Fall 2023 Exam 1

1. Throughout this problem, let

$$A = \begin{pmatrix} 6 & -4 \\ 7 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -7 & 3 \\ -1 & 0 & 4 \end{pmatrix} \qquad \mathbf{x} = (3, -5, 2) \qquad \mathbf{y} = (-2, -1, 4).$$

Compute each of the following expressions (unless the expression is nonsense, in which case you should indicate that it is nonsense):

(a) $3\mathbf{x} - 2\mathbf{y}$
(b) B^2
(c) BB^T
(d) $b_{21}\mathbf{x}$
(e) $2B\mathbf{y}$
(f) $tr(A + I)$

2. In each part of this problem, you are given a subset of $\mathbb{R}^3$. Sketch a crude picture of that set:

(a) $Span((2, 2, 0))$
(b) $Span((0, 2, 0), (0, 1, 1))$
(c) $Span((0, 2, 0)) + Span((0, 1, 1))$
(d) $Span((0, 2, 0)) + (0, 1, 1)$

3. In each part of this problem, you are given a set $\mathcal{S}$ of vectors in $\mathbb{R}^3$. Determine whether $\mathcal{S}$ is linearly dependent or linearly independent. Your answers should be appropriately justified, either by appealing to some theory or by doing some algebra.

(a) $\mathcal{S} = \{(7, 11, -19), (14, 5, 23)\}$
(b) $\mathcal{S} = \{(2, 1, -5), (3, -2, 6), (11, -8, 1), (4, 7, -2)\}$
(c) $\mathcal{S} = \{(5, 2, 8), (0, 0, 0), (17, 5, -3)\}$
(d) $\mathcal{S} = \{(1, 5, -6)\}$
(e) $\mathcal{S} = \{(4, 3, 0), (-2, 5, 0), (1, 5, 1)\}$

4. Consider the subset W of $\mathbb{R}^3$ defined by

$$W = \{(x, y, z) : 2x + y - 3z = 0\}.$$

Determine, with proof, whether or not W is a subspace of $\mathbb{R}^3$.

5. Let V be the set of differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let W be the subset of V consisting of all increasing functions (from calculus, this is the set of functions f such that $f'(x) \geq 0$ for all x; this includes all the constant functions, by the way). Determine, with proof, whether or not W is a subspace of V.

6. In each part of this problem, you are given a vector space V and a subspace W of V (you do not need to prove that W is a subspace). In each part, give a basis of W and the dimension of W.

(a) $V = \mathbb{R}^3$; $W = Span((1, 3, -1), (2, 1, 0))$
(b) $V = \mathbb{R}^3$; $W = Span((1, 3, -1), (2, 6, -2))$
(c) $V = \mathbb{R}^3$; $W = \{(x, y, z) \in \mathbb{R}^3 : 2x + y - z = 0 \text{ and } x - y + 3z = 0\}$
(d) $V = M_2(\mathbb{R})$; $W = \{A \in M_2(\mathbb{R}) : a_{12} = 3a_{11} \text{ and } a_{22} = -a_{21}\}$

7. Classify each statement as true or false (circle your answer, no justification is needed).

(a) A 5×7 matrix has more rows than columns.
(b) There is a basis of $\mathbb{R}^5$ consisting of six vectors.
(c) If five vectors in $\mathbb{R}^5$ are linearly independent, then they must also span $\mathbb{R}^5$.

(d) If five vectors in $\mathbb{R}^5$ are linearly dependent, then they cannot span $\mathbb{R}^5$.

(e) The only affine subspaces of $\mathbb{R}^3$ are lines, planes and all of $\mathbb{R}^3$.

(f) If W is a subspace of vector space V, then $W + W = W$.

8. (a) Write parametric equations for the plane in $\mathbb{R}^3$ containing the three points $(5, -1, 0)$, $(3, 2, 7)$ and $(-2, -4, 5)$.

   (b) Determine, with justification, whether the point $(8, 7, 5)$ also belongs to the plane described in part (a). (The equations you wrote in part (a) may be helpful here.)

