Fall 2023 Exam 2

1. Throughout this problem, let

$$\mathbf{v} = (1, 4, -3); \qquad \mathbf{w} = (-3, 7, 3); \qquad \mathbf{x} = (-4, 5); \qquad \mathbf{y} = (1, 3).$$

   In parts (a)-(f), compute each given expression (unless the expression is nonsense, in which case you should indicate that it is nonsense):

   (a) (4.2) the distance between $\mathbf{x}$ and $\mathbf{y}$
   (b) (4.6) the angle between $\mathbf{v}$ and $\mathbf{w}$
   (c) (4.2) a vector of magnitude 6 in the same direction as $\mathbf{v}$
   (d) (4.7) $\mathbf{v} \times \mathbf{w}$
   (e) (4.7) $\mathbf{x} \times \mathbf{y}$
   (f) (5.8) $\begin{pmatrix} 5 & 2 \\ 9 & 4 \end{pmatrix}^{-1} \mathbf{x}$
   (g) (4.1) If $A\mathbf{v} \cdot \mathbf{x}$ is a scalar, what must be the size of matrix A?

2. Let W be the subspace of $\mathbb{R}^5$ spanned by $(1, 0, 1, 0, 0)$, $(0, 1, 1, 1, 0)$ and $(1, 0, 1, 0, 1)$.

   (a) (4.4) Find a basis of $W^\perp$.
   (b) (4.5) Use the Gram-Schmidt procedure to find an orthornormal basis of W.

3. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the function $T(A) = A + A^T$.

   (a) (5.3) Prove that T is a linear transformation.
   (b) (5.5) Find a basis of the kernel of T.
   (c) (5.5) Find a basis of the image of T.

4. (5.2) Compute the standard matrix of each given linear transformation:

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotates the three-dimensional space $\mathbb{R}^3$ around the z-axis by $\frac{\pi}{2}$ radians counterclockwise (relative to the xy-plane).
   (b) $S = S_2 \circ S_1$, where $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ projects vectors onto the subspace spanned by $(5, 4)$ and $S_2 : \mathbb{R}^2 \to \mathbb{R}^3$ is $S_2(x, y) = (4x + y, 2x - y, y)$.

5. (a) (5.6) Give a specific example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ which is surjective. If no such transformation exists, briefly explain why this is the case.
   (b) (5.6) Give a specific example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ which is injective. If no such transformation exists, briefly explain why this is the case.
   (c) (5.5) Give a specific example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose image is spanned by the single vector $(1, 2, 3, 4)$. If no such transformation exists, briefly explain why this is the case.
   (d) (5.5) Give a specific example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ whose kernel is spanned by the two vectors $(1, 1, 1)$ and $(3, 1, 1)$. If no such transformation exists, briefly explain why this is the case.

6. Throughout this question, assume:

   - A is a 7×5 matrix with 3 linearly independent columns, and
   - $T : \mathbb{R}^3 \to \mathbb{R}^5$ is a transformation with rank 3.

   (a) (5.7) What is the dimension of the null space of A?

   (b) (5.7) What is the dimension of the row space of A?

   (c) (5.5) What is the dimension of the image of T?

   (d) (5.8) Is A invertible?

   (e) (5.7) How many vectors are there in a basis of $C(A)$?

   (f) (5.7) $C(A)$ is a subspace of what vector space?

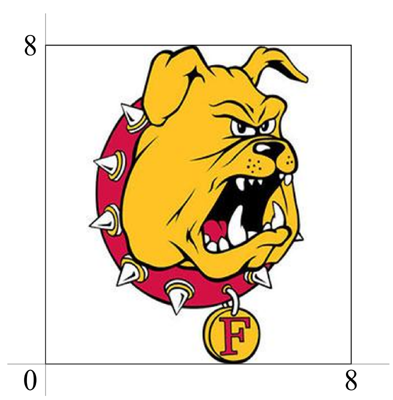
   (g) (5.5) Is there a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has no solution?

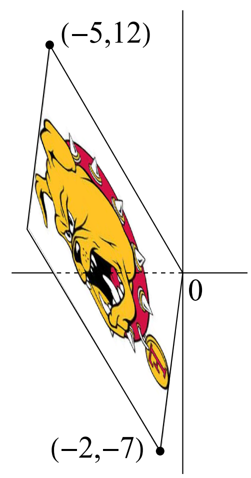
   (h) (5.5) Is there a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has exactly one solution?

   (i) (5.5) Is there a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has exactly two solutions?

