Old MATH 322 Final Exams

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Last updated to include exams from Fall 2023

Contents

1.1 General information about these exams

These are the final exams I have given in linear algebra courses at Ferris State. Each exam is given here, followed by what I believe are the solutions (there may be some number of computational errors or typos in these answers).

Each problem on these exams is marked with a section in parenthesis like, for example, "(3.5)"; this section refers to the section in my Fall 2023 version of my MATH 322 lecture notes to which this question best corresponds.

1.2 Fall 2023 Final Exam

1. Throughout this problem, let

$$A = \begin{pmatrix} -5 & -1 \\ 6 & 2 \end{pmatrix}; \qquad B = \begin{pmatrix} 3 & 0 & -2 \\ 4 & -5 & 7 \end{pmatrix}; \qquad C = \begin{pmatrix} 1 & 4 \\ 0 & -3 \end{pmatrix}.$$

   a) Compute $2A + 3C$.
   b) Which one of the matrices A^{-1} or B^{-1} exists? For the one that exists, compute it.
   c) Which one of the products AB or BA exists? For the one that exists, compute it.
   d) Which of the matrix exponentials e^A or e^B exists? For the one that exists, compute it.

2. Solve this system of equations (using row reductions and showing your steps):

$$\begin{cases} 2x & +y & -3z & = 5 \\ x & -y & +z & = -2 \\ -3x & +2y & -z & = 1 \end{cases}$$

3. Let T be a linear transformation whose standard matrix is A. That matrix, and its reduced row-echelon form, are given below:

$$A = \begin{pmatrix} 1 & 4 & -3 & 3 & 1 & 4 \\ 0 & 1 & 3 & -5 & 0 & 1 \\ 7 & 19 & -5 & 3 & 12 & 3 \\ 3 & 17 & -37 & 47 & 2 & 1 \end{pmatrix} \xrightarrow{\text{row ops}} \begin{pmatrix} 1 & 0 & 0 & \frac{44}{43} & 0 & \frac{704}{43} \\ 0 & 1 & 0 & -\frac{26}{43} & 0 & -\frac{29}{43} \\ 0 & 0 & 1 & -\frac{63}{43} & 0 & \frac{24}{43} \\ 0 & 0 & 0 & 0 & 1 & -8 \end{pmatrix} = \text{rref}(A)$$

   a) Find a basis of the kernel of T.
   b) Find a basis of the image of T.
   c) Is T injective?
   d) Is T surjective?
   e) What are the possible number of solutions to $T(\mathbf{x}) = \mathbf{b}$, for various choices of $\mathbf{b}$?
   f) If $T(1,1,1,1,1,1) = (10,0,39,33)$, describe the solution set of the equation

$$T(\mathbf{x}) = (10,0,39,33).$$

4. Throughout this problem, let $\mathbf{v} = (2,-1,0,1)$, let $\mathbf{w} = (0,1,4,-3)$ and let W be the subspace of $\mathbb{R}^4$ with orthonormal basis $\left\{\left(\frac{3}{5},0,-\frac{4}{5},0\right), \left(-\frac{3}{5},0,0,\frac{4}{5}\right)\right\}$.

a) Determine, with justification, whether or not $(1,0,0,1)$ belongs to W.

b) Compute the projection of $(2,-1,0,1)$ onto W.

c) Find a basis of $W^{\perp}$.

5. Throughout this problem, let $\mathbf{a} = (1,4,-2)$ and $\mathbf{b} = (-5,0,3)$.

a) Compute $2\mathbf{a} + 5\mathbf{b}$.

b) Compute $||\mathbf{a}||$.

c) Compute the projection of $\mathbf{a}$ onto $\mathbf{b}$.

d) If $(3,-1,z) \perp \mathbf{a}$, compute z.

e) Write parametric equations for the line containing $\mathbf{a}$ and $\mathbf{b}$.

6. a) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation where $T(1,4) = (-13,-9)$ and $T(2,5) = (-14,-12)$. Compute the standard matrix of T, and use your answer to compute $T(-2,-3)$.

b) Write the normal equation of the plane in $\mathbb{R}^3$ containing the points $(1,-4,-3)$, $(3,0,2)$ and $(5,-7,-4)$.

c) Describe all functions f so that $f''(x) - 4f'(x) - 21f(x) = 0$.

7. Compute the determinant of each matrix:

(a) $\begin{pmatrix} 3 & -1 & 4 \\ 2 & 0 & -5 \\ 7 & 1 & -3 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 3 & -1 & 7 & 2 \\ 0 & 2 & 5 & 1 & 8 \\ 0 & 0 & -2 & 3 & 10 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$

8. Throughout this problem, assume:

- A is a 4×2 matrix;
- B is a 3×3 matrix;
- $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^2$;
- $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^3$; and
- $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation.

Determine whether each of these quantities is a **scalar**, a **vector** (in which case you should give the vector space to which the vector belongs), a **matrix** (in which case you should give the size of the matrix), or **nonsense**:

a) $\mathbf{x}B\mathbf{y}$

c) $tr(A)$

b) $(\det A)B\mathbf{x}$

d) $\mathbf{v} \times \mathbf{w}$

e) $||A||$

f) $(\mathbf{x}^T\mathbf{x})\mathbf{x}$

g) $T(\mathbf{v} + 3\mathbf{w})$

h) $T(\mathbf{v}) + 3\mathbf{w}$

i) the standard matrix of T^{-1}

j) an eigenvector of B

9. In each part of this problem, a linear algebra question is described, together with a proposed answer from a student. In each part of this problem, a linear algebra question is described, together with a proposed answer. Your task is to briefly explain, based either on theory or a quick and easy computation, why the proposed answer must be wrong.

   a) *Question:* Compute $\mathbf{v} \cdot \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are some vectors in $\mathbb{R}^4$.
   Proposed answer: $\mathbf{v} \cdot \mathbf{w} = (1, 5, -7, 2)$.

   b) *Question:* Compute $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v} = (1, 0, -1)$ and $\mathbf{w}$ is some vector in $\mathbb{R}^3$.
   Proposed answer: $\mathbf{v} \times \mathbf{w} = (3, 5, 2)$.

   c) *Question:* Find a basis of W, where W is some subspace.
   Proposed answer: W has basis $\{\mathbf{0}\}$.

   d) *Question:* Use the Gram-Schmidt procedure to compute an orthonormal basis of some subspace.
   Proposed answer: $\{\left(\frac{1}{2}, \frac{1}{2}, 0\right), (0, 0, 1)\}$.

   e) *Question:* Compute the standard matrix of some linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.
   Proposed answer: $A = \begin{pmatrix} 1 & -3 \\ 4 & 0 \\ -5 & 2 \end{pmatrix}$.

   f) *Question:* Find a basis of $\ker(T)$, where $T : M_2(\mathbb{R}) \to \mathbb{R}^3$ is some linear transformation.
   Proposed answer: $\ker(T)$ has basis $\{(1, 2, 5), (0, 0, 1)\}$.

   g) *Question:* Find the dimensions of the kernel and image of $T : \mathbb{R}^6 \to \mathbb{R}^5$.
   Proposed answer: $\dim \ker(T) = 2$ and $\dim im(T) = 2$.

   h) *Question:* Find bases for the row space and null space of some matrix.
   Proposed answer: $R(A)$ has basis $\{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 1)\}$;
   $N(A)$ has basis $\{(0, 0, 0, 1, 0), (1, 0, 0, 1, 0)\}$.

   i) *Question:* Find bases for the row space and null space of some matrix.
   Proposed answer: $R(A)$ has basis
   $$\{(1, 5, -7, 3, 2), (0, 4, -3, 8, 1), (6, -2, 3, 6, -5)\};$$
   $N(A)$ has basis
   $$\{(9, 1, 6, 0, 14), (-22, -81, -52, 21, 0), (109, -145, -2, 42, 238)\}.$$

j) *Question:* Compute the eigenvalues of $A = \begin{pmatrix} 3 & 1 & -2 & 0 \\ 5 & 6 & 3 & 1 \\ -1 & -5 & -3 & 2 \\ 2 & 4 & -1 & 2 \end{pmatrix}$.

Proposed answer: $\lambda = 5$, $\lambda = 1$, $\lambda = -4$, $\lambda = 2$.

k) *Question:* Let $\mathbf{v}$ and $\mathbf{w}$ be some vectors. Compute the norms of $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$.

Proposed answer: $||\mathbf{v}|| = 6$, $||\mathbf{w}|| = 7$, $||\mathbf{v} + \mathbf{w}|| = 15$.

l) *Question:* Give a basis for the subspace W of points in $\mathbb{R}^4$ satisfying some equation $aw + bx + cy + dz = 0$ (where a, b, c and d are some constants).

Proposed answer: $\{(1, 0, 1, 0), (0, 3, -1, 3), (1, 2, 3, 4), (0, 0, 5, -1)\}$.

Solutions

1. a) $2A + 3C = \begin{pmatrix} -10 & -2 \\ 12 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 12 \\ 0 & -9 \end{pmatrix} = \boxed{\begin{pmatrix} -7 & 10 \\ 12 & -5 \end{pmatrix}}.$

   b) A is square and $\det A = (-5)2 - 6(-1) = -4 \neq 0$, so A^{-1} exists and

$$A^{-1} = \frac{1}{(-5)2 - 6(-1)} \begin{pmatrix} 2 & 1 \\ -6 & -5 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{1}{2} & -\frac{1}{4} \\ \frac{3}{2} & \frac{5}{4} \end{pmatrix}}.$$

   c) $A_{2\times2}B_{2\times3}$ exists and equals $AB = \boxed{\begin{pmatrix} -19 & 5 & 3 \\ 26 & -10 & 2 \end{pmatrix}}.$

   d) A is square, so e^A exists. To compute it, use eigenvalues and eigenvectors:

$$0 = \det(A - \lambda I) = (-5-\lambda)(2-\lambda) + 6 = \lambda^2 + 3\lambda - 4 = (\lambda+4)(\lambda-1)$$

   so the eigenvalues are $\lambda = -4$ and $\lambda = 1$. Next, eigenvectors; write $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$:

$$\lambda = -4 : \begin{cases} -5x - y &= -4x \\ 6x + 2y &= -4y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1)$$

$$\lambda = 1 : \begin{cases} -5x - y &= x \\ 6x + 2y &= y \end{cases} \Rightarrow y = -6x \Rightarrow \mathbf{v} = (1, -6)$$

Now, we compute the matrix exponential:

$$\begin{aligned} e^A = e^{S\Lambda S^{-1}} = Se^{\Lambda}S^{-1} &= \begin{pmatrix} 1 & 1 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} e^{-4} & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -6 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} e^{-4} & 0 \\ 0 & e \end{pmatrix} \frac{1}{-5} \begin{pmatrix} -6 & -1 \\ 1 & 1 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} e^{-4} & e \\ -e^{-4} & -6e \end{pmatrix} \begin{pmatrix} -6 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \boxed{-\frac{1}{5} \begin{pmatrix} -6e^{-4} + e & -e^{-4} + e \\ 6e^{-4} - 6e & e^{-4} - 6e \end{pmatrix}}. \end{aligned}$$

2. Write the augmented matrix and perform Gaussian elimination as usual:

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 5 \\ 1 & -1 & 1 & -2 \\ -3 & 2 & -1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & 1 & -3 & 5 \\ -3 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2, 3R_1+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 3 & -5 & 9 \\ 0 & -1 & 2 & -5 \end{array}\right) \xrightarrow{-1 \cdot R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 3 & -5 & 9 \\ 0 & 1 & -2 & 5 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & -2 & 5 \\ 0 & 3 & -5 & 9 \end{array}\right) \xrightarrow{-3R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -6 \end{array}\right)$$

$$\xrightarrow{2R_3+R_2, -R_3+R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -6 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -6 \end{array}\right).$$

Therefore the solution set is $\boxed{\{(-3, -7, -6)\}}$.