Fall 2019 Exam 1

1. Throughout this problem, let $A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ and let $\mathbf{x} = (4, -1)$. Compute each of the following expressions (unless the expression is nonsense, in which case you should indicate that it is nonsense):

   (a) the trace of A

   (b) $A \begin{pmatrix} 1 & -3 & 4 \\ 2 & 0 & -3 \end{pmatrix}$

   (c) $3A - A^T + 4I$

   (d) $3a_{12}\mathbf{x}$

   (e) $\mathbf{x}^T A^2 \mathbf{x}$

   (f) $\dim(Span(\mathbf{x}))$

2. Let $\mathbf{v} = (-2, -1)$. In each part of this problem, you are given a set S. If the set S makes sense, sketch a picture of S; however, if S is nonsense, indicate that it is nonsense.

   (a) $S = Span(\mathbf{v})$

   (b) $S = (-1, 2) + Span(\mathbf{v})$

   (c) $S = Span(\mathbf{v}, \mathbf{v} + (0, 1))$

   (d) $S = Span(\mathbf{v}, (0, 0, 1))$

3. In each part of this problem, you are given a vector space V and a set $\mathcal{S}$ of vectors in V. Determine whether $\mathcal{S}$ is linearly dependent or linearly independent. Your answers should be appropriately justified, either by appealing to some theory or by doing some algebra.

   (a) $V = \mathbb{R}^3$; $\mathcal{S} = \{(1, 2, -3), (5, -1, 4)\}$

   (b) $V = \mathbb{R}^5$; $\mathcal{S} = \{(3, 0, 1, 0, 0), (2, 4, -3, 0, 1), (-6, 0, -2, 0, 0), (4, 1, 3, -2, 1)\}$

   (c) $V = M_2(\mathbb{R})$; $\mathcal{S} = \{\begin{pmatrix} 1 & 2 \\ 5 & 7 \end{pmatrix}, \begin{pmatrix} 3 & 8 \\ 15 & 11 \end{pmatrix}, \begin{pmatrix} 2 & 9 \\ 6 & 19 \end{pmatrix}, \begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 15 & 32 \end{pmatrix}\}$

   (d) $V = \mathbb{R}^3$; $\mathcal{S} = \{(a, b, c), (0, d, e), (0, 0, f))\}$
   Note: in this problem, letters a, b, c, d, e and f represent unknown nonzero constants.

4. (a) Suppose you know W_1 is a subspace of $\mathbb{R}^6$.
      i. What is the largest possible dimension of W_1?
      ii. What is the smallest possible dimension of W_1?
      iii. Is it possible that W_1 is spanned by four vectors?
      iv. Is it possible that W_1 contains a set of four linearly independent vectors?

   (b) Suppose you know W_2 is a subspace of $\mathbb{R}^6$ containing three linearly independent vectors.
      i. What is the largest possible dimension of W_2?
      ii. What is the smallest possible dimension of W_2?
      iii. Is it possible that W_2 is spanned by four vectors?
      iv. Is it possible that W_2 contains a set of four linearly independent vectors?

   (c) Suppose you know W_3 is a subspace of $\mathbb{R}^6$ spanned by three vectors.
      i. What is the largest possible dimension of W_3?
      ii. What is the smallest possible dimension of W_3?
      iii. Is it possible that W_3 is spanned by four vectors?
      iv. Is it possible that W_3 contains a set of four linearly independent vectors?

   (d) Suppose you know W_4 is a subspace of $\mathbb{R}^6$ spanned by three linearly independent vectors.
      i. What is the largest possible dimension of W_4?
      ii. What is the smallest possible dimension of W_4?
      iii. Is it possible that W_4 is spanned by four vectors?

iv. Is it possible that W_4 contains a set of four linearly independent vectors?

5. Let $V = \mathbb{R}^3$ and suppose that $W \subseteq V$ is the plane containing the three points $(1, 1, -5)$, $(2, -1, 2)$ and $(-2, 1, -2)$.
   (a) Write parametric equations for the plane W.
   (b) Write parametric equations of the plane passing through the point $(-5, 3, -2)$ that is parallel to W.
   (c) Is W a subspace of V? Prove or disprove your answer.

6. Let $V = \mathbb{R}^4$ and suppose that

$$W = \{(w, x, y, z) \in \mathbb{R}^4 : x = 2w, z = w + x + y\}.$$

   Prove W is a subspace of V; find the dimension of W; and give a basis of W.

7. Let $B = \begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$ and suppose that

$$W = \{A \in M_2(\mathbb{R}) : AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}.$$

   Determine, with proof, whether or not W is a subspace of $M_2(\mathbb{R})$.