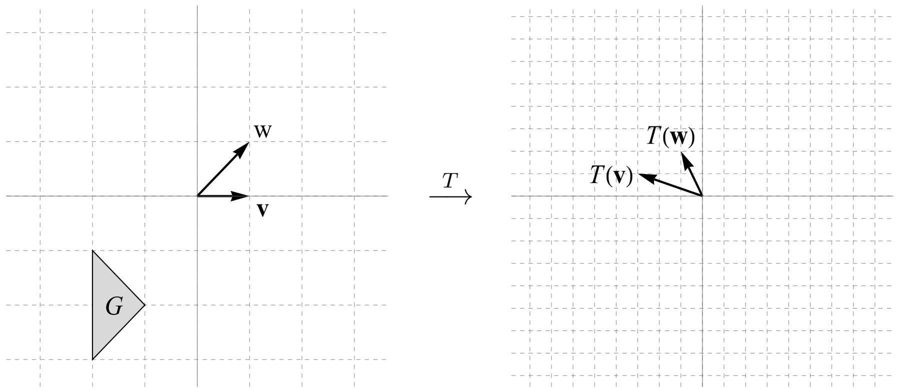
   (j) (5.5) Is there a $\mathbf{b} \in \mathbb{R}^5$ such that the equation $T(\mathbf{x}) = \mathbf{b}$ has infinitely many solutions?

7. (a) (5.2) An image originally occupies the square in the xy-plane as shown below at left. If the image is warped so that it fits in the parallelogram shown below at right, at what point does the pixel that is originally at position $(3, 5)$ end up?

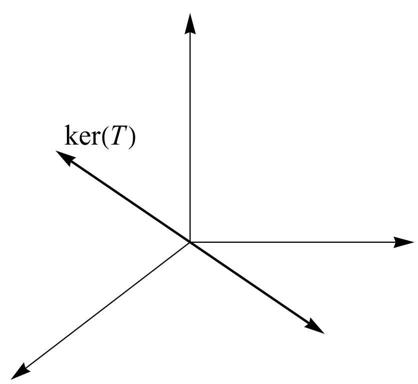




   (b) (5.2) Suppose $T : V_1 \to V_2$ is some linear transformation where vectors $\mathbf{v}, \mathbf{w} \in V_1$ are shown below at left, and vectors $T(\mathbf{v}), T(\mathbf{w}) \in V_2$ are shown below at right. If G is the shaded triangle shown in the left-hand picture, sketch the set $T(G)$ in the right-hand picture.



(c) (5.5) Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation whose kernel is the line in $\mathbb{R}^3$ indicated in the picture below. If $T(0,-4,0) = (1,3)$, sketch the solution set of the equation $T(\mathbf{x}) = (1,3)$ on this picture.



8. **(Bonus)** Compute the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that reflects points in $\mathbb{R}^3$ through the plane $x + y - 2z = 0$.

Fall 2019 Exam 2

1. The four parts of this problem are not related to one another.

   (a) Let $A = \begin{pmatrix} 8 & -3 \\ 1 & 2 \end{pmatrix}$. Compute A^{-1} (if it exists).

   (b) Let $B = \begin{pmatrix} 3 & -2 \\ 6 & x \end{pmatrix}$. For what value(s) of x (if any) is B not invertible?

   (c) Find a vector in $\mathbb{R}^6$ that has length 1 and is parallel to $\mathbf{a} = (3, -5, 4, 4, 5, -3)$.

   (d) Find a normal equation of the plane containing the three points $(5, -3, 0)$, $(2, 1, -3)$ and $(-3, 7, 1)$.

2. Throughout this problem, let $\mathbf{v} = (1, -2, 4)$, $\mathbf{w} = (2, -2, 1)$ and $\mathbf{x} = (3, 1, 1)$.

   (a) Compute $\mathbf{v} \cdot \mathbf{w}$.

   (b) Compute the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   (c) Compute $\mathbf{v} \times \mathbf{w}$.

   (d) Compute the projection of $\mathbf{v}$ onto $\mathbf{x}$.

   (e) Compute the projection of $\mathbf{x}$ onto the plane spanned by $\mathbf{v}$ and $\mathbf{w}$.
   Hint: The number 105 should appear somewhere in your answer.

   (f) Let $W = Span(\mathbf{w})$. Find a basis of $W^{\perp}$.

3. In each part of this problem, you are given a function $T : V_1 \to V_2$, where V_1 and V_2 are vector spaces. Determine, with proof, whether or not T is a linear transformation.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (3x - y + 5z, z - x)$.

