Old MATH 330 Exams

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Last updated to include exams from Fall 2023

Contents

Chapter 1

General information about these exams

These are the exams I have given in differential equations courses. Each exam is given here, followed by what I believe are the solutions (there may be some number of computational errors or typos in these answers).

Typically speaking (from Fall 2017 onward), tests labelled "Exam 1" cover Chapters 1 and 2 in my differential equations lecture notes; tests labelled "Exam 2" cover Chapters 3 and 4.

Each problem on these exams is marked with a section in parenthesis like, for example, "(3.5)"; this section refers to the section in my Fall 2023 version of my MATH 330 lecture notes to which this question best corresponds.

Chapter 2

Exams from Fall 2016

2.1 Fall 2016 Exam 1

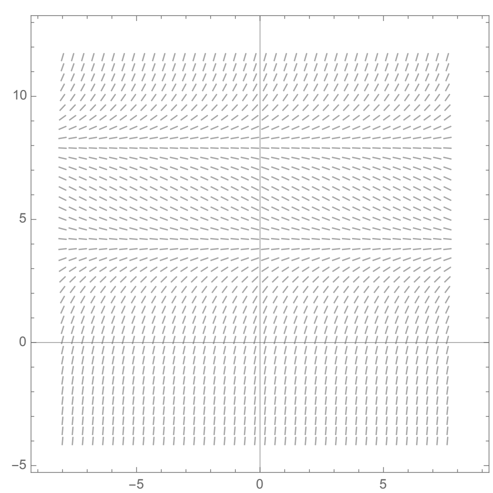
1. (1.6) Briefly explain what is meant by "existence / uniqueness" in the context of ordinary differential equations.

2. (1.5) Let $y = y(t)$ be the solution of the initial value problem

$$\begin{cases} y' = y + 2t \\ y(1) = -2 \end{cases}.$$

   Suppose you wanted to estimate $y(31)$ using Euler's method with 10 steps. Find the points (t_1, y_1) and (t_2, y_2) obtained by this method.

3. (1.4) Here is the picture of the slope field associated to an autonomous ODE $y' = \phi(y)$:



a) Suppose $y(0) = 7$. Estimate $y(2)$.

b) Suppose $y(-1) = 2$. Find $\lim_{t \to \infty} y(t)$.

c) On the picture above, sketch the graph of the solution satisfying the initial condition $y(0) = 6$. Label the graph "(d)".

d) Find all equilibria of this equation, and classify them as stable, semistable or unstable.

4. (2.2 or 2.3) Find the general solution of the following ODE:

$$\frac{dy}{dt} - 2y = 14e^{4t}$$

5. (2.4) Find the particular solution of the following initial value problem:

$$\begin{cases} t^2 y' = y^2 \\ y(1) = 2 \end{cases}$$

Write your answer as a function $y = f(t)$.

6. (no longer in MATH 330 as of Fall 2023) Find the particular solution of the following initial value problem:

$$\begin{cases} 2ty\frac{dy}{dt} = t^2 - y^2 \\ y(3) = 1 \end{cases}$$

7. (2.4) Find the general solution of the following ODE:

$$ty'' = y'$$

Write your answer as a function $y = f(t)$.

Solutions

1. The Existence/Uniqueness Theorem for first-order ODEs says if the function ϕ is "nice" (i.e. ϕ and $\frac{\partial \phi}{\partial y}$ are continuous), then the initial value problem

$$\begin{cases} y' = \phi(t, y) \\ y(t_0) = y_0 \end{cases}$$

has one and only one solution, which is of the form $y = f(t)$.

2. First, $\Delta t = \frac{1}{n}(t_n - t_0) = \frac{1}{10}(31 - 1) = 3$. Next, we are given $(t_0, y_0) = (1, -2)$. Now $\phi(t_0, y_0) = -2 + 2(1) = 0$ so

$$\begin{aligned} t_1 &= t_0 + \Delta t = 1 + 3 = 4 \\ y_1 &= y_0 + \phi(t_0, y_0)\Delta t = -2 + 0(3) = -2. \end{aligned}$$

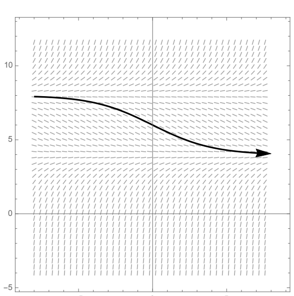
Therefore $(t_1, y_1) = (4, -2)$. Now $\phi(t_1, y_1) = -2 + 2(4) = 6$ so

$$\begin{aligned} t_2 &= t_1 + \Delta t = 4 + 3 = 7 \\ y_2 &= y_1 + \phi(t_1, y_1)\Delta t = -2 + 6(3) = 16. \end{aligned}$$

Therefore $(t_2, y_2) = (7, 16)$.