3. a) If we set $\text{rref}(A)\mathbf{x} = \mathbf{0}$, we get the system of equations

$$\begin{cases} x_1 + \frac{44}{43}x_4 + \frac{704}{43}x_6 &= 0 \\ x_2 - \frac{26}{43}x_4 - \frac{29}{43}x_6 &= 0 \\ x_3 - \frac{63}{43}x_4 + \frac{24}{43}x_6 &= 0 \\ x_5 - 8x_6 &= 0 \end{cases} \Rightarrow \begin{cases} x_1 &= -\frac{44}{43}x_4 - \frac{704}{43}x_6 \\ x_2 &= \frac{26}{43}x_4 + \frac{29}{43}x_6 \\ x_3 &= \frac{63}{43}x_4 - \frac{24}{43}x_6 \\ x_5 &= 8x_6 \end{cases}$$

Therefore

$$\begin{aligned} N(A) &= (x_1, x_2, x_3, x_4, x_5, x_6) \\ &= \left(-\frac{44}{43}x_4 - \frac{704}{43}x_6, \frac{26}{43}x_4 + \frac{29}{43}x_6, \frac{63}{43}x_4 - \frac{24}{43}x_6, x_4, 8x_6, x_6\right) \\ &= x_4\left(-\frac{44}{43}, \frac{26}{43}, \frac{63}{43}, 1, 0, 0\right) + x_6\left(-\frac{703}{43}, \frac{29}{43}, -\frac{24}{43}, 0, 8, 1\right) \end{aligned}$$

so a basis of $\ker(T) = N(A)$ is

$$\boxed{\left\{\left(-\frac{44}{43}, \frac{26}{43}, \frac{63}{43}, 1, 0, 0\right), \left(-\frac{703}{43}, \frac{29}{43}, -\frac{24}{43}, 0, 8, 1\right)\right\}}.$$

b) A basis of the $im(T) = C(A)$ is the pivot columns of A:

$$\boxed{\{(1, 0, 7, 3), (4, 1, 19, 17), (-3, 3, -5, -37), (1, 0, 12, 2)\}}.$$

(Alternatively, since $\dim C(A) = 4$ but $C(A)$ is a subspace of $\mathbb{R}^4$, $im(T) = C(A)$ must be all of $\mathbb{R}^4$, so any basis of $\mathbb{R}^4$ works. For instance, we can use the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\} = \boxed{\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}}$.)

c) Since $r = 4 \neq 6 = n$ (or because $\ker(T) \neq \{\mathbf{0}\}$) T is $\boxed{\textbf{not}}$ injective.

d) Since $r = m(= 4)$ (or because $im(T) = \mathbb{R}^4$), T $\boxed{\textbf{is}}$ surjective.

e) Since T is surjective but not injective, $T(\mathbf{x}) = \mathbf{b}$ always has $\boxed{\textbf{infinitely many}}$ solutions.

f) If $T(1,1,1,1,1,1) = (10,0,39,33)$, describe
The solution set of the equation $T(\mathbf{x}) = (10,0,39,33)$ is $\mathbf{x}_p+\ker(T)$, which given the answer to part (a) is

$$\boxed{(1,1,1,1,1,1) + Span(\left(-\frac{44}{43},\frac{26}{43},\frac{63}{43},0,0,0\right),\left(-\frac{703}{43},\frac{29}{43},-\frac{24}{43},0,0,8\right))}.$$

4. a) $(1,0,0,1) \in W$ if and only if there are scalars x and y so that

$$x\left(\frac{3}{5},0,-\frac{4}{5},0\right) + y\left(-\frac{3}{5},0,0,\frac{4}{5}\right) = (1,0,0,1),$$

i.e.

$$\begin{cases} \frac{3}{5}x - \frac{3}{5}y &= 1 \\ 0 &= 0 \\ -\frac{4}{5}x &= 0 \\ \frac{4}{5}y &= 1 \end{cases}.$$

From the last two equations, $x = 0$ and $y = \frac{5}{4}$, but this doesn't work in the first equation. So there are no such x and y, meaning $(1,0,0,1)$ is $\boxed{\textbf{not}}$ in W.

b) Call the given vectors in the orthonormal basis of W $\mathbf{x}_1$ and $\mathbf{x}_2$. Therefore, by the projection formula,

$$\begin{aligned} \pi_W(2,-1,0,1) &= ((2,-1,0,1)\cdot\mathbf{x}_1)\mathbf{x}_1 + ((2,-1,0,1)\cdot\mathbf{x}_2)\cdot\mathbf{x}_2 \\ &= \frac{6}{5}\left(\frac{3}{5},0,-\frac{4}{5},0\right) - \frac{2}{5}\left(-\frac{3}{5},0,0,\frac{4}{5}\right) \\ &= \boxed{\left(\frac{24}{25},0,-\frac{24}{25},-\frac{8}{25}\right)}. \end{aligned}$$

c) $\mathbf{v} = (w,x,y,z) \in W^\perp$ if and only if $\mathbf{v} \perp \mathbf{x}_1$ and $\mathbf{v} \perp \mathbf{x}_2$, i.e. $\mathbf{v}\cdot\mathbf{x}_1 = 0$ and $\mathbf{v}\cdot\mathbf{x}_2 = 0$, i.e.

$$\begin{cases} \frac{3}{5}w - \frac{4}{5}y &= 0 \\ -\frac{3}{5}w + \frac{4}{5}z &= 0 \end{cases} \Rightarrow y = \frac{3}{4}w, z = \frac{3}{4}w$$

So every vector in $W^\perp$ has the form $(w,x,\frac{3}{4}w,\frac{3}{4}w) = w(1,0,\frac{3}{4},\frac{3}{4})+x(0,1,0,0)$, so a basis of $W^\perp$ is

$$\boxed{\{\left(1,0,\frac{3}{4},\frac{3}{4}\right),(0,1,0,0)\}}.$$

5. a) $2\mathbf{a} + 5\mathbf{b} = (2, 8, -4) + (-25, 0, 15) = \boxed{(-23, 8, 11)}$.

b) $||\mathbf{a}|| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{1^2 + 4^2 + (-2)^2} = \boxed{\sqrt{21}}$.

c) $\pi_{\mathbf{b}}\mathbf{a} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}}\mathbf{b} = \dfrac{-11}{34}(-5, 0, 3) = \boxed{\left(\dfrac{55}{34}, 0, -\dfrac{33}{34}\right)}$.

d) If $(3, -1, z) \perp \mathbf{a}$, then $(3, -1, z) \cdot \mathbf{a} = 0$, i.e. $3 - 4 - 2z = 0$ so $2z = -1$ so $z = \boxed{-\dfrac{1}{2}}$.

e) A direction vector for the line is $\mathbf{v} = \mathbf{a} - \mathbf{b} = (6, 4, -5)$. So one set of parametric equations for the line are

$$\mathbf{x} = \mathbf{a} + t\mathbf{v} \Leftrightarrow \boxed{\begin{cases} x = 1 + 6t \\ y = 4 + 4t \\ z = -2 - 5t \end{cases}}.$$

6. a) Write the standard matrix as $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then,

$$T(1, 4) = (-13, 9) \Rightarrow \begin{cases} a + 4b &= -13 \\ c + 4d &= -9 \end{cases}$$

$$T(2, 5) = (-14, -12) \Rightarrow \begin{cases} 2a + 5b &= -14 \\ 2c + 5d &= -12 \end{cases}$$

Solving the two equations above containing a and b, we get $a = 3$, $b = -4$. Solving the two equations above that contain c and d, we get $c = -1$, $d = -2$. Therefore the standard matrix of T is $A = \boxed{\begin{pmatrix} 3 & -4 \\ -1 & -2 \end{pmatrix}}$.

Finally, $T(-2, -3) = A(-2, -3) = \boxed{(6, 8)}$.

b) Two vectors in the plane are $\mathbf{v} = (1, -4, -3) - (3, 0, 2) = (-2, -4, -5)$ and $(5, -7, -4) - (3, 0, 2) = (2, -7, -6)$. Therefore a normal vector to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (-2, -4, -5) \times (2, -7, -6) = (-11, -22, 22)$. Since normal vectors can be taken to have any nonzero length, I'll use $\mathbf{n} = (1, 2, -2)$ instead (divide through the previous $\mathbf{n}$ by -11). Finally,

$$d = \mathbf{n} \cdot (\text{any point in the plane}) = (1, 2, -2) \cdot (3, 0, 2) = -1$$

so the normal equation of the plane is $\mathbf{n} \cdot \mathbf{x} = d$, i.e. $\boxed{x + 2y - 2z = -1}$.

c) Define $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ by $T(f) = f'' - 4f' - 21f$, so that the given equation becomes $T(f) = 0$. Thus the solution set is the kernel of T, which we compute using the characteristic equation:

$$\lambda^2 - 4\lambda - 21 = 0 \Rightarrow (\lambda - 7)(\lambda + 3) = 0 \Rightarrow \lambda = 7, \lambda = -3$$

so the solution set is $\ker(T) = \boxed{Span(e^{7t}, e^{-3t})} = \boxed{\left\{C_1e^{7t} + C_2e^{-3t} : C_1, C_2 \in \mathbb{R}\right\}}$.

7. a) Use the Rule of Sarrus: write $\begin{pmatrix} 3 & -1 & 4 \\ 2 & 0 & -5 \\ 7 & 1 & -3 \end{pmatrix} \begin{matrix} 3 & -1 \\ 2 & 0 \\ 7 & 1 \end{matrix}$ and multiply along the diagonals to get

$$(0 + 35 + 8) - (0 - 15 + 6) = 43 - (-9) = \boxed{52}.$$

b) This matrix is upper triangular, so its determinant is the product of the diagonal entries, which is $4(2)(-2)(1)(5) = \boxed{-80}$.

