, \frac{12}{\sqrt{14}} \right\rangle.$$

(b) There are two possible answers; you could have taken the unit vector and multiplied it by -12.

2. Let $\overrightarrow{v} = \langle 1, -3, -1 \rangle \times \langle -2, 3, 1 \rangle = \langle 0, 1, 3 \rangle$; $\overrightarrow{v}$ is orthogonal to both of the given vectors. To find a vector of length 3 in the direction of $\overrightarrow{v}$, first write

$$\overrightarrow{v} = ||\overrightarrow{v}|| \left(\frac{\overrightarrow{v}}{||\overrightarrow{v}||} \right) = \sqrt{10} \left\langle 0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle$$

and then take the unit vector from this decomposition and multiply by 3:

$$3 \left\langle 0, \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \left\langle 0, \frac{3}{\sqrt{10}}, \frac{9}{\sqrt{10}} \right\rangle.$$

3. Let $\overrightarrow{a} = \langle -2, 4 \rangle$ and let $\overrightarrow{b} = \langle 3, -5 \rangle$. Then

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{b} &= ||\overrightarrow{a}|| \, ||\overrightarrow{b}|| \cos\theta \\ -26 &= \sqrt{20}\sqrt{34} \cos\theta \\ \Rightarrow \cos\theta &= \frac{-26}{\sqrt{20}\sqrt{34}} < 0 \end{aligned}$$

Since $\cos\theta$ is negative, θ has measure greater than $\pi/2$.

4. We have $||\overrightarrow{v}|| = \sqrt{21}$, $||\overrightarrow{w}|| = \sqrt{14}$ and $\overrightarrow{v} \cdot \overrightarrow{w} = 0$. So, from the formula $\overrightarrow{v} \cdot \overrightarrow{w} = ||\overrightarrow{v}|| \, ||\overrightarrow{w}|| \cos\theta$ we see that

$$0 = \sqrt{21}\sqrt{14} \cos\theta$$

so $\cos\theta = 0$, i.e. $\theta = \pi/2$.

5. Let $\overrightarrow{a} = \langle -1, -1, 4\rangle$ and let $\overrightarrow{b} = \langle -2, 1, -2\rangle$. Then

$$\begin{aligned}
\mathbf{proj}_{\overrightarrow{b}}\overrightarrow{a} &= \left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\overrightarrow{b}\cdot\overrightarrow{b}}\right)\overrightarrow{b} \\
&= \left(\frac{\langle -1,-1,4\rangle\cdot\langle -2,1,-2\rangle}{\langle -2,1,-2\rangle\cdot\langle -2,1,-2\rangle}\right)\langle -2,1,-2\rangle \\
&= \frac{-7}{9}\langle -2,1,-2\rangle \\
&= \langle 14/9, -7/9, 14/9\rangle .
\end{aligned}$$

6. (a) FALSE: $\overrightarrow{v}\times\overrightarrow{w} = -(\overrightarrow{w}\times\overrightarrow{v})$.
   (b) TRUE: Both sides of the equation are 0 since $\overrightarrow{v}\perp(\overrightarrow{v}\times\overrightarrow{w})$ and $\overrightarrow{v}\perp(\overrightarrow{w}\times\overrightarrow{v})$.
   (c) TRUE:

$$\begin{aligned}
\mathbf{proj}_{\overrightarrow{b}}(\overrightarrow{v}+\overrightarrow{w}) &= \frac{(\overrightarrow{v}+\overrightarrow{w})\cdot\overrightarrow{b}}{\overrightarrow{b}\cdot\overrightarrow{b}}\overrightarrow{b} \\
&= \frac{\overrightarrow{v}\cdot\overrightarrow{b}}{\overrightarrow{b}\cdot\overrightarrow{b}}\overrightarrow{b} + \frac{\overrightarrow{w}\cdot\overrightarrow{b}}{\overrightarrow{b}\cdot\overrightarrow{b}}\overrightarrow{b} \\
&= \mathbf{proj}_{\overrightarrow{b}}\overrightarrow{v} + \mathbf{proj}_{\overrightarrow{b}}\overrightarrow{w}
\end{aligned}$$

   (d) FALSE: $||c\overrightarrow{v}|| = |c|\,||\overrightarrow{v}||$
   (e) FALSE: $\overrightarrow{v}\times\overrightarrow{w}$ is a vector but $||\overrightarrow{v}||\,||\overrightarrow{w}||\sin\theta$ is a number so they can't be equal. A statement that is true is $||\overrightarrow{v}\times\overrightarrow{w}|| = ||\overrightarrow{v}||\,||\overrightarrow{w}||\sin\theta$
   (f) TRUE: $\overrightarrow{v}\cdot\overrightarrow{v} = ||\overrightarrow{v}||^2 \geq 0$.
   (g) FALSE: Cross-product is not associative; to evaluate something like $\overrightarrow{u}\times(\overrightarrow{v}\times\overrightarrow{w})$ you use the vector triple-product formula.