3. a) Starting at the point $(0, 7)$ and following the vector field, we come to the point $(2, 6)$ so $y(2) \approx 6$.

   b) Starting at the point $(-1, 2)$ and moving to the extreme right, we see that $\lim_{t\to\infty} y(t) = 4$.

   c)

   d) $y = 4$ is stable; $y = 8$ is unstable.

4. *Method 1 (integrating factors):* The integrating factor is $\mu(t) = \exp\left[\int_0^t (-2)\, ds\right] =$

e^{-2t}. After multiplying through by $\mu(t)$, the equation becomes

$$\begin{aligned}
\frac{dy}{dt}e^{-2t} - 2e^{-2t}y &= 14e^{4t}(e^{-2t}) \\
\frac{d}{dt}\left(ye^{-2t}\right) &= 14e^{2t} \\
ye^{-2t} &= \int 14e^{2t}\,dt \\
ye^{-2t} &= 7e^{2t} + C \\
y &= e^{2t}\left(7e^{2t} + C\right) \\
y &= 7e^{4t} + Ce^{2t}.
\end{aligned}$$

Method 2 (undetermined coefficients): The corresponding homogeneous equation is $\frac{dy}{dt} - 2y = 0$ which has solution $y_h = e^{2t}$ (exponential growth model).

Now, guess $y_p = Ae^{4t}$ and plug into the left-hand side of the equation to get $4Ae^{4t} - 2Ae^{4t} = 14e^{4t}$. That means $4A - 2A = 14$, i.e. $A = 7$. Therefore $y_p = 7e^{4t}$ so $y = y_p + Cy_h$, i.e.

$$y = 7e^{4t} + Ce^{2t}.$$

5. Start with the ODE, which is separable:

$$\begin{aligned}
t^2\frac{dy}{dt} &= y^2 \\
y^{-2}\,dy &= t^{-2}\,dt \\
\int y^{-2}\,dy &= \int t^{-2}\,dt \\
-\frac{1}{y} &= -\frac{1}{t} + C
\end{aligned}$$

Next, I will solve for C using the initial condition (you could have solved for y first):

$$-\frac{1}{2} = -\frac{1}{1} + C \quad \Rightarrow \quad C = \frac{1}{2}$$

Therefore the particular solution is

$$-\frac{1}{y} = -\frac{1}{t} + \frac{1}{2}.$$

Solve for y by first multiplying through by -1 and then taking reciprocals to get

$$y = \frac{1}{\frac{1}{t} - \frac{1}{2}} \quad \text{(this answer is fine)}$$

which, if you multiply through the numerator and denominator by $2t$ simplifies to

$$y = \frac{2t}{2-t}.$$

6. Rewrite this equation as $(y^2 - t^2) + 2ty\frac{dy}{dt}$. Letting $M = y^2 - t^2$ and $N = 2ty$, we see that

$$M_y = 2y = N_t$$

so the equation is exact. Now

$$\begin{aligned}\psi(t,y) &= \int M\,dt = \int (y^2 - t^2)\,dt = y^2 t - \frac{1}{3}t^3 + A(y)\\ &= \int N\,dy = \int 2ty\,dy = ty^2 + B(t).\end{aligned}$$

By setting $B(t) = -\frac{1}{3}t^3$ and $A(y) = 0$, we reconcile these integrals to obtain $\psi(t,y) = y^2 t - \frac{1}{3}t^3$. Thus the general solution is $\psi(t,y) = C$, i.e. $y^2 t - \frac{1}{3}t^3 = C$. Plugging in the initial condition and solving for C, we see $1^2(3) - \frac{1}{3}(3^3) = C$, i.e. $C = 3 - 9 = -6$. So the particular solution is

$$y^2 t - \frac{1}{3}t^3 = -6.$$

7. This equation is second-order with no y. To solve it, let $v = \frac{dy}{dt}$ so that the equation becomes

$$tv' = v \quad \text{i.e.} \quad t\frac{dv}{dt} = v.$$

This is separable: rewrite it as $\frac{1}{v}\,dv = \frac{1}{t}\,dt$ and integrate both sides to obtain $\ln v = \ln t + C$. Solving for v, we get

$$v = e^{\ln t + C} = e^{\ln t}e^C = te^C = Ct.$$

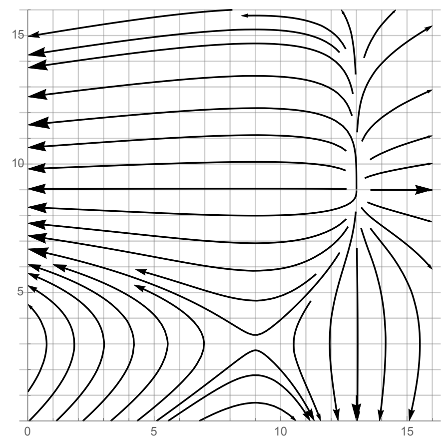
Last, since $v = \frac{dy}{dt}$, integrate to obtain y:

$$y = \int v(t)\,dt = \int Ct\,dt = \frac{1}{2}Ct^2 + D.$$

Renaming the first constant, this can be written as $y = Ct^2 + D$.