8. a) $\mathbf{x}B\mathbf{y} = \mathbf{x}_{3\times 1}B_{3\times 3}\mathbf{y}_{3\times 1}$ is $\boxed{\textbf{nonsense}}$.

b) A isn't square, so $\det A$ is nonsense, so the whole thing is $\boxed{\textbf{nonsense}}$.

c) $tr(A)$ is a $\boxed{\textbf{scalar}}$.

d) $\mathbf{v} \times \mathbf{w}$ is $\boxed{\textbf{nonsense}}$ since cross product is only defined for vectors in $\mathbb{R}^3$.

e) $||A||$ is a $\boxed{\textbf{scalar}}$.

f) $(\mathbf{x}^T\mathbf{x})\mathbf{x} = (\mathbf{x}^T_{1\times 3}\mathbf{x}_{3\times 1})\mathbf{x}_{3\times 1} = (\text{scalar})\mathbf{x}_{3\times 1}$ is a 3×1 matrix which is a $\boxed{\textbf{vector in } \mathbb{R}^3}$.

g) $T(\mathbf{v} + 3\mathbf{w}) = T(\text{vector in } \mathbb{R}^2)$ which is a $\boxed{\textbf{vector in } \mathbb{R}^3}$.

h) $T(\mathbf{v}) + 3\mathbf{w} = T(\text{vector in } \mathbb{R}^2) +$ vector in $\mathbb{R}^2 =$ vector in $\mathbb{R}^3 +$ vector in $\mathbb{R}^2$ which is $\boxed{\textbf{nonsense}}$.

i) The domain and codomain of T have different dimension, so T is not invertible. So T^{-1} (and therefore its standard matrix) is $\boxed{\textbf{nonsense}}$.

j) An eigenvector of the 3×3 matrix B is a $\boxed{\textbf{vector in } \mathbb{R}^3}$.

9. a) $\mathbf{v} \cdot \mathbf{w}$ is a <u>scalar</u>, not a vector.

b) $\mathbf{v} \times \mathbf{w}$ must be <u>orthogonal to $\mathbf{v}$</u> (and $\mathbf{w}$), but $\mathbf{v} \cdot (3, 5, 2) = 1 \neq 0$ so $\mathbf{v}$ and the proposed answer are not orthogonal.

c) $\{\mathbf{0}\}$ is <u>never part of a basis</u> (since $\mathbf{0}$ is never part of a linearly independent set).

d) An orthonormal basis is, by definition, made up of <u>unit vectors</u>. The first vector in the answer has norm $\sqrt{\frac{1}{2}} \neq 1$, so it is not a unit vector.

e) Since $T : \mathbb{R}^3 \to \mathbb{R}^2$, its standard matrix is 2×3, but the proposed answer is 3×2.

f) $\ker(T)$ is a subspace of the domain $M_2(\mathbb{R})$, so its basis must consist of a list of 2×2 matrices, not vectors in $\mathbb{R}^3$.

g) This violates the Image-Kernel Theorem, which says $\dim \ker(T) + \dim im(T) = \dim($domain of $T)$. Here, $2 + 2 = 4 \neq 6 = \dim($domain of $T) = \dim \mathbb{R}^6$.

h) By the Fundamental Theorem of Linear Algebra, $N(A) = [R(A)]^\perp$, but the first vector in the proposed basis of $R(A)$ is not orthogonal to the second vector in the proposed basis of $N(A)$, because $(1, 0, 0, 0, 0) \cdot (1, 0, 0, 1, 0) = 1 \neq 0$.

i) This violates the Rank-Nullty Theorem. Notice A must have 5 columns since the vectors in $R(A)$ have 5 components, so $n = 5$. Let r be the rank of A. The proposed answer means $\dim R(A) = r = 3$ so $\dim N(A) = n - r = 5 - 3 = 2$. But there are 3 vectors given in the basis of $N(A)$.

j) The proposed eigenvalues do not add to the trace of A ($tr(A) = 3 + 6 + (-3) + 2 = 8$, but the proposed eigenvalues add to $5 + 1 + (-4) + 2 = 4$).

k) The Cauchy-Schwarz Inequality says $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$, but this is incompatible with the proposed answer $6 + 7 \not\leq 15$).

l) W is not all of $\mathbb{R}^4$, so its dimension must be at most 3, so at most 3 vectors can be in any basis of W.

1.3 Fall 2019 Final Exam

1. a) Use the Gauss-Jordan method to find the inverse of this matrix. Show all the steps in your row reductions.

$$A = \begin{pmatrix} 3 & 4 & -2 \\ 1 & 0 & 1 \\ -2 & -3 & 2 \end{pmatrix}$$

   b) Let A be as in part (a). Use your answer to part (a) to find the solution set of the system $A\mathbf{x} = (3, -2, -5)$. (To receive credit, it must be clear how you are using your answer to part (a).)

2. Below, you are given the augmented matrix corresponding to a linear system of equations, together with the reduced row-echelon form of that augmented matrix:

$$\left(\begin{array}{ccccc|c} 3 & -1 & 2 & 4 & -3 & 2 \\ 2 & 1 & 7 & -4 & -1 & -3 \\ -1 & -8 & -29 & 32 & -4 & 21 \\ 4 & -3 & -3 & 12 & -5 & 7 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccccc|c} 1 & 0 & \frac{9}{5} & 0 & -\frac{4}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{17}{5} & -4 & \frac{3}{5} & -\frac{13}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

   a) If you think of this system of equations as a matrix equation $A\mathbf{x} = \mathbf{b}$, what is $\mathbf{b}$?

   b) If you think of this system of equations as a functional equation $T(\mathbf{x}) = \mathbf{b}$, what is the domain of T?

   c) How many linearly independent columns does A have?

   d) Find the solution set of $A\mathbf{x} = \mathbf{b}$.

   e) Find a basis for the row space of A.

   f) Find a basis for the null space of A.

3. Let x_n and y_n denote the number of female geese and male geese living in a pond at time n. Suppose that for every n,

$$\begin{cases} x_{n+1} &= \frac{8}{5}x_n + \frac{1}{5}y_n \\ y_{n+1} &= \frac{6}{5}x_n + \frac{7}{5}y_n \end{cases}.$$

   If at time 0, there are 2 female geese and 5 male geese in the pond, find the number of male geese living in the pond at time 100.

4. Throughout this problem, let $\mathbf{v} = (4, 1, -3)$ and let $\mathbf{w} = (2, 0, 1)$.

   a) Compute $4\mathbf{v} + 5\mathbf{w}$.

   b) Compute $\mathbf{v} \cdot (\mathbf{v} + \mathbf{w})$.

c) Find a unit vector in the same direction as $\mathbf{v}$.

d) Compute $\mathbf{w} \times \mathbf{v}$.

e) Compute the distance between $\mathbf{v}$ and $\mathbf{w}$.

f) Find parametric equations of the line containing $\mathbf{v}$ and $\mathbf{w}$.

g) Find the normal equation of the plane containing $\mathbf{v}$, $\mathbf{w}$ and $(1, 6, -2)$.

5. Throughout this problem, let W be the subspace of $\mathbb{R}^6$ which has orthonormal basis
$$\left\{ \left(\frac{1}{3}, 0, \frac{-2}{3}, \frac{2}{3}, 0, 0\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 0, 0, 0\right), (0, 0, 0, 0, 0, 1) \right\}$$
and let $\mathbf{v} = (9, 12, -6, 3, -7, 11)$.

a) Compute the projection of $\mathbf{v}$ onto W.

b) Compute the projection of $\mathbf{v}$ onto $W^\perp$.

6. Throughout this problem, let S and T be the following linear transformations:

- $S : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies $S(1, 0) = (1, 2, 1)$ and $S(0, 1) = (-3, 1, 0)$;
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $T(1, 0) = (3, -4)$ and $T(0, 1) = (2, -5)$.

a) Compute $T(3, -2)$.

b) Is S surjective? Explain.

c) Is S injective? Explain.

d) Is T invertible? If so, find the standard matrix of T^{-1}. If not, explain why not.

e) Which of the two transformations $T \circ S$ or $S \circ T$ is defined? For the transformation that is defined, find its standard matrix.

7. In this problem, let A, B and M be the following matrices:
$$A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \; B = \begin{pmatrix} 5 & -2 \\ -6 & 1 \end{pmatrix} \; M = \begin{pmatrix} 1 & 4 & -2 \\ -3 & 1 & -5 \\ 2 & 0 & -3 \end{pmatrix}$$

a) Compute $A^2 B$.

b) Compute $\det M$.

c) Compute $\det 10M$.

d) Compute the eigenvalues and eigenvectors of B.

8. Classify the following statements as true or false:

a) If a 3×3 matrix A has eigenvalues 3, 4 and -2, then A is diagonalizable.

b) If a 3×3 matrix A has eigenvalues 3, 4 and -2, then the equation $A\mathbf{x} = (-5, 7, 11)$ has exactly one solution.

c) If $A \in M_{mn}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$, then the least-squares solution of $A\mathbf{x} = \mathbf{b}$ is given by $\widehat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$

d) If A is an $m \times n$ matrix, then the row space of A and the null space of A are orthogonal complements.

e) If W is a subspace of V, then $\dim W \leq \dim V$.

f) If A and B are square matrices of the same size, then $tr(AB) = tr(A)\, tr(B)$.

g) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then for any matrix $A \in M_2(\mathbb{R})$, $T(A\mathbf{x}) = AT(\mathbf{x})$.

h) If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, then $(3\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (3\mathbf{w})$.

i) If A and B are invertible matrices of the same size, then $(AB)^{-1} = A^{-1}B^{-1}$.

j) If $\mathbf{v}$, $\mathbf{w}_1$ and $\mathbf{w}_2$ are vectors in $\mathbb{R}^n$, then $\pi_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{v}) = \pi_{\mathbf{w}_1}(\mathbf{v}) + \pi_{\mathbf{w}_2}(\mathbf{v})$.

9. In each part of this problem, a set W is described.

- If W is a subspace of $\mathbb{R}^n$ for some n, say so, identify the vector space W is a subspace of, and find $\dim W$.
- If W is not a subspace, but is an affine subspace of $\mathbb{R}^n$ for some n, say so, identify the vector space W is an affine subspace of, and find $\dim W$.
- If W is not an affine subspace of $\mathbb{R}^n$ for any n, say so.

a) $W = Span(1, 2, 3, 4)$.

b) $W = Span((1, 2, 3, 4))$.

c) W is the set of vectors orthogonal to both $(6, 7, 3)$ and $(-2, 4, -5)$.

d) W is a hyperplane in $\mathbb{R}^6$ which does not contain $\mathbf{0}$.

e) W is the null space of A, where A is a 7×9 matrix with rank 4.

f) $W = \{(x, y) : 3x + 4y = 7\}$.

g) $W = \{(x, y, z) : x = y = z\}$.

h) W is the solution set of $A\mathbf{x} = (3, 5, 8)$, where $A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

i) $W = Span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is a basis of $\mathbb{R}^5$.

j) $W = ker(T)$, where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\mathbf{v}) =$ the projection of $\mathbf{v}$ onto $(1, -6, 4)$.