7. The first given line has direction vector $\overrightarrow{a} = \langle 1, -3, 1\rangle$. The second line has direction vector $\langle -2, 8, 1\rangle$. Since the line you want is perpendicular to both of these lines, its direction vector is given by the cross product:

$$\langle 1,-3,1\rangle \times \langle -2,8,1\rangle = \langle -11,-3,2\rangle .$$

Finally, the symmetric equations for the line you want are

$$\frac{x-1}{-11} = \frac{y+2}{-3} = \frac{z-4}{2}.$$

8. We need to find a point on the line (given) and a direction vector $\overrightarrow{a}$ for the line. Now if the line lies in the given plane, its direction vector is orthogonal to the normal vector to that plane, which is $\langle -2, 8, 1\rangle$. $\overrightarrow{a}$ is also orthogonal to $\langle 1, -3, 1\rangle$, the direction vector of the line perpendicular to the line you want. So the direction vector is

$$\overrightarrow{a} = \langle -2, 8, 1\rangle \times \langle 1, -3, 1\rangle = \langle 11, 3, -2\rangle$$

and the line has symmetric equations

$$\frac{x-1}{11} = \frac{y+2}{3} = \frac{z-4}{-2}.$$

9. First write parametric equations for the lines, using s as the parameter for the first line and t for the parameter for the second line:

$$\begin{cases} x = 3s - 1 \\ y = s + 2 \\ z = -2s - 2 \end{cases} \qquad \begin{cases} x = 3t + 8 \\ y = 4t + 8 \\ z = -2t - 4 \end{cases}$$

Now set the expressions for x, y, z equal and try to solve for s, t:

$$\begin{cases} 3t + 8 = 3s - 1 \\ 4t + 8 = s + 2 \\ -2t - 4 = -2s - 2 \end{cases}$$

From the second equation, we have $s = 4t + 6$; plugging this into the first equation we get $3t + 8 = 3(4t + 6) - 1$. Solve this for t and get $t = -1$. Then $s = 4(-1) + 6 = 2$. Check $s = 2, t = -1$ in all three equations; we see that it is a solution for all equations so the lines intersect. (Although this was not asked, to find the point of intersection find x, y, z when $s = 2$; we get $x = 5, y = 4, z = -6$, i.e. the point $(5, 4, -6)$.)

10. The given line passes through the point $(4, -2, -1)$, so the vector which starts at $(4, -2, -1)$ and ends at the given point $(2, -2, 3)$ is in the plane. This vector has coordinates $\langle -2, 0, 4\rangle$. Another vector in the plane is the direction vector for the given line, namely $\langle -3, -1, 2\rangle$. So to find a normal vector to the plane, take the cross product:

$$\langle -2, 0, 4\rangle \times \langle -3, -1, 2\rangle = \langle 4, 8, 2\rangle$$

The plane has equation $4(x-2) + 8(y+2) + 2(z-3) = 0$ (other answers are possible).

11. (a) $z = x^2 - y^2$ is a hyperbolic paraboloid (saddle).

(b) $z^2 = x^2 - y^2$ should be written $x^2 - y^2 - z^2 = 0$; this is a cone opening in the $x-$direction.

(c) $z^2 = x^2 + y^2$ should be written $x^2 + y^2 - z^2 = 0$; this is a cone opening in the $z-$direction.

(d) $z = x^2 + y^2$ is an elliptic paraboloid opening in the positive $z-$direction.

(e) $z^2 - y^2 - x^2 = 1$ is a hyperboloid of two sheets opening in the $z-$direction.

(f) $z^2 - y^2 - x^2 = 0$ is a cone opening in the $z-$direction.

(g) Multiply through by (-1) to get $x^2 + y^2 - z^2 = 1$, a hyperboloid of one sheet opening in the $z-$direction.

(h) $x^2 + y^2 = 1$ is a cylinder parallel to the $z-$axis.

(i) $x^2 - y^2 = z$ is a hyperbolic paraboloid (saddle); from the origin the graph goes up in the $x-$ direction and down in the $y-$direction.