   (b) $T : M_3(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = tr(A)$.

4. The parts of this problem are not related to each other.

   (a) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation which first rotates the plane counterclockwise by $\frac{\pi}{3}$, then reflects points across the y-axis, then rotates the plane clockwise by $\frac{\pi}{3}$. Compute $T(4, 6)$.

   (b) Suppose $S : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with $S(1, 0, 0) = (3, -4)$; $S(1, 2, 0) = (-1, -4)$; and $S(3, 0, 1) = (0, 7)$. Compute $S(-2, 3, -2)$.

5. In this problem, assume T_1, T_2 and T_3 are as follows (you may assume that each of these are linear transformations, without proof):

   (a) $T_1 : \mathcal{P}^2 \to \mathbb{R}^2$ defined by $T_1(f) = (f(0), f'(0))$;
   (*Note:* $\mathcal{P}^2$ is the space of polynomials of degree ≤ 2)

   (b) $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_2(\mathbf{x}) = \pi_{(1,2)}(\mathbf{x})$;

   (c) $T_3 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_3(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{pmatrix} -2 & 3 \\ -5 & 7 \end{pmatrix}$.

   For each of these transformations,

- find the dimension of $ker(T_j)$;
- find the dimension of $im(T_j)$;
- classify T_j as one of these four types:
  - injective but not surjective;
  - surjective but not injective;
  - bijective;
  - neither injective nor surjective;
- determine how many solutions there are to the equation $T_j(\mathbf{x}) = (1,1)$;
  (*Note:* for T_1, "$\mathbf{x}$" means "f", so the equation is really $T_1(f) = (1,1)$.)
- describe the solution set of $T_j(\mathbf{x}) = (1,1)$.
  (*Note:* for T_1, "$\mathbf{x}$" means "f", so the equation is really $T_1(f) = (1,1)$.)

6. (a) Suppose $T_1(\mathbf{x}) = A_1\mathbf{x}$, where A is a 7×3 matrix.
   i. What vector space is the domain of T_1?
   ii. What vector space is the codomain (i.e. space of outputs) of T_1?
   iii. Is it possible for T_1 to be injective?
   If the answer to this question is "No", ignore parts (iv) and (v).
   iv. If T_1 is injective, do you know how many linearly independent rows that A_1 has? If so, how many?
   v. If T_1 is injective, do you know what the dimension of $im(T_1)$ is? If so, what is that dimension?

   (b) Suppose $T_2(\mathbf{x}) = A_2\mathbf{x}$, where A_2 is a 3×7 matrix.
   i. Is it possible for T_2 to be injective?
   If the answer to this question is "No", ignore parts (ii) and (iii).
   ii. If T_2 is injective, do you know how many linearly independent rows that A_2 has? If so, how many?
   iii. If T_2 is injective, do you know what the dimension of $im(T_2)$ is? If so, what is that dimension?

   (c) Suppose $T_3(\mathbf{x}) = A_3\mathbf{x}$, where A_3 is a 7×3 matrix.
   i. Is it possible for T_3 to be surjective?
   If the answer to this question is "No", ignore parts (ii) and (iii).
   ii. If T_3 is surjective, do you know how many linearly independent rows that A_3 has? If so, how many?
   iii. If T_3 is surjective, do you know what the dimension of $ker(T_3)$ is? If so, what is that dimension?

   (d) Suppose $T_4(\mathbf{x}) = A_4\mathbf{x}$, where A_4 is a 3×7 matrix.
   i. Is it possible for T_4 to be surjective?
   If the answer to this question is "No", ignore parts (ii) and (iii).
   ii. If T_4 is surjective, do you know how many linearly independent rows that A_4 has? If so, how many?
   iii. If T_4 is surjective, do you know what the dimension of $ker(T_4)$ is? If so, what is that dimension?

Fall 2016 Exam 2

1. Let $\mathbf{v} = (2, -3, 5)$ and $\mathbf{w} = (1, 4, -2)$. Compute the following quantities:
   (a) $\mathbf{v} \cdot \mathbf{w}$
   (b) the norm of $\mathbf{v}$
   (c) $\mathbf{v} \times \mathbf{w}$
   (d) a vector which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$
   (e) the projection of $\mathbf{v}$ onto $\mathbf{w}$
   (f) the normal equation of the plane containing the points $\mathbf{v}$, $\mathbf{w}$ and $(5, 2, 0)$

2. In each part of this problem, you are given a function $T : V_1 \to V_2$, where V_1 and V_2 are vector spaces. Determine, with proof, whether or not T is a linear transformation.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(x, y) = (x + y, x - y, x + y, y)$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}$ where $T(\mathbf{v}) = dist(\mathbf{v}, \mathbf{0})$.