Solutions

1. a) Perform row reductions on the augmented matrix $(A\,|\,I)$:

$$(A\,|\,I) = \left(\begin{array}{ccc|ccc} 3 & 4 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 4 & -2 & 1 & 0 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[2R_1+R_3]{-3R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & -5 & 1 & -3 & 0 \\ 0 & -3 & 4 & 0 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & -3 & 4 & 0 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{3R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 3 & -1 & 4 \end{array}\right)$$

$$\xrightarrow[-R_3+R_1]{R_3+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & -4 \\ 0 & 1 & 0 & 4 & -2 & 5 \\ 0 & 0 & 1 & 3 & -1 & 4 \end{array}\right)$$

This last matrix is $(I\,|A^{-1})$, so $A^{-1} = \begin{pmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ 3 & -1 & 4 \end{pmatrix}$.

b) Since A is invertible, the one and only solution to $A\mathbf{x} = (3, -2, -5)$ is

$$\mathbf{x} = A^{-1}(3, -2, -5) = \begin{pmatrix} -3 & 2 & -4 \\ 4 & -2 & 5 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ -9 \end{pmatrix}.$$

2. a) $\mathbf{b} = (2, -3, 21, 7)$.

b) $T : \mathbb{R}^5 \to \mathbb{R}^4$, so the domain is $\mathbb{R}^5$.

c) This is the number of pivots, which is 2.

d) Solve $A\mathbf{x} = \mathbf{b}$ using the rref form. Writing $\mathbf{x} = (v, w, x, y, z)$, we have the system

$$\begin{cases} v + \frac{9}{5}x - \frac{4}{5}z = -\frac{1}{5} \\ w + \frac{17}{5}x - 4y + \frac{3}{5}z = -\frac{13}{5} \end{cases} \Rightarrow \begin{cases} v = -\frac{1}{5} - \frac{9}{5}x + \frac{4}{5}z \\ w = -\frac{13}{5} - \frac{17}{5}x + 4y - \frac{3}{5}z \end{cases}$$

Substituting, we obtain the solution set

$$\begin{aligned} \mathbf{x} &= \{\left(-\frac{1}{5} - \frac{9}{5}x + \frac{4}{5}z, -\frac{13}{5} - \frac{17}{5}x + 4y - \frac{3}{5}z, x, y, z\right) : x, y, z \in \mathbb{R}\} \\ &= \{\left(-\frac{1}{5}, -\frac{13}{5}, 0, 0, 0\right) + x\left(\frac{-9}{5}, \frac{-17}{5}, 1, 0, 0\right) + y(0, 4, 0, 1, 0) + z\left(\frac{4}{5}, \frac{-3}{5}, 0, 0, 1\right) : x, y, z \in \mathbb{R}\} \\ &= \left(-\frac{1}{5}, -\frac{13}{5}, 0, 0, 0\right) + Span(\left(\frac{-9}{5}, \frac{-17}{5}, 1, 0, 0\right), (0, 4, 0, 1, 0), \left(\frac{4}{5}, \frac{-3}{5}, 0, 0, 1\right)). \end{aligned}$$

e) A basis for $R(A)$ consists of the pivot rows of $rref(A)$:

$$\{\left(1, 0, \frac{9}{5}, 0, -\frac{4}{5}\right), \left(0, 1, \frac{17}{5}, -4, \frac{3}{5}\right)\}$$

f) From the work in part (d), we can conclude

$$N(A) = Span(\left(\frac{-9}{5}, \frac{-17}{5}, 1, 0, 0\right), (0, 4, 0, 1, 0), \left(\frac{4}{5}, \frac{-3}{5}, 0, 0, 1\right)).$$

The three vectors in the spanning set form a basis, since we know $\dim N(A) = n - r = 5 - 2 = 3$.

3. Writing $A = \begin{pmatrix} \frac{8}{5} & \frac{1}{5} \\ \frac{6}{5} & \frac{7}{5} \end{pmatrix}$ and $\mathbf{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, we have $\mathbf{x}_{100} = A^{100}\mathbf{x}_0 = A^{100}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$. To compute this, diagonalize A by finding eigenvalues and eigenvectors. The characteristic polynomial of A is

$$\det(A - \lambda I) = (\frac{8}{5} - \lambda)(\frac{7}{5} - \lambda) - \frac{6}{25} = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

so the eigenvalues are $\lambda = 2$ and $\lambda = 1$. To find the corresponding eigenvectors, set $\mathbf{x} = (x, y)$ and solve $A\mathbf{x} = \lambda\mathbf{x}$ to get

$$\lambda = 2: \begin{cases} \frac{8}{5}x + \frac{1}{5}y &= 2x \\ \frac{6}{5}x + \frac{7}{5}y &= 2y \end{cases} \Rightarrow \frac{1}{5}y = \frac{2}{5}x \Rightarrow y = 2x \Rightarrow (1, 2)$$

$$\lambda = 1: \begin{cases} \frac{8}{5}x + \frac{1}{5}y &= x \\ \frac{6}{5}x + \frac{7}{5}y &= y \end{cases} \Rightarrow y = -3x \Rightarrow y = -3x \Rightarrow (1, -3)$$

Thus $A = S\Lambda S^{-1}$ where $S = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. That means

$$\begin{aligned}
\mathbf{x}_{100} &= A^{100}\mathbf{x}_0 \\
&= S\Lambda^{100}S^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&= \begin{pmatrix} 2^{100} & 1 \\ 2 \cdot 2^{100} & -3 \end{pmatrix}\frac{1}{-5}\begin{pmatrix} -3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\
&= -\frac{1}{5}\begin{pmatrix} 2^{100} & 1 \\ 2 \cdot 2^{100} & -3 \end{pmatrix}\begin{pmatrix} -11 \\ 1 \end{pmatrix} \\
&= -\frac{1}{5}\begin{pmatrix} -11 \cdot 2^{100} + 1 \\ -22 \cdot 2^{100} - 3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{5}(11 \cdot 2^{100} - 1) \\ \frac{1}{5}(22 \cdot 2^{100} + 3) \end{pmatrix}.
\end{aligned}$$

The number of female geese at time 100 is therefore $\frac{1}{5}(22 \cdot 2^{100} + 3)$.

4. a) $4\mathbf{v}+5\mathbf{w}=(16,4,-12)+(10,0,5)=(26,4,-7)$.

   b) $\mathbf{v}\cdot(\mathbf{v}+\mathbf{w})=(4,1,-3)\cdot(6,1,-2)=24+1+6=31$.

   c) The unit vector is $\frac{1}{||\mathbf{v}||}\mathbf{v}=\frac{1}{\sqrt{4^2+1^2+(-3)^2}}\mathbf{v}=\frac{1}{\sqrt{26}}(4,1,-3)=\left(\frac{4}{\sqrt{26}},\frac{1}{\sqrt{26}},\frac{-3}{\sqrt{26}}\right)$.

   d) $\mathbf{w}\times\mathbf{v}=((-3)0-1(1),1(4)-2(-3),2(1)-0(4))=(-1,10,2)$.

   e) $dist(\mathbf{v},\mathbf{w})=||\mathbf{v}-\mathbf{w}||=||(2,1,-4)||=\sqrt{2^2+1^2+(-4)^2}=\sqrt{21}$.

   f) The line has direction vector $\mathbf{v}-\mathbf{w}=(2,1,-4)$ and passes through the point $\mathbf{v}=(4,1,-3)$. Thus one set of parametric equations for the line is

$$\mathbf{x}=\mathbf{v}+t(\mathbf{v}-\mathbf{w})\Leftrightarrow\begin{cases}x=4+2t\\y=1+t\\z=-3-4t\end{cases}$$

   g) Two vectors in the plane are $\mathbf{v}-\mathbf{w}=(2,1,-4)$ and $\mathbf{v}-(1,6,-2)=(3,-5,-1)$. So a normal vector to the plane is $\mathbf{n}=(2,1,-4)\times(3,-5,-1)=(-21,-10,-13)$. Set $d=\mathbf{n}\cdot\mathbf{w}=(-21,-10,13)=-55$; then the plane has normal equation $\mathbf{n}\cdot\mathbf{x}=d$, i.e. $(-21,-10,-13)\cdot(x,y,z)=-55$. Writing this out, the plane has equation $-21x-10y-13z=-55$.

5. a) Denote the given orthonormal basis of W by $\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$. Using the projection formula, we get

$$\begin{aligned}\pi_W(\mathbf{v})&=(\mathbf{v}\cdot\mathbf{x}_1)\mathbf{x}_1+(\mathbf{v}\cdot\mathbf{x}_2)\mathbf{x}_2+(\mathbf{v}\cdot\mathbf{x}_3)\mathbf{x}_3\\&=9\left(\frac{1}{3},0,\frac{-2}{3},\frac{2}{3},0,0\right)+12\left(\frac{2}{3},\frac{2}{3},\frac{1}{3},0,0,0\right)+11(0,0,0,0,0,1)\\&=(3,0,-6,6,0,0)+(8,8,4,0,0,0)+(0,0,0,0,0,11)\\&=(11,8,-2,6,0,11).\end{aligned}$$

   b) Subtract the answer from part (a) from $\mathbf{v}$:

$$\begin{aligned}\pi_{W^\perp}(\mathbf{v})&=\mathbf{v}-\pi_W(\mathbf{v})\\&=(9,12,-6,3,-7,11)-(11,8,-2,6,0,11)\\&=(-2,4,-4,-3,-7,0).\end{aligned}$$

6. a) $T(3,-2)=3T(1,0)-2T(0,1)=3(3,-4)-2(2,-5)=(9,-12)-(4,-10)=(5,-2)$.

   b) Since S maps a 2-dimensional space into a space of dimension greater than 2, S cannot be surjective.

   c) Note that $im(S)=Span((1,2,1),(-3,1,0))$, so $im(S)$ contains two linearly independent vectors. Thus $rank(S)=\dim im(S)\geq 2$. That means $\dim ker(S)\leq 2-2=0$, meaning $\dim ker(S)=0$, meaning S is injective.