12. First, find the derivatives of the component functions:

$$\frac{d}{dt}(\cosh(t^2)) = 2t\sinh(t^2); \frac{d}{dt}(4\sqrt{t}) = 2t^{-1/2}; \frac{d}{dt}(2\tan t) = 2\sec^2 t$$

So then the length s is given by:

$$\begin{aligned} s &= \int_2^5 ||\overrightarrow{r}'(t)||\, dt \\ &= \int_2^5 \sqrt{[2t\sinh(t^2)]^2 + [2t^{-1/2}]^2 + [2\sec^2 t]^2}\, dt \\ &= \int_2^5 \sqrt{4t^2\sinh^2(t^2) + 4t^{-1} + 4\sec^4 t}\, dt. \end{aligned}$$

13. This graph is a helix which winds around in the $x-$ and $z-$direction and moves to the right as t increases; the graph only lies above the negative part of the $y-$axis; the graph gets more and more "crowded" as $t \to \infty$, i.e. as $y \to 0$.

14. (a) Differentiate componentwise to get $\left\langle (\cos t)e^{\sin t}, e^t\cos(e^t), e^t\cos t + e^t \sin t \right\rangle$ (use the chain rule on the first two components and the product rule on the third component).

(b) $\left\langle 5\ln t + c_1, \frac{1}{2}e^{2t-3} + c_2, \sin^{-1} t + c_3 \right\rangle$

(c) The limit is $\langle 1, 5, 1\rangle$. The last two limits are done by simply substituting in $t = 0$; use L'Hopital's rule or power series or some other trick on the first limit.

15. (a) This is a helix which rotates in the $x-$ and $y-$ directions and goes up as t increases.

(b) We find the length of C between $t = 0$ and $t = \pi$:

$$\begin{aligned} s &= \int_0^\pi ||\overrightarrow{r}(t)||\, dt \\ &= \int_0^\pi \sqrt{[-2\sin 2t]^2 + [2\cos 2t]^2 + 1^2}\, dt \\ &= \int_0^\pi \sqrt{4[\sin^2(2t) + \cos^2(2t)] + 1}\, dt \\ &= \int_0^\pi \sqrt{4(1) + 1}\, dt \\ &= \int_0^\pi \sqrt{5}\, dt = \pi\sqrt{5}. \end{aligned}$$

16. First find parametric equations for the second surface; the equation for this surface, rewritten, is

$$\frac{x^2}{1/2} + \frac{y^2}{1} = 1$$

so it has parametric equations $x = \sqrt{(1/2)}\cos t$, $y = \sin t$, $0 \le t < 2\pi$. Then from the first surface we know $z^2 = x^2 + 2y^2$ which in terms of t is $(1/2)\cos^2 t + 2\sin^2 t$. Therefore $z = \pm\sqrt{(1/2)\cos^2 t + 2\sin^2 t}$ and we have the two sets of parametric equations

$$\left\{ \begin{array}{l} x = \sqrt{(1/2)}\cos t \\ y = \sin t \\ z = \sqrt{(1/2)\cos^2 t + 2\sin^2 t} \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} x = \sqrt{(1/2)}\cos t \\ y = \sin t \\ z = -\sqrt{(1/2)\cos^2 t + 2\sin^2 t} \end{array} \right.$$

which describe the intersection.

17. (a) $\overrightarrow{r}(t) = \langle t\cos t + \sin t, 8\sec^t \tan t, 3e^{3t-1}\rangle$ so $\overrightarrow{r}(0) = \langle 0, 0, 3\rangle$.
(b) $\int \overrightarrow{r}(t) = \left\langle -t\cos t + \sin t + c_1, 4\tan t + c_2, \frac{1}{3}e^{3t-1} + c_3 \right\rangle$.

Quizzes after Exam 1 - Answers

18. (a) First, the length of the velocity vector is $||\overrightarrow{r}'(t)|| = \frac{1}{\cos t \sin t}$ (this work is omitted). Next, find the unit tangent vector:

$$\begin{aligned} \overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{||\overrightarrow{r}'(t)||} &= (\cos t \sin t)\left\langle \frac{\cos t}{\sin t}, \frac{-\sin t}{\cos t}, 1 \right\rangle \\ &= \left\langle \cos^2 t, -\sin^2 t, \cos t \sin t \right\rangle. \end{aligned}$$

Next, find the derivative of the unit tangent vector:

$$\overrightarrow{T}'(t) = \left\langle -2\cos t\sin t, -2\sin t\cos t, \cos^2 t - \sin^2 t\right\rangle.$$

