# Fall 2023 Exam 3

1. Let $A = \begin{pmatrix} -3 & 5 \\ 2 & 6 \end{pmatrix}$.
   (a) (7.2) Compute the determinant of $A$.
   (b) (8.4) Compute the eigenvalues and eigenvectors of $A$.
   (c) (8.4) Compute and simplify $A^{200}$.

2. (8.1) Find all functions $y(t)$ which satisfy the differential equation
$$y''(t) - 2y'(t) - 8y(t) = -9\cos t + 2\sin t.$$

3. (6.5) Use the Gauss-Jordan method to compute the inverse of this matrix. Show all the steps in your row reductions.
$$\begin{pmatrix} 0 & 2 & 3 \\ 2 & -4 & -5 \\ 1 & 3 & 6 \end{pmatrix}.$$

4. The augmented matrix of a system $A\mathbf{x} = \mathbf{b}$ is given below, along with its reduced row echelon form:
$$(A\,|\,\mathbf{b}) = \left(\begin{array}{cccccc|c} -2 & 4 & 4 & -1 & 0 & 1 & 3 \\ 1 & -2 & 3 & 2 & 1 & 5 & -1 \\ 0 & 0 & 0 & 3 & 4 & -3 & 2 \\ -7 & 14 & 9 & 1 & 7 & -8 & 14 \\ 4 & -8 & -5 & 4 & 3 & -4 & 6 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & -2 & \frac{21}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{9}{4} \\ 0 & 0 & 0 & 1 & 0 & 5 & -15 \\ 0 & 0 & 0 & 0 & 1 & -\frac{9}{2} & \frac{47}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
   (a) (6.1) How many equations are in the system $A\mathbf{x} = \mathbf{b}$?
   (b) (6.1) How many variables are being solved for in the system $A\mathbf{x} = \mathbf{b}$?
   (c) (6.4) How many solutions does the system $A\mathbf{x} = \mathbf{b}$ have?
   (d) (6.4) What is the dimension of $R(A)$?
   (e) (6.3) Find a basis for the column space of $A$.
   (f) (6.3) Find a basis for $\ker(T)$, where $T(\mathbf{x}) = A\mathbf{x}$.

5. (6.3) Solve each system. If your answers are correct, I do not need to see your work.
   (a) $\begin{cases} 5x + 4y - z = -14 \\ x + 3y - 2z = -11 \\ -4x - 7y + 2z = 12 \\ 3x + y - z = -14 \end{cases}$

(b) $\begin{cases} 4x & -3y & +2z & = -16 \\ x & +y & -4z & = -14 \\ 2x & -6y & +5z & = -5 \\ x & & -z & = -4 \end{cases}$

(c) $\begin{cases} x & -3y & -z & = 2 \\ 2x & -7y & -z & = 5 \\ x & & -4z & = -1 \end{cases}$

6. (6.6) Find the model of the form

$$z = ax^2 + by^2$$

that best fits the data points

$$(1,-1,2) \qquad (2,1,8) \qquad (2,3,-5) \qquad (0,3,-14) \qquad (3,1,22)$$

and use your model to predict the value of $z$ when $x = 2$ and $y = 5$.

7. Fill in each blank with the word "ALWAYS", "SOMETIMES" or "NEVER" so that the statement is true.

   (a) (6.3) The first four columns of a $7 \times 8$ matrix with rank $4$ are ________________ linearly independent.

   (b) (6.3) The first five columns of a $7 \times 8$ matrix with rank $4$ are ________________ linearly independent.

   (c) (6.2) If $A$ is a matrix with $N(A^T) = \{\mathbf{0}\}$, then the equation $A\mathbf{x} = \mathbf{b}$ ________________ has at least one solution.

   (d) (7.1) If $A$ is an invertible matrix, then $\det A$ ________________ equals $0$.

   (e) (8.2) If $A$ is a $6 \times 6$ matrix, then $A$ ________________ has six eigenvalues.

   (f) (8.5) If $A$ and $B$ are square matrices, then $e^{A+B}$ ________________ equals $e^A e^B$.

   (g) (6.2) A linear transformation from $\mathbb{R}^8$ to $\mathbb{R}^5$ is ________________ injective.