d) The standard matrix of T is $\begin{pmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}$. Since the determinant of this matrix is $3(-5) - 2(-4) = -7 \neq 0$, this matrix is invertible, meaning T is invertible. The standard matrix of T^{-1} is

$$\begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix}^{-1} = \frac{1}{-7}\begin{pmatrix} -5 & -2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{7} & \frac{2}{7} \\ -\frac{4}{7} & -\frac{3}{7} \end{pmatrix}.$$

e) Since T is given by a 2×2 matrix and S is given by a 3×2 matrix, $S \circ T$ is defined. Its standard matrix is

$$\begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 15 & 17 \\ 2 & -1 \\ 3 & 2 \end{pmatrix}.$$

7. a) By usual matrix multiplication,

$$A^2B = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -6 & 1 \end{pmatrix} = \begin{pmatrix} -49 & 7 \\ -3 & 4 \end{pmatrix}.$$

b) Using the Rule of Sarrus,

$$\begin{aligned} \det M &= [1(1)(-3) + (4)(-5)2 + (-2)(-3)0] - [2(1)(-2) + 0(-5)1 + (-3)(-3)4] \\ &= [-3 - 40] - [-4 + 36] \\ &= -43 - 32 \\ &= -75. \end{aligned}$$

c) Since M is 3×3, $\det 10M = 10^3 \det M = 1000(-75) = -75000$.

d) Start with the eigenvectors. The characteristic polynomial is $p_B(\lambda) = \det(B - \lambda I) = (5-\lambda)(1-\lambda) - 12 = \lambda^2 - 6\lambda - 7 = (\lambda - 7)(\lambda + 1)$ so the eigenvalues are $\lambda = 7$ and $\lambda = -1$. Now for the eigenvectors. Set $\mathbf{x} = (x, y)$ and solve $A\mathbf{x} = \lambda\mathbf{x}$ to get

$$\begin{aligned} \lambda = 7: &\quad \begin{cases} 5x - 2y &= 7x \\ -6x + y &= 7y \end{cases} \Rightarrow -y = x \Rightarrow (1, -1) \\ \lambda = -1: &\quad \begin{cases} 5x - 2y &= -x \\ -6x + y &= -y \end{cases} \Rightarrow -2y = -6x \Rightarrow y = 3x \Rightarrow (1, 3). \end{aligned}$$

8. a) **TRUE**. Any $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

b) **TRUE**. $\det A = 3(4)(-2) = -24 \neq 0$, so A is invertible, meaning $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$.

c) **TRUE**. The formula is indeed $\widehat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$.

d) **TRUE**. This is one part of the Fundamental Theorem of Linear Algebra.

e) **TRUE**. This follows from the Exchange Lemma.

f) **FALSE**. For a counterexample, set $A = B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $tr(AB) = 2$ but $tr(A)\,tr(B) = 0 \cdot 0 = 0$.

g) **FALSE**. For a counterexample, set $T(x, y) = (x, 2y)$ and let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T(A(1, 0)) = T(1, 1) = (1, 2)$ but $AT(1, 1) = A(1, 2) = (3, 3)$.

h) **TRUE**. This is what is called "bilinearity of dot product".

i) **FALSE**. The order reverses: $(AB)^{-1} = B^{-1}A^{-1}$, not $A^{-1}B^{-1}$.

j) **FALSE**. For a counterexample, set $\mathbf{v} = \mathbf{w}_1 = \mathbf{w}_2 = (1, 0)$. Then if $\pi_{\mathbf{w}_1+\mathbf{w}_2}(\mathbf{v}) = \pi_{(2,0)}(1, 0) = (1, 0)$ but $\pi_{\mathbf{w}_1}(\mathbf{v}) + \pi_{\mathbf{w}_2}(\mathbf{v}) = \pi_{(1,0)}(1, 0) + \pi_{(1,0)}(1, 0) = (1, 0) + (1, 0) = (2, 0)$.

9. a) W is a subspace of $\mathbb{R}$ with $\dim W = 1$. (This W is the span of four elements of $\mathbb{R}$, all of which are parallel to one another.)

b) W is a subspace of $\mathbb{R}^4$ with $\dim W = 1$. (This W is the span of one nonzero element of $\mathbb{R}^4$.)

c) W is a subspace of $\mathbb{R}^3$ with $\dim W = 1$, since $W = Span((6, 7, 3), (-2, 4, -5))^{\perp}$.

d) W is an affine subspace of $\mathbb{R}^6$ with $\dim W = 6 - 1 = 5$.

e) W is a subspace of $\mathbb{R}^9$, with $\dim W = n - r = 9 - 4 = 5$.

f) W is an affine subspace of $\mathbb{R}^2$ with $\dim W = 1$. (W is a line in $\mathbb{R}^2$ not passing through $\mathbf{0}$.)

g) W is a subspace of $\mathbb{R}^3$ with $\dim W = 1$. ($W = \{(x, x, x)\} = Span((1, 1, 1))$.)

h) W is an affine subspace of R^4 with $\dim W = 4 - 3 = 1$. (In general, $\dim W = \dim N(A) = n - r$ where A is $m \times n$ and has rank r.)

i) W is a subspace of $\mathbb{R}^5$ with $\dim W = 3$ (since the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent).

j) W is a subspace of $\mathbb{R}^3$ of dimension 2. (The rank of this T is 1, since $im(T) = Span((1, -6, 4))$, so $\dim ker(T) = n - r = 3 - 2 = 1$.)

1.4 Fall 2016 Final Exam

1. Solve the following system of equations (by hand, using row reductions, showing your steps).

$$\begin{cases} 2x & -y & +3z & = -1 \\ -x & +4y & +2z & = -3 \\ 2x & & +4z & = -2 \end{cases}$$

2. Below, you are given the augmented matrix corresponding to a linear system of equations, together with the reduced row-echelon form of that augmented matrix:

$$(A\,|\,\mathbf{b}) = \left(\begin{array}{cccc|c} 3 & -2 & 0 & 1 & -4 \\ 1 & 1 & -2 & 0 & -3 \\ 3 & -7 & 6 & 2 & 1 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & \frac{-4}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{5}{6} \end{array}\right) = \text{rref}(A\,|\,\mathbf{b})$$

   a) How many equations are in this system of equations?
   b) How many variables are in this system of equations?
   c) Solve the system $A\mathbf{x} = \mathbf{b}$.
   d) Find a basis for the column space of A.
   e) Find a basis for the null space of A.
   f) Suppose A is the standard matrix of linear transformation T.
      i. What is the domain of T?
      ii. Is T injective?
      iii. Is T surjective?
      iv. Is T bijective?
   g) Give a vector $\mathbf{y}$ for which the system $A\mathbf{x} = \mathbf{y}$ has no solution.

3. Let $A = \begin{pmatrix} -1 & -2 \\ -12 & -3 \end{pmatrix}$.

   a) Find the eigenvalues and eigenvectors of A.
   b) Diagonalize A.
   c) Compute the matrix exponential of A.

4. Throughout this problem, let $\mathbf{v} = (1, 5)$ and let $\mathbf{w} = (-3, 4)$.

   a) Compute $3\mathbf{v} - 5\mathbf{w}$.
   b) Find a unit vector in the same direction as $\mathbf{w}$.

c) Find k so that the vector $(3, k)$ is orthogonal to $\mathbf{v}$.

d) Find a nonzero vector which is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$.

e) Compute the projection of $\mathbf{v}$ onto $\mathbf{w}$.

f) Compute $||\mathbf{v} + \mathbf{w}||$.

g) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation such that $T(\mathbf{e}_1) = (3, 2)$ and $T(\mathbf{e}_2) = \mathbf{w}$, find $T(\mathbf{v})$.

5. a) Find the trace of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

   b) Find AB^2 if $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$.

   c) Find the determinant of the matrix $\begin{pmatrix} 1 & 2 & -5 \\ 3 & 0 & -4 \\ -2 & -3 & 1 \end{pmatrix}$.

   d) Find the determinant of the matrix $\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$.

6. a) Write the normal equation of the plane in $\mathbb{R}^3$ passing through the points $(1, -5, -4)$, $(-2, -3, 2)$ and $(5, 3, 4)$.

   b) The two planes in $\mathbb{R}^3$ whose normal equations are $x - 4y + z = 7$ and $2x - 7y + 3z = 3$ intersect in a line. Find the parametric equations of this line.

7. Suppose:

   - A is an invertible 3×3 matrix;
   - B is a 3×4 matrix;
   - C is a 4×3 matrix;
   - D is a 1×3 matrix;
   - $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vectors in $\mathbb{R}^3$;
   - $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^4$; and
   - k, l and m are scalars.

   Determine whether the following expressions are a **matrix** (in which you should give its size), a **vector** (in which case you should give the vector space to which the vector belongs), a **scalar**, or **nonsense**:

a) B^T
b) $\mathbf{v} \cdot (k\mathbf{v})$
c) $\mathbf{x} \times B\mathbf{v}$
d) $\mathbf{w}\mathbf{w}^T$
e) $\det(mA)A$
f) $Bk\mathbf{x}$
g) $(CB)^{-1}$
h) $2 + klDB\mathbf{w}$
i) C^2
j) the largest eigenvalue of A
k) $(\mathbf{x}^T\mathbf{x})\mathbf{z} - \pi_{\mathbf{y}}\mathbf{x}$
l) $||\mathbf{v}||CA\mathbf{x}D||\mathbf{w}||$

8. Classify the following statements as true or false:

   a) If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in $\mathbb{R}^3$, then $\mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{v}$.
   b) If A and B are square matrices of the same size, then $\det(AB) = \det A \det B$.
   c) The function $T : \mathbb{R}^4 \to R^4$ which projects points onto the span of $(1, 2, -1, 3)$ and $(-3, 0, 1, 2)$ is a linear transformation.
   d) If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in $\mathbb{R}^4$, then $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.
   e) The line $y = 2x$ is a subspace of $\mathbb{R}^2$.
   f) The following set of vectors is linearly independent:
   $$\{(-1, 1, -1), (3, 0, 4), (2, -2, 2)\}$$
   g) If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors in $\mathbb{R}^5$, then $|\mathbf{v} \cdot \mathbf{w}| \leq ||\mathbf{v}||\,||\mathbf{w}||$.
   h) The function $T : M_3(\mathbb{R}) \to \mathbb{R}$ defined by $T(A) = \det A$ is a linear transformation.
   i) The distance from vector $\mathbf{v}$ to subspace W is the length of the projection of $\mathbf{v}$ onto W.
   j) Given any set of linearly independent vectors in a finite-dimensional vector space V, that set can be extended to form a basis of V.