Evaluated at $t = \pi/4$, we have

$$\overrightarrow{T}'(\pi/4) = \left\langle -2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}, (\frac{\sqrt{2}}{2})^2 - (\frac{\sqrt{2}}{2})^2\right\rangle = \langle -1, -1, 0\rangle.$$

Now the principal unit normal vector is

$$\overrightarrow{N}(\pi/4) = \frac{\overrightarrow{T}(\pi/4)}{||\overrightarrow{T}(\pi/4)||} = \frac{\langle -1,-1,0\rangle}{||\langle -1,-1,0\rangle||} = \frac{\langle -1,-1,0\rangle}{\sqrt{2}} = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right\rangle.$$

(b) The tangential component is:

$$a_T = \frac{d}{dt}(||\overrightarrow{r}'(t)||) = \frac{d}{dt}\left(\frac{1}{\cos t\sin t}\right) = \frac{\sin^2 t - \cos^2 t}{\cos^2 t\sin^2 t}$$

which at time $t = \pi/3$ is

$$a_T = \frac{(\frac{\sqrt{3}}{2})^2 - (\frac{1}{2})^2}{(\frac{1}{2})^2(\frac{\sqrt{3}}{2})^2} = \frac{1/2}{3/16} = \frac{8}{3}.$$

19. (a) We have:

$$\begin{aligned} x &= \rho\sin\phi\cos\theta = 18\sin(\pi/3)\cos(\pi/6) = 18(\sqrt{3}/2)(\sqrt{3}/2) = 27/2 \\ y &= \rho\sin\phi\sin\theta = 18\sin(\pi/3)\sin(\pi/6) = 18(\sqrt{3}/2)(1/2) = 9\sqrt{3}/2 \\ z &= \rho\cos\phi = 18\cos(\pi/3) = 18(1/2) = 9 \end{aligned}$$

so the rectangular coordinates are $(27/2, 9\sqrt{3}/2, 9)$.

(b) Multiply both sides by $\cos\phi$ to get $\rho\cos\phi = 2$, i.e. $z = 2$.

(c) We have:

$$\begin{aligned} r &= \sqrt{x^2+y^2} = \sqrt{12+4} = 4 \\ \theta &= \tan^{-1}(y/x) = \tan^{-1}(-1/\sqrt{3}) = -\pi/6 \\ z &= z = 2 \end{aligned}$$

so the cylindrical coordinates are $(4, -\pi/6, 2)$.

(d) We have

$$x = 12\sin(2\pi/3)\cos\pi = 12(\sqrt{3}/2)(-1) = -6\sqrt{3};$$

$$y = 12\sin(2\pi/3)\sin(\pi) = 0;$$

$$z = 12\cos(2\pi/3) = 12(-1/2) = -6.$$

So the rectangular coordinates are $(-6\sqrt{3}, 0, -6)$.

(e) Use cylindrical coordinates: $r^2 = x^2 + y^2$ so $z = -r^2$ is the equation. (The use of spherical coordinates here is not recommended but you can write the equation in some complicated fashion using spherical coordinates.)

20. Three points on the plane are $(0,0,6)$, $(0,6,0)$, and $(6,0,0)$. So two nonparallel vectors lying in the plane are

$$(0,0,6)-(6,0,0) = \langle -6,0,6\rangle \text{ and } (0,0,6)-(0,6,0) = \langle 0,-6,6\rangle .$$

Using these two vectors and the point $(0,0,6)$, we have the parametrization

$$\begin{cases} x = -6u \\ y = -6v \\ z = 6u + 6v + 6 \end{cases}$$

Other answers are of course possible.

21. The sphere has equation $\rho = 3$ in spherical coordinates so you can let $u = \phi$ and $v = \theta$ to obtain

$$\begin{cases} x = 3\sin u \cos v \\ y = 3 \sin u \sin v \\ z = 3\cos u \end{cases} .$$

Other answers are possible.

22. The domain consists of all points (x, y) with $y - x \geq 0$. To draw a picture of this, draw the line $y = x$ and indicate all points lying above or on this diagonal line.

23. The level curves are all circles; the radii of the circles decreases as z increases. The graph is a "horn" which gets narrower in the $z-$direction.

24. The domain consists of all points (x, y) with $y + x^2 > 0$, that is, points lying outside (and not on) the parabola $y = -x^2$.