2.2 Fall 2016 Exam 2

1. Here is a picture of the phase plane of an autonomous, first-order 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$, where as usual, $\mathbf{y} = (x, y)$:



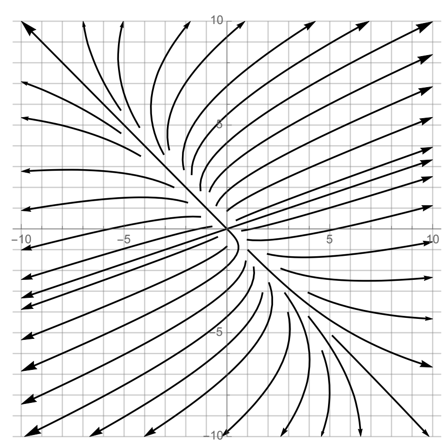
a) (4.7) Find the two equilibria of this system and classify each of them as a center, node, spiral or saddle.

b) (3.7) Suppose $x(0) = 10$ and $y(0) = 13$. Find the following four limits:

$$\lim_{t\to\infty} x(t) = \qquad \lim_{t\to\infty} y(t) = \qquad \lim_{t\to-\infty} x(t) = \qquad \lim_{t\to-\infty} y(t) =$$

c) (3.7) Suppose $\mathbf{y}(0) = (3, 0)$. Estimate the maximum value obtained by $x(t)$.

2. Here is the phase plane of a first-order, constant-coefficient, homogeneous linear 2×2 system of ODEs $\mathbf{y}' = A\mathbf{y}$:



a) (4.1) Find two eigenvectors of A (corresponding to distinct eigenvalues).

b) (4.7) How many positive, real eigenvalues does A have?

c) (4.7) How many negative, real eigenvalues does A have?

d) (4.7) How many non-real eigenvalues does A have?

3. (4.6) Find the general solution of the following system of ODEs:

$$\begin{cases} x' = & 2x & + & 4y & + & 2e^{2t} \\ y' = & x & - & y & + & e^{2t} \end{cases}$$

4. (4.5) Find the general solution of the following system of ODEs:

$$\begin{cases} x' = -3x + y \\ y' = -x - 5y \end{cases}$$

Write your final answer coordinate-wise.

5. (4.5) Find the particular solution of the following initial value problem:

$$\begin{cases} \mathbf{y}' = (x - 4y, 2x + 5y) \\ \mathbf{y}(0) = (3, -1) \end{cases}$$

Solutions

1. a) From looking at the picture, the two equilibria are $(9,3)$ (which is a **saddle**) and $(13,9)$ (which is a (unstable) **node**).

   b) By following the curve passing through $(10,13)$ forwards and backwards, we see that

$$\lim_{t\to\infty} x(t) = -\infty \qquad \lim_{t\to\infty} y(t) = 9 \qquad \lim_{t\to-\infty} x(t) = 13 \qquad \lim_{t\to-\infty} y(t) = 9.$$

   c) The maximum value obtained by $x(t)$ is the right-most point on the curve passing through $(3,0)$, which is approximately 5.

2. a) The eigenvectors of A go in the direction of the straight-line solutions; from the picture these are $(-1,1)$ (or any multiple of $(-1,1)$) and $(3,1)$ (or any multiple of $(3,1)$).

   b) Since **0** is an unstable node (from the picture), both eigenvalues of A are real and positive, so the answer is **two**.

   c) **None** (since the eigenvalues of A are both positive).

   d) **None** (since the eigenvalues of A are both positive and real).