3. Find the standard matrix of each of these linear transformations (you may assume that each transformation is linear, without proof):
   (a) $T : \mathbb{R}^4 \to \mathbb{R}^2$, where $T(1, 0, 0, 0) = (2, 1)$; $T(0, 1, 0, 0) = (-1, 3)$; $T(0, 0, 1, 0) = (4, 0)$ and $T(0, 0, 0, 1) = (-3, 7)$.
   (b) $S : \mathbb{R}^2 \to \mathbb{R}^3$, where $S(1, 5) = (3, -4, 0)$ and $S(-2, 1) = (0, 3, 1)$.
   (c) $S \circ T$, where S and T are as described in parts (a) and (b) of this problem.

4. In this problem, you are given these linear transformations (you do not need to prove that these transformations are linear):
   (a) $T_1 : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T_1(w, x, y, z) = (w + 2x, 2w + 4x)$
   (b) $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_1(x, y, z) = (y, z + 3y, z - y)$
   (c) $T_3 : \mathbb{R} \to \mathbb{R}^3$ defined by $T_1(x) = (4x, -x, 3x)$

   For each of these transformations,

   - if the kernel has a basis, give a basis of the kernel (otherwise, write "NO BASIS").
   - if the image has a basis, give a basis of the image (otherwise, write "NO BASIS").
   - Is the transformation injective?
   - Is the transformation surjective?
   - Is the transformation invertible?

5. In this problem, assume T_1, T_2 and T_3 are linear transformations whose standard matrices are A_1, A_2 and A_3 respectively. Assuming the information that is given in this chart, fill out the rest of the chart (no justification is required):

	T_1	T_2	T_3
domain of T_j	$\mathbb{R}^4$		$\mathbb{R}^3$
codomain of T_j (space of outputs)	$\mathbb{R}^5$		$\mathbb{R}^7$
size of A_j		7×7	
rank of T_j		7	
$\dim C(A_j)$			
$\dim R(A_j)$			
$\dim N(A_j)$	2		
$\dim N(A_j^T)$			
Does $T(\mathbf{x}) = \mathbf{b}$ have a solution?	No		
How many solutions does $T(\mathbf{x}) = \mathbf{b}$ have?	0		1

6. In each part of this problem, you are given a vector space V and a subspace W of V. Find a basis of the orthogonal complement $W^\perp$ (your answers should be appropriately justified).

   (a) $V = \mathbb{R}^4$; $W = \{(w, x, y, z) \in \mathbb{R}^4 : x + 3y = 0 \text{ and } w - 7z = 0\}$.

   (b) $V = \mathbb{R}^3$; $W = Span((2, -3, 5))$.

Spring 2014 Exam 2

1. Let $\mathbf{v} = (1, 2, -1, 1, 1, -1)$ and $\mathbf{w} = (2, 3, -1, 0, 0, 2)$. Compute:

   (a) $\mathbf{v} \cdot \mathbf{w}$
   (b) a vector of length 4 which is parallel to $\mathbf{v}$
   (c) the projection of $\mathbf{w}$ onto $\mathbf{v}$
   (d) the distance from $\mathbf{v}$ to $\mathbf{w}$
   (e) the norm of $\mathbf{v} + \mathbf{w}$

2. Given each vector space V and each subspace W of V, find the dimension of $W^\perp$ and a basis of $W^\perp$.

   (a) $V = \mathbb{R}^3$; $W = Span(1, -5, 2)$
   (b) $V = \mathbb{R}^6$, $W = Span((1, 0, 0, 0, 0, 0), (2, 1, 0, 0, 0, 0), (-3, 4, 5, 0, 0, 0))$
   (c) $V = \mathbb{R}^4$; $W = Span((2, 1, -3, 0), (4, 2, -6, 0), (1, 1, 1, 1))$

3. Circle the letter of the correct answer to each question.

   (a) If A is a matrix with 9 columns and 5 rows, then $C(A)$ is...
   A. ... a subspace of $\mathbb{R}^9$
   B. ... a subspace of $\mathbb{R}^5$
   C. ... a subspace of $\mathbb{R}$
   D. ... not a subspace