25. The partial derivatives are:

$$\begin{array}{rl} f_x = y\cos(xy)\ln z & \qquad f_x(\pi,1,e) = 1(-1)(1) = -1. \\ f_y = x\cos(xy)\ln z & \Rightarrow \; f_y(\pi,1,e) = \pi(-1)(1) = -\pi. \\ f_z = \frac{\sin(xy)}{z} & \qquad f_z(\pi,1,e) = 0. \end{array}$$

26. (a) Use polar coordinates to rewrite the limit as

$$\lim_{r\to 0} \frac{r\sin\theta}{r} = \lim_{r\to 0} \sin\theta = \sin\theta.$$

This expression depends on θ so the original limit does not exist.

(b) First notice that only even powers of x and y exist in the function so the function is bounded below by zero, i.e.

$$0 \leq \frac{x^4y^6}{x^4+y^2}.$$

Next, notice that $y^2 \geq 0$ so $x^4+y^2 \geq x^4$ and therefore $\frac{1}{x^4+y^2} \leq \frac{1}{x^4}$. By multiplying this inequality through by the nonnegative number x^4y^6, we get

$$\frac{x^4y^6}{x^4+y^2} \leq \frac{x^4y^6}{x^4} = y^6.$$

Now it is clear by substitution that

$$\lim_{(x,y)\to(0,0)} 0 = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} y^6 = 0,$$

so by the Squeeze theorem together with the inequality

$$0 \leq \frac{x^4y^6}{x^4+y^2} \leq y^6$$

(which was derived above), we see that the desired limit is zero.

(c) This can be done by factoring and cancelling:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x-y} = \lim_{(x,y)\to(0,0)} (x+y) = 0.$$

(You could also do this problem with polar coordinates.)

(d) By using polar coordinates we get

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = \lim_{r\to 0} \frac{r^4\cos^4\theta + r^4\sin^4\theta}{r^2} = \lim_{r\to 0} r^2(\cos^4\theta + \sin^4\theta)$$

which is 0 by substitution.

(e) We show this limit does not exist using limits along paths. Along the $x-$axis ($y=0$), we get

$$\lim_{(x,0)\to(0,0)} \frac{x-0}{x+0} = 1$$

and along the $y-$axis ($x=0$), we get

$$\lim_{(0,y)\to(0,0)} \frac{0-y}{0+y} = -1.$$

Since the limits along these two paths do not match, the original limit does not exist.

27. By the Chain Rule we get

$$f_x = \frac{1}{x \sin y}(\sin y) = \frac{1}{x}$$

and

$$f_y = \frac{1}{x \sin y}(x \cos y) = \cot y.$$

28. First find the value of the function: $f(0, \frac{1}{2}\ln 2) = 1e^{\ln 2} = 2$. Next find partial derivatives:

$$f_x = 4x^3 e^{x+2y} + (x^4+1)e^{x+2y} \;\Rightarrow\; f_x(0, \frac{1}{2}\ln 2) = 0 + 2 = 2.$$

$$f_y = 2(x^4+1)e^{x+2y} \;\Rightarrow\; f_y(0, \frac{1}{2}\ln 2) = 2 \cdot 1 \cdot 2 = 4.$$

So the tangent plane is

$$\begin{aligned} z &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= 2 + 2(x-0) + 4(y - \frac{1}{2}\ln 2). \end{aligned}$$

29. First, $f(x,y)$ is a polynomial so all of its partial derivatives (of any order) are continuous. So you can interchange the order of the partial derivatives without a problem:

$$\frac{\partial^{100} f}{\partial x^{20}\partial y^{40}\partial x^{10}\partial y^{10}\partial x^{20}}(2007, 2007) = \frac{\partial^{100} f}{\partial y^{50}\partial x^{50}}(2007, 2007)$$

Now in order to calculate this partial derivative, you would need to calculate the 50th partial derivative with respect to x. Every time you calculate such a partial derivative however, the power on the x goes down by 1. So you will eventually "run out" of powers of x and the 41st partial derivative will be 0. So all further partial derivatives are 0, so

$$\frac{\partial^{100} f}{\partial x^{20}\partial y^{40}\partial x^{10}\partial y^{10}\partial x^{20}}(2007, 2007) = \frac{\partial^{100} f}{\partial y^{50}\partial x^{50}}(2007, 2007) = 0.$$

30. (a) First, find the partial derivatives: $f_x(x,y) = 8x^3y^2 - 3y^3$ and $f_y(x,y) = 4x^4y - 9xy^2$. The total differential dz is therefore

$$\begin{aligned} dz &= f_x(x,y)dx + f_y(x,y)dy \\ &= (8x^3y^2 - 3y^3)dx + (4x^4y - 9xy^2)dy. \end{aligned}$$