   (h) (6.2) An injective linear transformation from $\mathbb{R}^7$ to $\mathbb{R}^7$ is ________________ surjective.

   (i) (8.2) If $\lambda$ is an eigenvalue of square matrix $A$, then $2\lambda$ is ________________ an eigenvalue of $2A$.

   (j) (8.2) If $\lambda$ is an eigenvalue of square matrix $A$, then $\lambda^2$ is ________________ an eigenvalue of $A^2$.

# Fall 2019 Exam 3

1. Suppose $A$ is a $4 \times 4$ matrix whose trace is $2$ and whose determinant is $4$. Suppose further that $A$ has two different real eigenvalues, both of which are of multiplicity $2$.

   (a) Suppose that matrix $B$ is formed from $A$ by multiplying the second row by $3$, then swapping the first and third columns. What is the determinant of $B$?

   (b) What is $\det(-3A)$?

   (c) How many solutions does the equation $A\mathbf{x} = (2, 3, -5, 1)$ have?

   (d) What are the eigenvalues of $A$?

2. Parts (a) and (b) of this question are related, but (c) has nothing to do with (a) or (b).

   (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 2 \\ 6 & -3 \end{pmatrix}$. Be sure to tell me which eigenvector goes with which eigenvalue. (To get credit, you must show your work.)

   (b) Compute and simplify $A^{2019}$, where $A$ is the matrix given in part (a) of this problem. (To get credit, you must show enough work so that I can tell how your answer comes from your work in part (a).)

   (c) Find all functions $y(t)$ which satisfy this differential equation:

$$y''(t) + 6y'(t) - 16y(t) = 72e^{-2t}$$

3. Compute the determinant of each matrix. (You may (and should) use technology to check your answers, but to get credit, I need to see appropriate work.)

   (a) $\begin{pmatrix} 2 & -1 & -2 \\ -3 & 7 & 4 \\ 1 & 5 & 3 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & -2 & 0 & -3 \\ 0 & 8 & 0 & 0 & 0 \end{pmatrix}$

4. Consider the following system of equations:

$$A\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \left\{ \begin{array}{rrrrrl} -2v & +w & -2x & +4y & +5z & = 2 \\ v & +2w & & -y & +3z & = 4 \\ 7v & +4w & 4x & -11y & -z & = 8 \end{array} \right.$$

   (a) Solve the system using row reductions. You must show all your steps.

   (b) Find a basis of the null space of $A$.

   (c) Find a basis of the column space of $A$.

   (d) Find a basis of the row space of $A$.

5. For each given system of equations:
    - If the system has a solution, find its solution set.
    - If the system has no solution, say so, and find the least-squares solution of the system.

(a) $\begin{cases} 2x & -y & +3z & = 4 \\ -5x & & +2z & = -1 \\ x & -3y & +11z & = 2 \\ 4x & +y & & = 5 \end{cases}$

(b) $\begin{cases} 3x & -7y & +2z & = -26 \\ 2x & -y & +4z & = -1 \\ x & & -3z & = -9 \end{cases}$

(c) $\begin{cases} x & -3y & & = -8 \\ -4x & +2y & +z & = 7 \\ -3x & & +2z & = 7 \\ 2x & +7y & -4z & = 3 \end{cases}$

6. Each column of this chart corresponds to a matrix $A$ whose entries are real numbers. Use the given information to fill in all the blank spaces:

| | (a) | (b) | (c) |
|---|---|---|---|
| Size of $A$ | | $8 \times 6$ | $5 \times 7$ |
| Number of equations in system $A\mathbf{x} = \mathbf{b}$ | 3 | | |
| Number of components in each solution of the system $A\mathbf{x} = \mathbf{b}$ | | | |
| Domain of $T$, where $T(\mathbf{x}) = A\mathbf{x}$ | | | |
| $\dim C(A)$ | | | |
| $\dim N(A)$ | | | 4 |
| Number of pivots in an echelon form of $A$ | | 6 | |
| Number of solutions of the system $A\mathbf{x} = \mathbf{0}$ | | | |
| Possible number of solutions of the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is arbitrary | always 1 solution | | |
| Is $A$ invertible? | | | |

# Fall 2016 Exam 3

1. Suppose that $A$ is a $3 \times 3$ matrix whose determinant is $5$.

   (a) Suppose that a matrix is made from $A$ by switching the first two rows of $A$, then switching the first two columns. Find the determinant of this new matrix.