3. Let $A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$. We start by finding the solution of the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. First, the eigenvalues of A:

$$\begin{aligned}\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{pmatrix} &= (2-\lambda)(-1-\lambda) - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda-3)(\lambda+2)\end{aligned}$$

so the eigenvalues are $\lambda = 3$ and $\lambda = -2$. Now for the eigenvectors (let $\mathbf{v} = (x,y)$):

$$\lambda = 3 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 2x+4y = 3x \\ x-y = 3y \end{cases} \Rightarrow x = 4y \Rightarrow (4,1)$$

$$\lambda = -2 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 2x+4y = -2x \\ x-y = -2y \end{cases} \Rightarrow x = -y \Rightarrow (1,-1)$$

Therefore the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, we find a particular solution $\mathbf{y}_p$ using undetermined coefficients. Since $\mathbf{q} = (2e^{2t}, e^{2t}$, we guess $\mathbf{y}_p = (Ae^{2t}, Be^{2t}$. Plugging this into the system, we get

$$\begin{cases} 2Ae^{2t} & = 2(Ae^{2t}) + 4(Be^{2t}) + 2e^{2t} \\ 2Be^{2t} & = Ae^{2t} - Be^{2t} + e^{2t} \end{cases}$$

Dividing through by e^{2t}, we get

$$\begin{cases} 2A & = 2A + 4B + 2 \\ 2B & = A - B + 1 \end{cases}$$

In the first equation, the As cancel, so we can solve for B to get $B = \frac{-1}{2}$. From the second equation, we have $A = 3B - 1 = \frac{-5}{2}$ so

$$\mathbf{y}_p = \begin{pmatrix} Ae^{2t} \\ Be^{2t} \end{pmatrix} = \begin{pmatrix} \frac{-5}{2}e^{2t} \\ \frac{-1}{2}e^{2t} \end{pmatrix}.$$

Last, the solution is

$$\begin{aligned} \mathbf{y} &= \mathbf{y}_p + \mathbf{y}_h \\ &= \begin{pmatrix} \frac{-5}{2}e^{2t} \\ \frac{-1}{2}e^{2t} \end{pmatrix} + C_1e^{3t}\begin{pmatrix} 4 \\ 1 \end{pmatrix} + C_2e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-5}{2}e^{2t} + 4C_1e^{3t} + C_2e^{-2t} \\ \frac{-1}{2}e^{2t} + C_1e^{3t} - C_2e^{-2t} \end{pmatrix}. \end{aligned}$$

4. Let $A = \begin{pmatrix} -3 & 1 \\ -1 & -5 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -5-\lambda \end{pmatrix} &= (-3-\lambda)(-5-\lambda) + 1 \\ &= \lambda^2 + 8\lambda + 16 = (\lambda+4)^2 \end{aligned}$$

so the only eigenvalue is $\lambda = -4$ (repeated twice). Now for the eigenvector(s); let $\mathbf{v} = (x, y)$:

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -3x + y = -4x \\ -3x + y = -4y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1)$$

We will also need a generalized eigenvector $\mathbf{w}$, which satisfies $(A-\lambda I)\mathbf{w} = \mathbf{v}$:

$$
\begin{aligned}
(A - \lambda I)\mathbf{w} &= \mathbf{v} \\
\Rightarrow \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\
\Rightarrow \begin{cases} x + y = & 1 \\ -x - y = & -1 \end{cases}
\end{aligned}
$$

Any $\mathbf{w} = (x, y)$ satisfying $x + y = 1$ works in both these equations; let's use $x = 1$, $y = 0$ so that $\mathbf{w} = (1, 0)$.

Now, applying the formula from Theorem 2.68 from the lecture notes (which should be on your index card), we see that the solution has the form

$$
\begin{aligned}
\mathbf{y} &= C_1 e^{\lambda t}\mathbf{v} + C_2 \left[e^{\lambda t}\mathbf{w} + t e^{\lambda t}\mathbf{v} \right] \\
&= C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left[e^{4t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \\
&= \begin{pmatrix} C_1 e^{-4t} + C_2 e^{-4t} + C_2 t e^{-4t} \\ -C_1 e^{-4t} - C_2 t e^{-4t} \end{pmatrix} \\
&= \begin{pmatrix} (C_1 + C_2) e^{-4t} + C_2 t e^{-4t} \\ -C_1 e^{-4t} - C_2 t e^{-4t} \end{pmatrix}.
\end{aligned}
$$

Writing this coordinate-wise as requested, we have the solution

$$
\begin{cases} x(t) = (C_1 + C_2) e^{-4t} + C_2 t e^{-4t} \\ y(t) = -C_1 e^{-4t} - C_2 t e^{-4t} \end{cases}
$$

5. Let $A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$. We start by finding the solution of the homogeneous equation $\mathbf{y}' = A\mathbf{y}$. First, the eigenvalues of A:

$$
\begin{aligned}
\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -4 \\ 2 & 5-\lambda \end{pmatrix} &= (1-\lambda)(5-\lambda) + 8 \\
&= \lambda^2 - 6\lambda + 13.
\end{aligned}
$$