(b) The tangent plane $L(x,y)$ at $(a,b) = (1,-1)$ is given by

$$\begin{aligned} L(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= [2(1)1 - 3(1)(-1)] + [8(1)1 - 3(-1)](x-1) + \\ &\qquad [4(1)(-1) - 9(1)1](y+1) \\ &= 5 + 11(x-1) - 13(y+1). \end{aligned}$$

(c) The linear approximation is given by

$$\begin{aligned} L(.998, -.997) &= 5 + 11(.998 - 1) - 13(-.997 + 1) \\ &= 5 + 11(-.002) - 13(.003) \\ &= 5 - .022 - .039 \\ &= 5 - .061 = 4.939. \end{aligned}$$

31. Setting $x = r\cos\theta$ and $y = r\sin\theta$, we get $f(r,\theta) = \sin(r^2\cos^2\theta + r\sin\theta)$. Then the partial derivatives are found as usual:

$$\frac{\partial f}{\partial r} = \cos(r^2\cos^2\theta + r\sin\theta)[2r\cos^2\theta + \sin\theta]$$

$$\frac{\partial f}{\partial \theta} = \cos(r^2\cos^2\theta + r\sin\theta)[-2r^2\cos\theta\sin\theta + r\cos\theta]$$

This problem could also be done using a Chain Rule.

32. First let $F(x,y,z) = x^2y^2z^2 - 16$ (which is equal to zero). Then

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2xy^2z^2}{2yx^2z^2} = \frac{-y}{x} = \frac{-(-1)}{2} = \frac{1}{2}.$$

33. The total differential is

$$\begin{aligned} dz &= f_x(x,y)\,dx + f_y(x,y)\,dy \\ &= 2x\sin y\,dx + x^2\cos y\,dy. \end{aligned}$$

34. (a) The gradient is

$$\begin{aligned} \nabla f(x,y) &= \langle f_x, f_y \rangle = \left\langle 6xy + 3yx^2, 3x^2 + x^3 \right\rangle \\ \nabla f(1,2) &= \langle 6(1)2 + 3(2)1, 3(1) + 1 \rangle = \langle 18, 4 \rangle. \end{aligned}$$

(b) First notice $\overrightarrow{u}$ is a unit vector. So

$$D_{\overrightarrow{u}} f(1,2) = \nabla f(1,2) \cdot \overrightarrow{u} = \langle 18, 4 \rangle \cdot \left\langle -\frac{2}{3}, \frac{\sqrt{5}}{3} \right\rangle = -12 + \frac{4\sqrt{5}}{3}.$$

35. By the Chain Rule,

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ &= (3x^2y^2)(-3s\sin(3st)) + (2x^3y)(-\cos(s-t))\\ &= 3\cos^2(3st)\sin^2(s-t)[-3s\sin(3st)] + 2\cos^3(3st)\sin(s-t)[-\cos(s-t)]\\ &= -9s\cos^2(3st)\sin(3st)\sin^2(s-t) - 2\sin(s-t)\cos(s-t)\cos^3(3st).\end{aligned}$$

This problem could also have been done by substituting in the original equation to obtain $z = \cos^3(3st)\sin^2(s-t)$ and taking a partial derivative as usual.

36. This surface is the level surface of the function $w = f(x,y,z) = \sin z + \cos y + x^4$ when $w = 2$. So a normal vector to the tangent plane to this level surface is given by the gradient:

$$\nabla(f)(1,0,\pi) = \left\langle 4x^3, -\sin y, \cos z\right\rangle|_{(1,0,\pi)} = \langle 4,0,-1\rangle.$$

So the tangent plane has equation $4(x-1) + 0(y-0) - 1(z-\pi) = 0$, i.e.

$$4(x-1) - (z-\pi) = 0.$$

37. Such a vector is given by $-\nabla f(-1,2,\pi)$. Here is the calculation:

$$\begin{aligned}-\nabla f(-1,2,\pi) &= -\left\langle 2xe^{y\sin x}, x^2y\sin ze^{y\sin z}, x^2\cos ze^{y\sin z}\right\rangle|_{(-1,2,\pi)}\\ &= -\left\langle 2(-1)e^0, 1(0)e^0, 1(2)(-1)e^0\right\rangle\\ &= -\langle -2,0,-2\rangle\\ &= \langle 2,0,2\rangle.\end{aligned}$$