   (b) Suppose that a matrix is made from $A$ by adding the first column to the second. Find the determinant of this new matrix.

   (c) Is $A$ invertible? If so, give the determinant of $A^{-1}$. If not, explain why not.

   (d) Find the determinant of $3A$.

   (e) If $A$ has eigenvalue $2$ (repeated twice), what is the third eigenvalue of $A$?

   (f) Find the determinant of $AB$, where

$$B = \begin{pmatrix} 4 & -3 & -5 \\ 2 & -1 & -2 \\ 1 & 7 & -3 \end{pmatrix}.$$

2. Suppose that a collection of data points $(x, y, z)$ is supposed to fit an equation of the form

$$axy + bxz + cyz = 1,$$

   where $a, b$ and $c$ are constants. Suppose also that the data points collected are:

$$(1, 2, 5) \quad (1, 7, 10) \quad (2, 3, 8) \quad (3, 1, 2) \quad (3, 5, 14) \quad (4, 2, 9)$$

   (a) Set up a matrix equation $A\mathbf{x} = \mathbf{b}$ which can be used to find the equation. In particular, what are $A$, $\mathbf{x}$ and $\mathbf{b}$?

   (b) Compute (using least-squares) the model which best fits the data.

3. Solve the following system of equations. In this problem, you must use row reductions and show all your steps.

$$\begin{cases} -3x & +y & +z & = -2 \\ 5x & -2y & & = -4 \\ x & -4y & -3z & = -17 \end{cases}$$

4. Solve the following systems of equations. Write your answers in the appropriate form.

   (a) $$\begin{cases} w & +x & -3y & +z & = -3 \\ 3w & -x & & -2z & = 7 \\ -2w & +2x & -3y & +3z & = -10 \end{cases}$$

(b) $\begin{cases} 4x & -3y & +7z & = 6 \\ -6x & +5y & -4z & = -8 \\ -6x & +7y & +22z & = 5 \end{cases}$

(c) $\begin{cases} 2x & +5y & -z & = 8 \\ -x & +4y & -3z & = 4 \\ 5x & -2y & +7z & = 3 \end{cases}$

5. Suppose the following matrices are row equivalent:

$$(A\,|\,\mathbf{b}) = \left(\begin{array}{ccccc|c} 2 & 1 & -2 & 1 & 4 & 2 \\ 5 & -1 & 5 & -4 & 7 & 0 \\ 0 & 7 & -22 & 13 & 6 & 10 \\ 1 & 4 & 3 & 0 & 1 & -3 \\ 1 & -17 & -6 & -4 & 3 & 12 \\ 8 & 4 & 7 & -3 & 11 & 9 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{-17}{105} & 0 & \frac{268}{15} \\ 0 & 1 & 0 & \frac{41}{105} & 0 & \frac{-4}{15} \\ 0 & 0 & 1 & \frac{-7}{15} & 0 & \frac{-49}{15} \\ 0 & 0 & 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) Find the rank of $A$.

(b) Find a basis of the column space of $A$.

(c) Find a basis of the row space of $A$.

(d) Find a basis of the null space of $A$.

(e) Find the dimension of the left nullspace of $A$.

(f) Classify the following statements as true or false:

i. $A\mathbf{x} = \mathbf{b}$ has at least one solution.

ii. $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

iii. The first, second and fourth columns of $A$ are linearly independent.

iv. The first, second and fifth columns of $A$ are linearly independent.

v. The first five rows of $A$ are linearly independent.

6. Find the eigenvalues and eigenvectors of the following matrix (work must be shown). Be sure to indicate which eigenvector goes with which eigenvalue.

$$\begin{pmatrix} 6 & 6 & 6 \\ -3 & 3 & 9 \\ 5 & -1 & -7 \end{pmatrix}$$

7. Suppose that the sequence of numbers $\{x_n\}$ is defined recursively by setting $x_0 = 1$, $x_1 = 4$ and by defining, for $n \geq 2$, $x_n = 2x_{n-1} + 3x_{n-2}$. Find the exact value of $x_{2016}$, the $2016^{th}$ number in this sequence.