Setting this equal to zero and solving with the quadratic formula, we get

$$
\lambda = \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2 \cdot 1} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i
$$

so the eigenvalues are $\lambda = 3 + 2i$ and $\overline{\lambda} = 3 - 2i$. Now for the eigenvector corresponding to one of the eigenvalues (let $\mathbf{v} = (x, y)$):

$$\begin{aligned}\lambda = 3 + 2i :\ A\mathbf{v} = \lambda\mathbf{v} &\Rightarrow \begin{cases} x - 4y = (3+2i)x \\ 2x + 5y = (3+2i)y \end{cases} \\ &\Rightarrow -4y = (2+2i)x \\ &\Rightarrow -2y = (1+i)x \\ &\Rightarrow \mathbf{v} = (-2, 1+i) = (-2, 1) + i(0, 1)\end{aligned}$$

We have $\alpha = 3$, $\beta = 2$, $\mathbf{a} = (-2, 1)$ and $\mathbf{b} = (0, 1)$. So applying the formula from Theorem 2.67 of the lecture notes (which should be on your index card), we obtain the solution

$$\begin{aligned}\mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{3t}\cos 2t\begin{pmatrix} -2 \\ 1 \end{pmatrix} - e^{3t}\sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + C_2\left[e^{3t}\cos 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t}\sin 2t\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} -2C_1e^{3t}\cos 2t - 2C_2e^{3t}\sin 2t \\ (C_1 + C_2)e^{3t}\cos 2t + (C_2 - C_1)e^{3t}\sin 2t \end{pmatrix}.\end{aligned}$$

Now we plug in the initial condition $\mathbf{y}(0) = (3, -1)$ to find C_1 and C_2: plugging in, we see

$$\begin{aligned}&\begin{cases} 3 = -2C_1e^0\cos 0 - 2C_2e^0\sin 0 \\ -1 = (C_1 + C_2)e^0\cos 0 + (C_2 - C_1)e^0\sin 0 \end{cases} \\ &\qquad\Rightarrow \begin{cases} 3 = -2C_1 \\ -1 = C_1 + C_2 \end{cases} \\ &\qquad\Rightarrow C_1 = \frac{-3}{2}, C_2 = \frac{1}{2}.\end{aligned}$$

Therefore the particular solution is

$$\begin{aligned}\mathbf{y} &= \begin{pmatrix} -2\left(\frac{-3}{2}\right)e^{3t}\cos 2t - 2\left(\frac{1}{2}\right)e^{3t}\sin 2t \\ (\frac{-3}{2} + \frac{1}{2})e^{3t}\cos 2t + (\frac{1}{2} - \frac{-3}{2})e^{3t}\sin 2t \end{pmatrix} \\ &= \begin{pmatrix} 3e^{3t}\cos 2t - e^{3t}\sin 2t \\ -e^{3t}\cos 2t + 2e^{3t}\sin 2t \end{pmatrix}.\end{aligned}$$

2.3 Fall 2016 Exam 3

1. (5.1) Suppose you are given a fourth-order, linear ODE. What is meant by an "initial value" of this ODE?

2. (5.1) Convert this third-order ODE to a first-order system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$.

$$e^{2t}y''' - 6ty'' + 4y' - y = 7\sin 3t$$

3. (5.2) Find the particular solution of this initial value problem:

$$\begin{cases} y'' + 9y' + 18y = 0. \\ y(0) = -1 \\ y'(0) = 7 \end{cases}$$

4. (5.2) Find the general solution of this ODE:

$$y^{(4)} - 14y^{(3)} + 49y'' = 0$$

5. (5.2) Find the general solution of this ODE:

$$y'' - 4y' - 12y = 48e^{6t}$$

6. (5.2) Find the general solution of this ODE:

$$y'' - 10y' + 34y = 0$$

7. (5.3) A 4 kg mass is attached to a fixed point by a spring whose spring constant is 40 N/m. The mass moves back and forth along a line, subject to friction where the damping coefficient is 24 N sec/m. Suppose also that initiallly, the mass is 3 m to the right of its equilibrium position, and moving to the right at 2 m/sec.

    a) Suppose the mass is not subject to any external force. Find the position of the mass at time t.

    b) Suppose that the mass is subject to an external force of $20\sin 2t$ newtons. Find the position of the mass at time t.