8. Let $V = C(\mathbb{R}, \mathbb{R})$ and suppose that W is the set of functions f in V that are equal to their second derivative.
   (a) Find a nonzero element of W.
   (b) Determine, with proof, whether or not W is a subspace of V.

Fall 2016 Exam 1

1. Let A, B and $\mathbf{x}$ be:

$$A = \left(\begin{array}{cccc} 1 & 4 & 2 & -5 \\ 0 & -3 & -2 & 1 \end{array} \right) \qquad B = \left(\begin{array}{cc} 3 & -1 \\ 2 & -3 \end{array} \right) \qquad \mathbf{x} = (-1, -3, 7, 0).$$

Compute each of the following expressions (unless the expression is nonsense, in which case you should indicate that it is nonsense):

(a) the size of A

(b) BA

(c) $B^T BA$

(d) $B^T BAB$

(e) $3A\mathbf{x}$

(f) $a_{12}\mathbf{x} + \left(\begin{array}{c} 0 \\ 2 \\ -5 \\ 1 \end{array} \right)$

2. Sketch a picture of the following subsets of $\mathbb{R}^3$:

(a) $Span((0, -1, 0))$

(b) $Span((0, -1, 0)) + (0, 0, 2)$

(c) $Span((1, 0, 0), (0, 0, 2))$

(d) $Span((1, 0, 0), (0, 0, 2)) + (0, 0, 3)$

3. (a) Find the parametric equations of the line in $\mathbb{R}^3$ passing through the points $(2, -3, 5)$ and $(1, 0, 4)$.

(b) Find the parametric equations of the plane in $\mathbb{R}^3$ passing through the points $(1, 4, -3)$, $(2, -3, 0)$ and $(-1, 5, -1)$.

(c) Does the line described in part (a) of this problem intersect the plane described in part (b), or is the line parallel to the plane? Explain.

4. Fill in each blank with the word ALWAYS, SOMETIMES or NEVER, so that the sentence is as accurate as possible.

(a) A set of six linearly independent vectors in $\mathbb{R}^6$ ________________ spans $\mathbb{R}^6$.

(b) A set of three linearly independent vectors in $\mathbb{R}^6$ ________________ spans $\mathbb{R}^6$.

(c) If W is a subspace of V, then the dimension of W is ________________ greater than the dimension of V.

(d) A square matrix ________________ has the same size as its transpose.

(e) A plane containing the origin is ________________ a subspace of $\mathbb{R}^3$.

(f) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is ________________ a linearly independent set.

(g) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is ________________ a linearly independent set.

(h) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors and $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set of vectors, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is ________________ a linearly independent set.

(i) If $\mathcal{B}$ and $\mathcal{B}'$ are two different bases of the same finite-dimensional vector space V, then the union $\mathcal{B} \cup \mathcal{B}'$ is ________________ a basis of V.

(j) If $\mathcal{B}$ and $\mathcal{B}'$ are two different bases of the same finite-dimensional vector space V, then the intersection $\mathcal{B} \cap \mathcal{B}'$ is ________________ a basis of V.

5. In each part of this problem, you are given a vector space V and a subset W of V. For each problem:

- Determine, with proof, whether or not W is a subspace of V.
- If W is a subspace of V, find a basis of W and compute $\dim W$.

(a) $V = \mathbb{R}^4$; $W = \{(w, x, y, z) \in \mathbb{R}^4 : w - 2x + 5y - z = 0\}$.

(b) $V = C(\mathbb{R}, \mathbb{R})$; $W = \{f \in C(\mathbb{R}, \mathbb{R}) : \int_0^1 f(x)\,dx = 2\}$.

(c) $V = M_2(\mathbb{R})$; $W = \{A \in M_2(\mathbb{R}) : tr(A) = 0\}$.

6. In each part of this problem, you are given a vector space V and a set $\mathcal{S}$ of vectors in V. Determine whether $\mathcal{S}$ is linearly dependent or linearly independent. Your answers should be appropriately justified, either by appealing to some theory or by doing some algebra.