   (b) If $T : \mathbb{R}^8 \to V_2$ is a surjective linear transformation, then $\dim(V_2)$...
   A. ... must be at least 8
   B. ... must be at most 8
   C. ... must be equal to 8
   D. ... could be anything

   (c) If $T : \mathbb{R}^4 \to V_2$ is an injective linear transformation, then $\dim(V_2)$...
   A. ... must be at least 4
   B. ... must be at most 4
   C. ... must be equal to 4
   D. ... could be anything

   (d) If A is a 11×14 matrix with 8 linearly independent columns, then $N(A)$ ...
   A. ... is a 3-dimensional subspace of $\mathbb{R}^{11}$
   B. ... is a 3-dimensional subspace of $\mathbb{R}^{14}$
   C. ... is a 6-dimensional subspace of $\mathbb{R}^{11}$
   D. ... is a 6-dimensional subspace of $\mathbb{R}^{14}$

(e) If the standard matrix of T is a 6×8 matrix with 6 linearly independent columns, then the equation $T(\mathbf{x}) = \mathbf{b}$ ...

A. ... either has no solution or infinitely many solutions, depending on what $\mathbf{b}$ is
B. ... always has one solution
C. ... always has no solution
D. ... always has infinitely many solutions

(f) If the standard matrix of T is a 7×7 matrix with 7 linearly independent rows, then the equation $T(\mathbf{x}) = \mathbf{b}$ ...

A. ... either has no solution or infinitely many solutions, depending on what $\mathbf{b}$ is
B. ... always has one solution
C. ... always has no solution
D. ... always has infinitely many solutions

4. (a) Let W be the subspace of $\mathbb{R}^4$ spanned by $(2,0,0,0)$ and $(2,-1,0,-2)$. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by setting $T(\mathbf{v}) = \mathbf{v}^W$, the projection of $\mathbf{v}$ onto W. Find the standard matrix of T.

   (b) Use your answer to part (a) to compute the projection of $(2,-5,6,1)$ onto W.

5. Let $\mathbf{v}$ and $\mathbf{w}$ be nonparallel vectors in some real vector space V. Prove that if $\mathbf{v}$ and $\mathbf{w}$ have the same length, then the angle between $\mathbf{v}$ and $\mathbf{v}+\mathbf{w}$ is the same as the angle between $\mathbf{w}$ and $\mathbf{v}+\mathbf{w}$.

6. (a) Find the normal equation of the plane in $\mathbb{R}^3$ which contains the point $(2,-1,4)$ and the line whose parametric equations are

$$\begin{cases} x = 1 - 3t \\ y = 1 + 7t \\ z = -3 - t \end{cases}$$

   (b) Write the normal equation of any plane in $\mathbb{R}^3$ which is parallel to the plane you found in part (a). (Two planes in $\mathbb{R}^3$ are called *parallel* if they don't intersect.)

7. In each part of this question you are given a function $T : V_1 \to V_2$, where V_1 and V_2 are real vector spaces. Determine, with proof, whether or not T is a linear transformation:

   (a) $V_1 = \mathbb{R}^4$; $V_2 = \mathbb{R}^2$; $T(x_1,x_2,x_3,x_4) = (x_2 - x_1, x_1 + x_4)$.

   (b) $V_1 = M_{33}(\mathbb{R})$; $V_2 = M_{23}(\mathbb{R})$; $T(A) = BA$ where B is the matrix $B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \end{pmatrix}$.

8. Here are three linear transformations (you do not need to prove that these transformations are linear).

   (a) $R : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $R(x, y) = (x + y, x - 2y)$
   (b) $S : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $S(x, y, z) = (x - 2y + z, x + y + z, -3x + 6y - 3z, 0)$
   (c) $T : \mathbb{R}^5 \to \mathbb{R}$ defined by $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$ where $\mathbf{v} = (1, 2, 3, 4, 5)$

   For each of these transformations,

   - if the kernel has a basis, write a basis of the kernel in the first row of the chart below (otherwise, write "NO BASIS").
   - if the image has a basis, write a basis of the image in the second row of the chart below (otherwise, write "NO BASIS").
   - Is the transformation injective? Answer "YES" or "NO" in the third row of the chart below.
   - Is the transformation surjective? Answer "YES" or "NO" in the fourth row of the chart below.
   - Is the transformation bijective? Answer "YES" or "NO" in the fifth row of the chart below.

	R	S	T
basis of kernel			
basis of image			
injective?			
surjective?			
bijective?			