Solutions

1. An "initial value" of a fourth-order ODE consists of values of y, y', y'' and y''' at the same value of t. In other words,
$$\mathbf{y}_0 = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ y''(t_0) \\ y'''(t_0) \end{pmatrix}.$$

2. First, solve for y''' to get $y''' = e^{-2t}y - 4e^{-2t}y' + 6te^{-2t}y'' + 7e^{-2t}\sin 3t$. Then let $\mathbf{y} = (y, y', y'')$; then
$$\mathbf{y}' = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0y + 1y' + 0y'' \\ 0y + 0y' + 1y'' \\ e^{-2t}y - 4e^{-2t}y' + 6te^{-2t}y'' + 7\sin 3t \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^{-2t} & -4e^{-2t} & 6te^{-2t} \end{pmatrix}\mathbf{y} + \begin{pmatrix} 0 \\ 0 \\ 7e^{-2t}\sin 3t \end{pmatrix}.$$
So by setting
$$\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e^{-2t} & -4e^{-2t} & 6te^{-2t} \end{pmatrix} \text{ and } \mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ 7e^{-2t}\sin 3t \end{pmatrix},$$
the system becomes $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$, as desired.

3. The characteristic equation is $\lambda^2 + 9\lambda + 18 = (\lambda + 6)(\lambda + 3)$, which has roots -6 and -3, so the general solution is $y = C_1e^{-6t} + C_2e^{-3t}$. Differentiating, we get $y' = -6C_1e^{-6t} - 3C_2e^{-3t}$, so by plugging in the given initial conditions we get
$$\begin{cases} y(0) = -1 \\ y'(0) = 7 \end{cases} \Rightarrow \begin{cases} -1 = C_1 + C_2 \\ 7 = -6C_1 - 3C_2 \end{cases} \Rightarrow C_1 = \frac{-4}{3}, C_2 = \frac{1}{3}$$
Therefore the particular solution is $y = \frac{-4}{3}e^{-6t} + \frac{1}{3}e^{-3t}$.

4. The characteristic equation is $l^4 - 14\lambda^3 + 49\lambda^2 = \lambda^2(\lambda - 7)^2$, which has roots 0 and 7 (both repeated twice) so the general solution is $y = C_1 + C_2t + C_3e^{7t} + C_4te^{7t}$.

5. The characteristic equation is $\lambda^2 - 4\lambda - 12 = (\lambda - 6)(\lambda + 2)$, which has roots 6 and -2, so the solution of the homogeneous is $y_h = C_1e^{6t} + C_2e^{-2t}$.

   Now for a particular solution y_p of the non-homogeneous equation. Since $q = 48e^{6t}$, we'd ordinarily guess $y_p = Ae^{6t}$, but since e^{6t} is part of the solution of the homogeneous, we need to multiply by t and guess $y_p = Ate^{6t}$. Thus by the Product Rule, $y_p' = Ae^{6t} + 6Ate^{6t}$ and $y_p'' = 12Ae^{6t} + 36Ate^{6t}$. Plugging in the original equation, we get

$$\begin{aligned} y_p'' - 4y_p' - 12y_p &= 48e^{6t} \\ 12Ae^{6t} + 36Ate^{6t} - 4(Ae^{6t} + 6Ate^{6t}) - 12Ate^{6t} &= 48e^{6t} \\ 8A &= 48 \\ A &= 6 \end{aligned}$$

Therefore $y_p = 6te^{6t}$, so the general solution is $y = y_p + y_h$, i.e.

$$y = 6te^{6t} + C_1e^{6t} + C_2e^{-2t}.$$

6. The characteristic equation is $\lambda^2 - 10\lambda + 34 = 0$ which has solutions

$$\lambda = \frac{10 \pm \sqrt{100 - 4(34)}}{2} = \frac{10 \pm \sqrt{-36}}{2} = \frac{10 \pm 6i}{2} = 5 \pm 3i.$$

Therefore the general solution is $y = C_1e^{5t}\cos 3t + C_2e^{5t}\sin 3t$.

7. Throughout this problem, let $x = x(t)$ be the position of the mass at time t. From the oscillator equation, we obtain the second-order ODE

$$\begin{aligned} mx'' + bx' + kx &= F_{ext}(t) \\ 4x'' + 24x' + 40x &= F_{ext}(t) \end{aligned}$$

Also, throughout the problem, we have the initial value $x(0) = 3, x'(0) = 2$.

a) In this part, assume $F_{ext}(t) = 0$. Then the characteristic equation is $4\lambda^2 + 24\lambda + 40\lambda = 4(\lambda^2 + 6\lambda + 10)$ which has solutions

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4(10)}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i.$$