# Spring 2014 Exam 3

1. (a) Find the determinant of the following matrix:

$$\begin{pmatrix} 2 & -3 & 1 \\ -4 & -1 & -2 \\ 5 & 1 & 2 \end{pmatrix}$$

   (b) Find the value of $k$ so that the following matrix is not invertible:

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 5 & -1 \\ 3 & 1 & k \end{pmatrix}$$

2. Suppose that a collection of data points is supposed to lie on a parabola of the form

$$y = a + bx + cx^2.$$

Suppose also that the data points collected are:

$$(-3,-13) \quad (-2,-8) \quad (-1,0) \quad (0,3) \quad (1,4) \quad (3,1) \quad (5,0) \quad (6,-7)$$

   (a) Set up a matrix equation $A\mathbf{x} = \mathbf{b}$ which can be used to find the best fitting parabola. In particular, what are $A$, $\mathbf{x}$ and $\mathbf{b}$?

   (b) Compute (using least-squares) the model which best fits the data.

   (c) Use your model to predict the value of $y$ when $x = 10$.

3. Find the inverse of the following matrix using the Gauss-Jordan method (please show all your steps).

$$\begin{pmatrix} 1 & 0 & -2 \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

4. Solve the following systems of equations:

   (a) $$\begin{cases} w - 4x + 2y + z = 1 \\ 2w - 3x + y + 2z = -4 \\ -5w + 2y - 5z = 15 \end{cases}$$

   (b) $$\begin{cases} x - 4y + 2z = 1 \\ 2x - 3y + z = -4 \\ -2x + y = 3 \end{cases}$$

   (c) $$\begin{cases} 3v - w - 3x + y + 2z = 1 \\ 2v - 3w + 4x - 2y + z = -4 \\ 4v - 5w - 10x + 4y + 3z = 6 \end{cases}$$

5. For each given $m \times n$ matrix $A$, fill out the rest of the chart, where the rows of the chart correspond to the following questions:
   (a) Give the dimension of the column space of $A$.
   (b) Give the dimension of the null space of $A$.
   (c) True or false: $A\mathbf{x} = \mathbf{b}$ has no solution for *some* $\mathbf{b} \in \mathbb{R}^m$.
   (d) True or false: $A\mathbf{x} = \mathbf{b}$ has no solution for *every* $\mathbf{b} \in \mathbb{R}^m$.
   (e) True or false: $A\mathbf{x} = \mathbf{b}$ has at least one solution for *some* $\mathbf{b} \in \mathbb{R}^m$.
   (f) True or false: $A\mathbf{x} = \mathbf{b}$ has at least one solution for *every* $\mathbf{b} \in \mathbb{R}^m$.
   (g) True or false: $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for *some* $\mathbf{b} \in \mathbb{R}^m$.
   (h) True or false: $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for *every* $\mathbf{b} \in \mathbb{R}^m$.

| | $A$ is an $18 \times 14$ matrix whose null space is six-dimensional | $A$ is a $7 \times 4$ matrix whose columns are linearly independent | $A$ is an $11 \times 22$ matrix that has 11 pivots | $A$ is an invertible $5 \times 5$ matrix |
|---|---|---|---|---|
| (a) | | | | |
| (b) | | | | |
| (c) | | | | |
| (d) | | | | |
| (e) | | | | |
| (f) | | | | |
| (g) | | | | |
| (h) | | | | |

6. Find the characteristic polynomial, the eigenvalues and corresponding eigenvectors of the following matrix:

$$\begin{pmatrix} 0 & -6 & 12 \\ -5 & 11 & -12 \\ -5 & 17 & -24 \end{pmatrix}$$

7. Suppose that the number $x(t)$ of sparrows at time $t$ and the number $y(t)$ of cardinals at time $t$ in a forest are modeled by this system of differential equations:

$$\begin{cases} x'(t) = \frac{4}{3}x(t) + \frac{2}{3}y(t) \\ y'(t) = \frac{1}{3}x(t) + \frac{5}{3}y(t) \end{cases}$$

If there are initially 30 sparrows and 120 cardinals in the forest, how many sparrows and how many cardinals will be in the forest at time $t = 20$? (I want an exact answer, not an approximation.)