9. In each part of this problem, a subset W of $\mathbb{R}^4$ is described. Determine whether the set W is a point, line, plane, hyperplane, or all of $\mathbb{R}^4$.

   a) $W = Span((1, 3, -7, 0))$.
   b) $W = Span((1, 3, -7, 0)) + (2, 1, -5, 1)$.
   c) $W = Span((1, 3, -7, 0), (2, 1, -5, 1))$.
   d) $W = Span((1, 3, -7, 0), (2, 1, -5, 1), (3, 4, -12, 1))$.
   e) $W = \{(w, x, y, z) : 2w - x + 5y - z = 0\}$.
   f) W is the orthogonal complement of a 2-dimensional subspace of $\mathbb{R}^4$.
   g) W is the set of solutions to $A\mathbf{x} = \mathbf{b}$, where A is a 4×4 matrix with 3 linearly independent columns and $\mathbf{b} \in C(A)$.

h) W is the row space of an invertible 4×4 matrix.

i) W is the null space of a 3×4 matrix whose rows are linearly independent.

j) W is the intersection of $Span(\mathbf{v}_1, \mathbf{v}_2)$ and $Span(\mathbf{v}_3, \mathbf{v}_4)$, where $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ form a basis of $\mathbb{R}^4$.

10. **(Bonus)** When teaching Math 120 (trigonometry), I do not make my students memorize the addition identities for sine and cosine, which go like this:

$\sin(\alpha + \beta) =$ something with $\sin$ and/or $\cos\alpha$, $\sin$ and/or $\cos\beta$, etc. in it

$\cos(\alpha + \beta) =$ something with $\sin$ and/or $\cos\alpha$, $\sin$ and/or $\cos\beta$, etc. in it

Use linear algebra to figure out what $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ must be equal to.

Solutions

1. Write the augmented matrix and perform row reductions:

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & -1 \\ -1 & 4 & 2 & -3 \\ 2 & 0 & 4 & -2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 4 & 2 & -3 \\ 2 & -1 & 3 & -1 \\ 2 & 0 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{2R_1+R_2, 2R_1+R_3} \left(\begin{array}{ccc|c} -1 & 4 & 2 & -3 \\ 0 & 7 & 7 & -7 \\ 0 & 8 & 8 & -8 \end{array}\right)$$

$$\xrightarrow{-1 \cdot R_1, \frac{-8}{7}R_2+R_3} \left(\begin{array}{ccc|c} 1 & -4 & -2 & 3 \\ 0 & 7 & 7 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{1}{7} \cdot R_2} \left(\begin{array}{ccc|c} 1 & -4 & -2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{4R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This leaves the system of equations

$$\begin{cases} x + 2z = -1 \\ y + z = -1 \end{cases} \Rightarrow \begin{cases} x = -2z - 1 \\ y = -z - 1 \end{cases}$$

Thus the solution set is $\{(-2z-1, -z-1, z)\} = \{(-1,-1,0) + z(-2,-1,1)\} = (-1,-1,0) + Span(-2,-1,1)$.

2. a) Since A has 3 rows, there are **3** equations in the system.

   b) Since A has 4 columns, there are **4** variables in the system.

   c) From the rref form, we have $w + \frac{1}{3}z = \frac{-4}{3}$, $x = 0$ and $y + \frac{1}{6}z = \frac{5}{6}$. Therefore $w = \frac{-1}{3}z - \frac{4}{3}$, $x = 0$, $y = \frac{-1}{6}z + \frac{5}{6}$ so the solution set is

$$\{(w,x,y,z)\} = \{\left(\frac{-1}{3}z - \frac{4}{3}, 0, \frac{-1}{6}z + \frac{5}{6}, z\right)\} = \left(\frac{-4}{3}, 0, \frac{5}{6}, 0\right) + Span\left(\frac{-1}{3}, 0, \frac{-1}{6}, 1\right).$$

   d) A basis for $C(A)$ is the pivot columns of A: $\{(3,1,3), (-2,1,-7), (0,-2,6)\}$. (As a side comment, since these are three linearly independent vectors in $\mathbb{R}^3$, $C(A) = \mathbb{R}^3$ so any three linearly independent vectors in $\mathbb{R}^3$ form a basis of $C(A)$.)

   e) From part (c), we know that $N(A) = Span\left(\frac{-1}{3}, 0, \frac{-1}{6}, 1\right)$ so a basis of $N(A)$ is the single vector $\left(\frac{-1}{3}, 0, \frac{-1}{6}, 1\right)$ (or any nonzero multiple of this such as $(-2,0,-1,6)$).

   f) i. The domain of T is $\mathbb{R}^4$.

      ii. T is not injective because the rank of T is $r = 3$ but $n = 4$.

iii. T is surjective because $r = m = 3$.

iv. T is not bijective because it is not injective.

g) $\mathbf{y}$ can be any vector which differs from $\mathbf{b}$ in the last coordinate, such as $(-4, -3, 2)$ or $(-4, -3, 0)$.

3. a) $\det(A - \lambda I) = (-1-\lambda)(-3-\lambda) - 24 = \lambda^2 + 4\lambda - 21 = (\lambda+7)(\lambda-3)$ so the eigenvalues are $\lambda = -7$ and $\lambda = 3$. Now for the eigenvectors: write $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ to find them:

$$\lambda = -7 : \begin{cases} -x - 2y = -7x \\ -12x - 3y = -7y \end{cases} \Rightarrow y = 3x \Rightarrow (1,3)$$

$$\lambda = 3 : \begin{cases} -x - 2y = 3x \\ -12x - 3y = 3y \end{cases} \Rightarrow y = -2x \Rightarrow (1,-2)$$

b) $A = S\Lambda S^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -7 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}^{-1}.$

c)

$$\begin{aligned} e^A &= \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} e^{-7} & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} e^{-7} & 0 \\ 0 & e^3 \end{pmatrix} \frac{-1}{5} \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{-1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -2e^{-7} & -e^{-7} \\ -3e^3 & e^3 \end{pmatrix} \\ &= \frac{-1}{5} \begin{pmatrix} -2e^{-7} - 3e^3 & -e^{-7} + e^3 \\ -6e^{-7} + 6e^3 & -3e^{-7} - 2e^3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2e^{-7}+3e^3}{5} & \frac{e^{-7}-e^3}{5} \\ \frac{6e^{-7}-6e^3}{5} & \frac{3e^{-7}+2e^3}{5} \end{pmatrix}. \end{aligned}$$

4. a) $3\mathbf{v} - 5\mathbf{w} = 3(1,5) - 5(-3,4) = (3,15) - (-15,20) = (18,-5)$.

b) $\frac{1}{||\mathbf{w}||}\mathbf{w} = \frac{1}{\sqrt{(-3)^2+4^2}}(-3,4) = \frac{1}{5}(-3,4) = \left(\frac{-3}{5}, \frac{4}{5}\right)$.

c) Set $(3,k) \cdot v = 0$ to get $3 + 5k = 0$, i.e. $k = \frac{-3}{5}$.

d) No such vector exists; since $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, their span is all of $\mathbb{R}^2$, and no nonzero vector is orthogonal to all vectors in $\mathbb{R}^2$.

e) $\pi_{\mathbf{w}}(\mathbf{v}) = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w} = \frac{17}{25}(-3,4) = \left(\frac{-51}{25}, \frac{68}{25}\right)$.

f) $||\mathbf{v} + \mathbf{w}|| = ||(-2,9)|| = \sqrt{(-2)^2 + 9^2} = \sqrt{85}$.

g) $T(\mathbf{v}) = T(1,5) = 1T(\mathbf{e}_1) + 5T(\mathbf{e}_2) = (3,2) + 5(-3,4) = (-12,22)$.

5. a) $tr \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9 = 15.$

b) By the usual method of matrix multiplication, $AB^2 = ABB = \begin{pmatrix} 0 & -1 \\ -5 & -8 \end{pmatrix}$.

c) $\det \begin{pmatrix} 1 & 2 & -5 \\ 3 & 0 & -4 \\ -2 & -3 & 1 \end{pmatrix} = (0 + 16 + 45) - (0 + 12 + 6) = 43.$

d) After switching rows 2 and 4 (which multiplies the determinant by -1), the matrix is upper triangular, so the determinant is (-1) times the product of the diagonal entries which is $(-1)(2)(3)(1)(1)(5)(2) = -60$.

6. a) Two vectors in the plane are $\mathbf{v} = (1, -5, -4) - (-2, -3, 2) = (3, -2, -6)$ and $\mathbf{w} = (5, 3, 4) - (1, -5, -4) = (4, 8, 8)$. Therefore a vector normal to the plane is $\mathbf{n} = \mathbf{v} \times \mathbf{w} = (32, -48, 32)$ (or by taking a multiple of this, $\mathbf{n} = (2, -3, 2)$). Thus the equation of the plane is $2x - 3y + 2z = d$. To find d, plug in a point on the plane like $(1, -5, -4)$: $2(1) - 3(-5) + 2(-4) = 2 + 15 - 8 = 9$ so the equation of the plane is $2x - 3y + 2z = 9$ (or any multiple of this).

b) The two planes in $\mathbb{R}^3$ whose normal equations are $x - 4y + z = 7$ and $2x - 7y + 3z = 3$ intersect in a line. Find the parametric equations of this line.
Start with the two equations and solve them as if they are a system:

$$\left(\begin{array}{ccc|c} 1 & -4 & 1 & 7 \\ 2 & -7 & 3 & 3 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -4 & 1 & 7 \\ 0 & 1 & 1 & -11 \end{array}\right) \xrightarrow{4R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 5 & -37 \\ 0 & 1 & 1 & -11 \end{array}\right)$$

Thus $x = -5z - 37$ and $y = -z - 11$, so the intersection of the two planes is given by

$$\{(-5z - 37, -z - 11, z)\} = (-37, -11, 0) + Span(-5, -1, 1).$$

That means the point $\mathbf{p} = (-37, -11, 0)$ is on the line and the line has direction vector $\mathbf{v} = (-5, -1, 1)$, so (one of many possible sets of) parametric equations of the line are $\mathbf{x} = \mathbf{p} + t\mathbf{v}$,i.e.

$$\begin{cases} x = -37 - 5t \\ y = -11 - t \\ z = t \end{cases}$$

7. a) B^T is a 4×3 **matrix**.

b) $\mathbf{v} \cdot (k\mathbf{v})$ is a **scalar**.

c) $\mathbf{x} \times B_{3\times4}\mathbf{v}_{4\times1}$ is a **vector** in $\mathbb{R}^3$.

d) $\mathbf{w}_{4\times1}\mathbf{w}^T_{1\times4}$ is a 4×4 **matrix**.

e) $\det(mA)A$ is a 3×3 **matrix**.

f) $B_{3\times 4}k\mathbf{x}_{3\times 1}$ is **nonsense**.

g) $(CB)^{-1}$ is **nonsense** (while CB is 4×4, it cannot have full rank because the column space of CB is a subspace of the column space of C, which is at most 3 dimensional, so CB cannot be invertible).

h) $2 + klD_{1\times 3}B_{3\times 4}\mathbf{w}_{4\times 1}$ is a **scalar**.

i) C^2 is **nonsense**.

j) the largest eigenvalue of A is a **scalar**.

k) $(\mathbf{x}^T\mathbf{x})\mathbf{z}_{3\times 1} - \pi_{\mathbf{y}}\mathbf{x}_{3\times 1}$ is a **vector** in $\mathbb{R}^3$.

l) $||\mathbf{v}||C_{4\times 3}A_{3\times 3}\mathbf{x}_{3\times 1}D_{1\times 3}||\mathbf{w}||$ is a 4×3 **matrix**.