Therefore the general solution of this ODE is $x = C_1e^{-3t}\cos t + C_2e^{-3t}\sin t$. Since $x(0) = 3$, we know $C_1 = 3$. Differentiating, we get

$$x' = -3C_1e^{3t}\cos t - C_1e^{3t}\sin t - 3C_2e^{-3t}\sin t + C_2e^{-3t}\cos t$$

and since $x'(0) = 2$, we get $-3C_1 + C_2 = 2$, i.e. $C_2 = 11$. Therefore the particular solution of this ODE is

$$x(t) = 3e^{-3t}\cos t + 11e^{-3t}\sin t.$$

b) In this part, assume $F_{ext}(t) = 20\sin 2t$. From part (a), the solution of the homogeneous is

$$x_h = C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t.$$

Now we need to find the x_p. Since $q = 20\sin 2t$, guess $x_p = A\sin 2t + B\cos 2t$. Differentiating, we get $x_p' = 2A\cos 2t - 2B\sin 2t$ and $x_p'' = -4A\sin 2t - 4B\cos 2t$, and by plugging in to the original equation we get

$$\begin{aligned}4x_p'' + 24x_p' + 40x_p &= 20\sin 2t\\ 4(-4A\sin 2t - 4B\cos 2t) + 24(2A\cos 2t - 2B\sin 2t) + 40(A\sin 2t + B\cos 2t) &= 20\sin 2t\\ (24A - 48B)\sin 2t + (24B + 48A)\cos 2t &= 20\sin 2t\end{aligned}$$

Therefore we get the system of equations

$$\begin{cases} 24A - 48B = 20 \\ 24B + 48A = 0 \end{cases} \Rightarrow B = -2A \Rightarrow A = \frac{1}{6}, B = \frac{-1}{3}$$

so the particular solution is $x_p = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$. That makes the solution of the ODE $x = x_p + x_h$, i.e.

$$x(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t + C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t.$$

Since $x(0) = 3$, we have $C_2 - \frac{1}{3} = 3$, i.e. $C_2 = \frac{10}{3}$. Differentiating, we obtain

$$x'(t) = \frac{1}{3}\cos 2t + \frac{2}{3}\sin 2t - 3C_1 e^{3t}\cos t - C_1 e^{3t}\sin t - 3C_2 e^{-3t}\sin t + C_2 e^{-3t}\cos t$$

and since $x'(0) = 2$, we get $\frac{1}{3} - 3C_1 + C_2 = 2$, i.e. $C_2 = \frac{35}{3}$. Therefore the particular solution is

$$x(t) = \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t + \frac{10}{3}e^{-3t}\cos t + \frac{35}{3}e^{-3t}\sin t.$$

2.4 Fall 2016 Final Exam

1. a) (1.1) Explain the difference between the terms "general solution" and "particular solution", in the context of ODEs.

   b) (4.3) What is Euler's formula? Why is this formula important, in the context of ODEs?

   c) (3.6) An evil professor tells a student to solve a 3×3 system of second-order, linear, homogeneous ODEs by hand. After hours of work, the student produces the following answer:

$$y = C_1 e^{t\sqrt{3}} + C_2 e^{-t\sqrt{3}} + C_3 e^{7t} + C_4 e^{-t}\cos(t\sqrt{2}) + C_5 e^{-t}\sin(t\sqrt{2}).$$

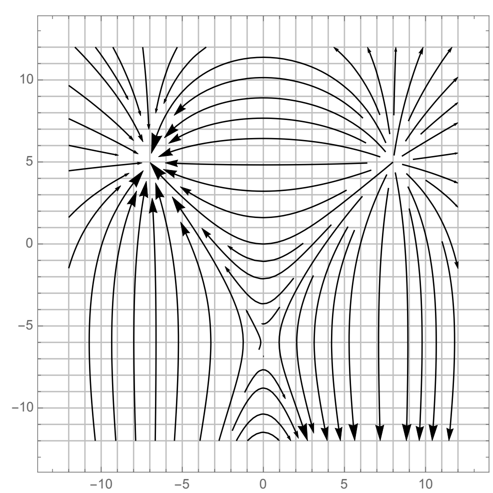
   Despite not having done the problem himself, the professor knows this answer is wrong, just by looking at it. Why?

2. (3.2) Consider the initial value problem

$$\begin{cases} \mathbf{y}' = (y + 2t, x - y) \\ \mathbf{y}(0) = (1, 2) \end{cases}.$$

   Suppose you wanted to estimate $\mathbf{y}(100)$ using Euler's method with 20 steps. Compute the first two points (other than the given initial condition) obtained by this method.

3. Here is a picture of the phase plane of a 2×2 first-order system $\mathbf{y}' = \Phi(\mathbf{y})$:



   a) (4.7) How many stable equilibria does this system have?

   b) (4.7) How many unstable equilibria does this system have?

   c) (3.7) Give the equation of any constant solution of the system.

d) (3.7) Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution to this system satisfying $\mathbf{y}(0) = (4, -1)$.

i. Which statement best describes the behavior of the function $x(t)$?
 A. $x(t)$ is increasing for all t.
 B. $x(t)$ is decreasing for all t.
 C. $x(t)$ is increasing for small t, but decreasing for large t.
 D. $x(t)$ is decreasing for small t, but increasing for large t.

ii. (3.7) Which statement best describes the behavior of the function $y(t)$?
 A. $y(t)$ is increasing for all t.
 B. $y(t)$ is decreasing for all t.
 C. $y(t)$ is increasing for small t, but decreasing for large t.
 D. $y(t)$ is decreasing for small t, but increasing for large t.

iii. (3.7) Find $\lim_{t\to\infty} x(t)$.

iv. (3.7) Find $\lim_{t\to-\infty} y(t)$.