(a) $V = \mathbb{R}^3$; $\mathcal{S} = \{(1, 2, 3), (4, -2, 7), (8, -1, 4), (5, 0, 0), (5, 2, -6)\}$

(b) $V = \mathbb{R}^6$; $\mathcal{S} = \{(1, 0, 2, -3, 1, 0), (2, 1, 5, -4, 0, 0)\}$

(c) $V = M_{34}(\mathbb{R})$; $\mathcal{S} = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\}$

(d) $V = \mathbb{R}^3$; $\mathcal{S} = \{(1, 2, 3), (1, -1, 1), (8, 7, 18)\}$

Spring 2014 Exam 1

1. Let A be the matrix
$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$
and let $\mathbf{x}$ be the vector $(-2, 3, 2)$. Compute each of the following expressions (unless the expression is nonsense, in which case you should indicate that it is nonsense):
   (a) $A\mathbf{x}$
   (b) $a_{23}\mathbf{x} + a_{12}(1, -1, 3)$
   (c) $(\mathbf{x}^T\mathbf{x})A$
   (d) $\mathbf{x}^T(\mathbf{x}A)$
   (e) $\begin{pmatrix} 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix} A$

2. (a) Find the parametric equations of the line in $\mathbb{R}^4$ passing through the points $(1, 3, 5, 0)$ and $(5, -1, 4, 1)$.
   (b) Does the line described in part (a) of this problem pass through the point $(13, -9, 2, 2)$? Explain your answer.

3. Sketch a picture of the following subsets of $\mathbb{R}^2$:
   (a) $Span((1, -3)) + (5, -1)$
   (b) $Span((2, 3), (4, 3))$

4. In this question, you are given various subsets of $\mathbb{R}^3$. Determine whether each given subset is a point, line, plane, or all of $\mathbb{R}^3$:

   (a) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((3, -2, 5))$

   (b) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $(2, -1, 5)$

   (c) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((1, 0, 0), (0, 0, 1), (3, 0, -2))$

   (d) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((2, 1, -3)) + (4, 0, 2)$

   (e) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((4, 0, 0), (0, 2, 0), (0, 0, -3))$

   (f) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $\{(x, y, z) : x = 2t, y = 4t, z = 12t\}$

   (g) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((2, 1, -1), (4, 2, -2), (-8, -4, 4))$

   (h) POINT LINE PLANE ALL OF $\mathbb{R}^3$ $Span((1, 0, 0), (0, 1, 0)) + (0, 0, 1)$

5. Classify the following statements as true or false.

   (a) TRUE FALSE The set of points (x, y) lying on the line $y = mx + b$ is a subspace of $\mathbb{R}^2$.

   (b) TRUE FALSE The following set of vectors in $\mathbb{R}^4$ is linearly independent: $\{(1, 2, -1, 5), (2, -1, 6, -9), (0, 4, -1, 3), (-1, 3, 6, 8), (0, 2, -1, 6)\}$

   (c) TRUE FALSE The following set of vectors in $C(\mathbb{R}, \mathbb{R})$ is linearly independent: $\{\sin x, \cos x\}$

   (d) TRUE FALSE There is a basis of $\mathbb{R}^3$ consisting of four vectors.

   (e) TRUE FALSE The $xz-$plane (i.e. the set of points whose $y-$coordinate is zero) is a subspace of $\mathbb{R}^3$.

   (f) TRUE FALSE Any five linearly independent vectors in $\mathbb{R}^5$ must also span all of $\mathbb{R}^5$.

   (g) TRUE FALSE If W_1 and W_2 are subspaces of the same vector space V, then the union $W_1 \cup W_2$ must also be a subspace of V.

   (h) TRUE FALSE If W_1 and W_2 are subspaces of the same vector space V, then the intersection $W_1 \cap W_2$ must also be a subspace of V.

6. (a) Let $V = \mathbb{R}^3$ and let $W = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$. Determine, with proof, whether or not W is a subspace of V.

   (b) Let $V = \mathbb{R}^5$ and let $W = \{(x_1, x_2, x_3, x_4, x_5) : x_1 = x_3 \text{ and } x_4 = 2x_2 - x_5\}$. Determine, with proof, whether or not W is a subspace of V.

7. Let $V = M_3(\mathbb{R})$ and let $W = \{A \in M_3(\mathbb{R}) : A = A^T\}$. W is a subspace of V (you do not need to prove that W is a subspace).

   (a) Find a basis of W. (You do not need to justify that your answer is a basis.)

   (b) Give the dimension of W.

8. Let $V = \mathbb{R}^3$ and let $W = \{(x, y, z) : 3x + 5y - z = 0\}$. W is a subspace of V (you do not need to prove that W is a subspace).

   (a) Find a basis of W; completely justify that your answer is in fact a basis of W.

   (b) Give the dimension of W.

   (c) What kind of geometric object is W?