8. a) **FALSE** ($\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$).

b) **TRUE** (this is a theorem from Chapter 7).

c) **TRUE** (projections are always linear transformations).

d) **TRUE** (this is the Triangle Inequality).

e) **TRUE** (it is the span of $(1, 2)$, and spans are always subspaces).

f) **FALSE** (the third vector is twice the first one).

g) **TRUE** (this is the Cauchy-Schwarz inequality).

h) **FALSE** ($T(2A) = \det(2A) = 2^3 \det A = 8 \det A \neq 2T(A)$ so T does not preserve scalar multiplication).

i) **FALSE** (the distance from vector $\mathbf{v}$ to subspace W is the length of $\pi_{W^\perp}\mathbf{v}$, not $\pi_W\mathbf{v}$.

j) **TRUE** (this is called the Basis Extension Theorem).

9. a) $W = Span((1, 3, -7, 0))$ is a **line**.

b) $W = Span((1, 3, -7, 0)) + (2, 1, -5, 1)$ is a **line**.

c) $W = Span((1, 3, -7, 0), (2, 1, -5, 1))$ is a **plane**.

d) Notice that the third vector is the sum of the first two, so it can be dropped from the span. Thus this W is the same as the one in part (c) which is a **plane**.

e) W is described by a normal equation, which must belong to a **hyperplane**.

f) $\dim W = 4 - \dim W^\perp = 4 - 2 = 2$, so W is a **plane**.

g) Call the matrix A; we have $m = n = 4$ but $r = 3$ so $\dim N(A) = 4 - 3 = 1$. The solution set is $\mathbf{x}_p + N(A)$ which has dimension 1, so it is a **line**.

h) Since the matrix is invertible, the row space of the matrix is **all of** $\mathbb{R}^4$.

i) Call the matrix A; we have $m = 3$ and $n = 4$ and $r = 3$ since the rows are linearly independent. That means $\dim N(A) = n - r = 4 - 3 = 1$ so $W = N(A)$ is a **line**.

j) These two subspaces intersect in a **point** since the dimensions of them add to the dimension of $\mathbb{R}^4$.

10. The rotation matrices for angles α, β and $\alpha + \beta$ are, respectively,

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \quad R_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \quad R_{\alpha+\beta} = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

Since rotating by β, then rotating by α is clearly the same as rotating by $\alpha+\beta$, we have the matrix equation

$$R_{\alpha+\beta} = R_\alpha R_\beta.$$

Writing this out and multiplying the matrices on the right-hand side, we get

$$\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & * \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & * \end{pmatrix}$$

where the entries indicated by the $*$s don't matter. Now by equating the upper-left entries of the matrices in this last line, we get the addition identity for cosine:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

By equating the lower-left entries of the matrices in the same line, we get the addition identity for sine:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

1.5 Spring 2014 Final Exam

1. Below, you are given the augmented matrix corresponding to a linear system of equations, together with the reduced row-echelon form of that augmented matrix:

$$(A\,|\,\mathbf{b}) = \left(\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 2 & -3 & 1 & 0 \\ 4 & -7 & 4 & -3 \\ -6 & 13 & -11 & 12 \end{array}\right) \xrightarrow{\text{row ops}} \left(\begin{array}{ccc|c} 1 & 0 & \frac{-5}{2} & \frac{9}{2} \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = \text{rref}(A\,|\,\mathbf{b})$$

   a) Write down the system of linear equations which correspond to the original matrix.
   b) Solve the system $A\mathbf{x} = \mathbf{b}$.
   c) Find a basis for the column space of A.
   d) Is $\mathbf{b}$ in the span of the columns of A? Why or why not?
   e) Find a basis for the null space of A.

2. Suppose that data obtained in an experiment is supposed to fit a model of the form

$$z = a + bx + cx^2 + dy$$

   where a, b, c and d are constants.

   a) Set up a linear system which can be used to solve for a, b, c and d if the data points (of the form (x, y, z) obtained are $(2, 1, 5)$, $(-3, 1, 2)$, $(-2, 0, 2)$, $(1, 4, 3)$ and $(3, 5, 10)$. In particular, what are A, $\mathbf{x}$ and $\mathbf{b}$?
   b) Write down the formula (in terms of A, $\mathbf{x}$ and/or $\mathbf{b}$) which computes the least-squares solution $\widehat{\mathbf{x}}$. (You do not actually have to compute $\widehat{\mathbf{x}}$.)

3. a) If $\mathbf{v} = (2, -1, 5)$ and $\mathbf{w} = (0, 1, -2)$, compute $2\mathbf{v} - \mathbf{w}$.
   b) Compute $(3, -1, 4) \cdot (2, 0, -5)$.
   c) Compute the projection of $(-11, 3)$ onto $(2, 7)$.
   d) Compute the distance between the vectors $(2, -1, 4, 3)$ and $(-4, 0, 7, -1)$.

4. a) Find the transpose of the matrix $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 4 \end{pmatrix}$.
   b) Find AB if $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 5 & 3 \end{pmatrix}$.
   c) Find the inverse of the matrix $\begin{pmatrix} 8 & -5 \\ -4 & 3 \end{pmatrix}$.

d) Find the determinant of the matrix $\begin{pmatrix} 2 & -5 & 1 \\ 0 & 4 & -1 \\ -3 & 3 & 2 \end{pmatrix}$.

5. Let $A = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$.

   a) Compute the exact value of A^{3000} (show all your work).

   b) Compute e^A.

6. a) Find the point of intersection (if there is one) of the two lines in $\mathbb{R}^3$ whose parametric equations are

$$\begin{cases} x = 3t \\ y = 1 + 2t \\ z = -1 - t \end{cases} \qquad \begin{cases} x = -2 + 4t \\ y = 7 - t \\ z = -3 + t \end{cases}$$

   b) Write the normal equation of the plane in $\mathbb{R}^3$ whose parametric equations are

$$\begin{cases} x = -1 + 2s - 3t \\ y = 1 - s + t \\ z = 3 - 3s + 5t \end{cases} .$$

7. Suppose:

   - A is an invertible 3×3 matrix;
   - B is a 3×4 matrix;
   - C is a 4×3 matrix;
   - D is a 1×3 matrix;
   - $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vectors in $\mathbb{R}^3$;
   - $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^4$; and
   - k, l and m are scalars.

   Determine whether the following expressions are a **matrix** (in which you should give its size), a **vector** (in which case you should give the vector space to which the vector belongs), a **scalar**, or **nonsense**:

   a) $tr(A)$

   b) $\mathbf{v} \cdot (\mathbf{v}\mathbf{w})$

   c) $\mathbf{x} \times \mathbf{y}$

   d) $\mathbf{v} \times \mathbf{w}$

   e) $||\mathbf{v} - \mathbf{w}|| \, ||\mathbf{x} + \mathbf{y}||$

   f) $||\mathbf{x}||\mathbf{x}$

   g) $kB^T A^{-1} m$

   h) $(D\mathbf{x})A^2(\mathbf{z} + 3k\mathbf{y})$

   i) $\det(A)B$

   j) $\det(AB)$

   k) $D\mathbf{z}m$

   l) $m\mathbf{z}D$

8. Classify the following statements as true or false:
   a) The vectors $(2, 1, -5)$ and $(3, 3, 2)$ are orthogonal.
   b) $\mathbb{R}^4$ is a four-dimensional subspace of $\mathbb{R}^5$.
   c) The function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (2x, x - y, x + y)$ is a linear transformation.
   d) The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which projects vectors onto the $y-$axis is injective.
   e) The set of vectors lying on the plane $3x - 2y + 4z = 2$ is a subspace of $\mathbb{R}^3$.
   f) The following set is linearly independent: $\{(1, 2, 1), (2, -5, 4), (3, -1, 2), (4, 4, -7)\}$
   g) For square matrices A and B of the same size, $\det(A + B) = \det(A) + \det(B)$.
   h) If $\mathbf{v}, \mathbf{w}$ and $\mathbf{x}$ are in $\mathbb{R}^n$, then $\mathbf{v} \cdot (\mathbf{w} + \mathbf{x}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{x}$.
   i) The vectors $(0, 0, 0, 0)$ and $(2, -3, 4, -1)$ are parallel.
   j) The transformation $T : C(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ defined by $T(f) = f(0)$ is a linear transformation.

9. Fill in the blank with the word “always”, “sometimes” or “never” to make the statement correct:
   a) A 2×3 matrix is _____ diagonalizable.
   b) If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are linear transformations, then the composition $S \circ T$ is _____ linear.
   c) A system of 5 linear equations in 3 variables _____ has exactly one solution.
   d) $|\mathbf{v} \cdot \mathbf{w}|$ is _____ less than or equal to $||\mathbf{v}||\,||\mathbf{w}||$.
   e) The zero vector is _____ part of a basis.
   f) Given a set of three vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   g) Given a set of four vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   h) Given a set of five vectors in $\mathbb{R}^4$, that set _____ spans $\mathbb{R}^4$.
   i) A matrix with eigenvalue 0 is _____ invertible.
   j) A set of one nonzero vector is _____ linearly independent.

10. Answer the following questions:
    a) If W is an eight-dimensional subspace of $\mathbb{R}^{13}$, what is the dimension of $W^\perp$?
    b) If a 10×7 matrix has 5 linearly independent columns, what is the dimension of the null space of this matrix?
    c) If a 5×8 matrix is the matrix of a surjective linear transformation, what is the rank of the matrix?
    d) If the eigenvalues of a matrix are $2, 2, -3$ and 1, what is the trace of the matrix?

e) How many vectors are there in a basis of a three-dimensional subspace of $\mathbb{R}^7$?

Solutions

1. a) If you call the variables x, y and z the system is

$$\begin{cases} y - 2z = 3 \\ 2x - 3y + z = 0 \\ 4x - 7y + 4z = -3 \\ -6x + 13y - 11z = 12 \end{cases}$$

   b) From the rref form, we see

$$\begin{cases} x - \frac{5}{2}z = \frac{9}{2} \\ y - 2z = 3 \end{cases} \Rightarrow \begin{cases} x = \frac{9}{2} + \frac{5}{2}z \\ y = 3 + 2z \end{cases}$$

   Thus the solution set is

$$(x, y, z) = (\frac{9}{2} + \frac{5}{2}z, 3 + 2z, z) = (\frac{9}{2}, 3, 0) + z(\frac{5}{2}, 2, 1) = \left(\frac{9}{2}, 3, 0\right) + Span\left(\frac{5}{2}, 2, 1\right).$$

   c) Such a basis consists of the pivot columns of A; it is $\{(0, 2, 4, -6), (1, -3, -7, 13)\}$.

   d) Yes, because $A\mathbf{x} = \mathbf{b}$ has at least one solution.

   e) From the answer to (b), we see that since the solution is always of the form $\mathbf{x}_p + N(A)$, $N(A) = Span(\frac{5}{2}, 2, 1)$ so a basis for $N(A)$ is the single vector $(\frac{5}{2}, 2, 1)$.

2. a) Plugging each point in for (x, y, z), we obtain the system

$$\begin{cases} a + 2b + 4c + 1 = 5 \\ a - 3b + 9c + d = 2 \\ a - 2b + 4c = 2 \\ a + b + c + 4d = 3 \\ a + 3b + 9c + 5d = 10 \end{cases}$$

   Therefore we are trying to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 4 & 1 \\ 1 & -3 & 9 & 1 \\ 1 & -2 & 4 & 0 \\ 1 & 1 & 1 & 4 \\ 1 & 3 & 9 & 5 \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \\ 3 \\ 10 \end{pmatrix}.$$

   Set up a linear system which can be used to solve for a, b, c and d if the data points (of the form (x, y, z) obtained are $(2, 1, 5)$, $(-3, 1, 2)$, $(-2, 0, 2)$, $(1, 4, 3)$ and $(3, 5, 10)$. In particular, what are A, $\mathbf{x}$ and $\mathbf{b}$?

   b) As always, $\widehat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$.

3. a) $2\mathbf{v} - \mathbf{w} = 2(2,-1,5) - (0,1,-2) = (4,-3,12)$.

   b) $(3,-1,4) \cdot (2,0,-5) = 3(2) - 1(0) + 4(-5) = 6 - 20 = -14$.

   c) $proj_{(2,7)}(-11,3) = \frac{(2,7)\cdot(-11,3)}{(2,7)\cdot(2,7)}(2,7) = \frac{-1}{53}(2,7) = \left(\frac{-2}{53}, \frac{-7}{53}\right)$.

   d) $||(2,-1,4,3)-(-4,0,7,-1)|| = ||(6,-1,-3,4)|| = \sqrt{(6,-1,-3,4)\cdot(6,-1,-3,4)} = \sqrt{36+1+9+16} = \sqrt{62}$.

4. a) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & -2 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -2 \\ 1 & 4 \end{pmatrix}$.

   b) $AB = \begin{pmatrix} 1(3)+(-3)(-2) & 1(0)-3(5) & 1(1)-3(3) \\ 2(3)-2 & 2(0)+5 & 2+3 \end{pmatrix} = \begin{pmatrix} 9 & -15 & -8 \\ 4 & 5 & 5 \end{pmatrix}$.

   c) $\begin{pmatrix} 8 & -5 \\ -4 & 3 \end{pmatrix}^{-1} = \frac{1}{8(3)-(-5)(-4)} \begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 5 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{5}{4} \\ 1 & 2 \end{pmatrix}$.

   d) Repeat the first two columns to the right of the matrix; multiply along the diagonals and then add/subtract to get

$$\det \begin{pmatrix} 2 & -5 & 1 \\ 0 & 4 & -1 \\ -3 & 3 & 2 \end{pmatrix} = 16 - 15 + 0 - (-12) - (-6) - 0 = 19.$$

5. a) First, find the eigenvalues of A. $\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & 4 \\ 1 & 6-\lambda \end{pmatrix} = (3-\lambda)(6-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-7)(\lambda-2)$ so the eigenvalues are $\lambda = 2$ and $\lambda = 7$.

   Next, eigenvectors: when $\lambda = 2$, $A\mathbf{x} = \lambda\mathbf{x}$ gives $3x + 4y = 2x$ and $x + 6y = 2y$, i.e. $x = -4y$, so an eigenvector is $(-4,1)$. When $\lambda = 7$, $A\mathbf{x} = \lambda\mathbf{x}$ gives $3x+4y = 7x$ and $x+6y = 7y$, i.e. $x = y$, so an eigenvector is $(1,1)$. Therefore

$$A = S\Lambda S^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

   Now

$$\begin{aligned} A^{3000} = S\Lambda^{3000}S^{-1} &= \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{3000} & 0 \\ 0 & 7^{3000} \end{pmatrix} \frac{1}{-5} \begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix} \\ &= \frac{-1}{5} \begin{pmatrix} -4\cdot 2^{3000} - 7^{3000} & 4\cdot 2^{3000} - 4\cdot 7^{3000} \\ 2^{3000} - 7^{3000} & -2^{3000} - 4\cdot 7^{3000} \end{pmatrix}. \end{aligned}$$

b) Using much of the work from part (a),

$$e^A = Se^\Lambda S^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^7 \end{pmatrix} \frac{1}{-5} \begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix}$$
$$= \frac{-1}{5} \begin{pmatrix} -4 \cdot e^2 - e^7 & 4e^2 - 4e^7 \\ e^2 - e^7 & -e^2 - 4e^7 \end{pmatrix}.$$

6. a) We need to change the t in one equation to s, then solve for the intersection point. We get

$$\begin{cases} 3t = -2 + 4s \\ 1 + 2t = 7 - s \\ -1 - t = -3 + s \end{cases}$$

Solving the second equation for s, we get $s = 6 - 2t$. Plugging this into the third equation, we get $-1 - t = -3 + 6 - 2t$, i.e. $t = 4$ (so by back-substitution in the equation $x = 6 - 2t$, $s = -2$). These values of s and t work in the last two equations, but not in the first one, so these lines do not intersect.

b) Two direction vectors for the plane are $(2, -1, -3)$ and $(-3, 1, 5)$. To obtain a normal vector, take the cross product: $\mathbf{n} = (2, -1, -3) \times (-3, 1, 5) = (-2, -1, -1)$. One point on the plane is $(-1, 1, 3)$; set $d = \mathbf{n} \cdot (-1, 1, 3) = -2$. So the equation of the plane is $\mathbf{n} \cdot \mathbf{x} = d$, i.e. $-2x - 1y - z = -2$. (Any multiple of this equation is also a valid solution.)

7. a) $tr(A)$ is the sum of the diagonal entries, which is a **scalar**.

b) $\mathbf{vw}$ is nonsense, so the whole thing is **nonsense**.

c) $\mathbf{x} \times \mathbf{y}$ is a **vector** in $\mathbb{R}^3$.

d) $\mathbf{v} \times \mathbf{w}$ is **nonsense** (there is no cross product of vectors in $\mathbb{R}^4$).

e) $||\mathbf{v} - \mathbf{w}||\, ||\mathbf{x} + \mathbf{y}||$ is the product of two scalars, hence a **scalar**.

f) $||\mathbf{x}||\mathbf{x}$ is a scalar times a vector which is a **vector** in $\mathbb{R}^3$.

g) $k(B^T)_{4\times3}(A^{-1})_{3\times3}m$ is a 4×3 **matrix**.

h) First, $\mathbf{z} + 3k\mathbf{y}$ is a vector in $\mathbb{R}^3$. Next, $D\mathbf{x} = D_{1\times3}\mathbf{x}_{3\times1}$ is a 1×1 matrix, hence a scalar. Then $(D\mathbf{x})_{scalar}(A^2)_{3\times3}(\mathbf{z} + 3k\mathbf{y})_{3\times1}$ is a 3×1 matrix, i.e. a **vector** in $\mathbb{R}^3$.

i) $\det(A)B$ is a scalar times a matrix which is a 3×4 **matrix**.

j) $\det(AB)$ is **nonsense** since nonsquare matrices do not have determinants.

k) $D_{1\times3}\mathbf{z}_{3\times1}m$ is a 1×1 matrix, i.e. a **scalar**.

l) $m\mathbf{z}_{3\times1}D_{1\times3}$ is a 3×3 **matrix**.

8. a) $(2,1,-5)\cdot(3,3,2) = 6+3-10 = -1 \neq 0$ so this is FALSE.
   b) $\mathbb{R}^4$ consist of vectors with four components, but subspaces of $\mathbb{R}^5$ are sets of vectors with five components. Therefore this is FALSE.
   c) You can check $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(r\mathbf{x}) = rT(\mathbf{x})$ so this T is linear; the answer is TRUE.
   d) $T(1,0) = 0$ so $\ker(T) \neq \{\mathbf{0}\}$ so this transformation is not injective, so this statement is FALSE.
   e) $\mathbf{0}$ is not in this set, so this statement is FALSE.
   f) These are four vectors in the three-dimensional space $\mathbb{R}^3$ (i.e. too many to be lin. indep.), so this statement is FALSE.
   g) If you try virtually any examples of matrices, you will see that this is FALSE.
   h) This is a property of dot products; it is TRUE.
   i) $\mathbf{0}$ is parallel to every vector, so this is TRUE.
   j) Evaluation of a function is linear, so this is TRUE.

9. a) A 2×3 matrix is NEVER diagonalizable (it isn't square).
   b) If $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are linear transformations, then the composition $S \circ T$ is ALWAYS linear (theorem from class).
   c) A system of 5 linear equations in 3 variables SOMETIMES has exactly one solution (because the null space could have dimension 0 or dimension greater than 0).
   d) $|\mathbf{v}\cdot\mathbf{w}|$ is ALWAYS less than or equal to $||\mathbf{v}||\,||\mathbf{w}||$ (this is the Cauchy-Schwarz Inequality).
   e) The zero vector is NEVER part of a basis (it is never part of a lin. indep. set).
   f) Given a set of three vectors in $\mathbb{R}^4$, that set NEVER spans $\mathbb{R}^4$ (there aren't enough vectors to span).
   g) Given a set of four vectors in $\mathbb{R}^4$, that set SOMETIMES spans $\mathbb{R}^4$ (depending on what those vectors are).
   h) Given a set of five vectors in $\mathbb{R}^4$, that set SOMETIMES spans $\mathbb{R}^4$ (it depends on what those vectors are).
   i) A matrix with eigenvalue 0 is NEVER invertible (because its determinant is the product of the eigenvalues which must be 0).
   j) A set of one nonzero vector is ALWAYS linearly independent (fact from class).

10. a) $\dim W^\perp = \dim \mathbb{R}^{13} - \dim W = 13 - 8 = 5$.

b) We have $m = 10$, $n = 7$ and $r = 5$. So $\dim N(A) = n - r = 2$.
c) If the transformation is surjective, we have $r = m = 5$.
d) The trace is the sum of the eigenvalues: $2 + 2 - 3 + 1 = 2$.
e) Since the dimension is 3, there are 3 vectors in any basis of that subspace.