4. (4.7) Find and classify all equilibria of the system

$$\begin{cases} x' = y - 2x \\ y' = y^2 - 2y - 80 \end{cases}.$$

5. (no longer in MATH 330 as of Fall 2023) Consider the parameterized family of ODEs $y' = \phi(y; r)$, where

$$\phi(y; r) = y^2 - r^2.$$

Find the location(s) of any bifurcation(s) occurring in this family, classify the bifurcation(s), and sketch the bifurcation diagram for the family.

6. a) (2.4) Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{y+1}{t+1} \\ y(2) = 3 \end{cases}$$

Write your answer as a function $y = f(t)$.

b) (2.4) Find the general solution of this ODE:

$$\frac{dy}{dt} = \sqrt{ty}$$

7. a) (2.2) Find the general solution of this ODE:

$$ty' + 2y = 4t^2$$

b) (5.2) Find the particular solution of this initial value problem:

$$\begin{cases} y'' - 9y' - 22y = 0 \\ y(0) = 9 \\ y'(0) = 8 \end{cases}$$

8. a) (5.2) Find the general solution of this ODE:

$$y''' + y'' - 20y' = 56e^{2t}$$

b) (4.5) Find the general solution of this system:

$$\begin{cases} x' = 3y \\ y' = 2x - y \end{cases}$$

9. (4.5) Find the particular solution of this initial value problem:

$$\begin{cases} x' = 6x - y \\ y' = x + 4y \\ \mathbf{y}(0) = (3, -1) \end{cases}$$

10. (2.4) Find the general solution of this ODE:

$$y'' = e^{-y}y'$$

Write your answer as a function $y = f(t)$.

11. (4.5) Find the particular solution of this initial value problem:

$$\begin{cases} x' = 5x + 2y \\ y' = -5x - y \\ \mathbf{y}(0) = (1, 1) \end{cases}$$

12. (2.5) A 40 L tank contains fresh water initially. A saline solution containing $.02$ kg of salt per liter is pumped into the tank at a rate of 4 L/min. At the same time, the tank drains through a pipe which removes solution from the tank at a rate of 4 L/min. Assuming the tank is kept well-stirred, how much salt is in the tank 3 minutes after this procedure starts?

13. (5.3) An RLC series circuit consists of a $12\ \Omega$ resistor, a 4 H inductor, and a $\frac{1}{25}$ F capacitor. Assume that at time 0, the charge across the resistor is 13 coulombs, and the current running through the system is 3 amperes. If an external power supply of $102\cos\frac{t}{2}$ V is applied to the circuit, find the charge in the circuit at time t.

Solutions

1. a) The **general solution** of an ODE (or system of ODEs) is a description of all its solutions (this description has arbitrary constants in it). Given an initial value problem, you can plug in the initial conditions to the general solution and solve for the constants, obtaining a **particular solution** of the IVP (which has no constants in it).

   b) **Euler's formula** says that for any complex number t, $e^{it} = \cos t + i \sin t$. This formula is important in solving systems of ODEs (and higher-order ODEs) because it tells you how to rewrite solutions obtained from complex eigenvalues and eigenvectors in terms of cosines and sines.

   c) If you reduce the order of a 3×3 second-order system, you will get a 6×6 first-order linear, homogeneous system (because $6 = 3\cdot 2$) The solution of any 6×6 first-order linear, homogeneous system is a 6-dimensional subspace, so it has to have six arbitrary constants in it. The student's answer only has five arbitrary constants, so it has to be wrong.

2. First, let $\Delta t = \frac{t_n - t_0}{n} = \frac{100-0}{20} = 5$. Next, to establish notation, let the system be $\mathbf{y}' = \Phi(t, \mathbf{y}) = (\phi_1(t,(x,y)), \phi_2(t,(x,y))$. We are given $(t_0, \mathbf{y}_0) = (0,(1,2))$. Therefore $t_1 = t_0 + \Delta t = 0 + 5 = 5$ and

$$\begin{cases} x_1 & = x_0 + \phi_1(0,(1,2))\Delta t \\ & = 1 + (2+2(0))5 \\ & = 1 + 10 = 11 \\ y_1 & = y_0 + \phi_2(0,(1,2))\Delta t \\ & = 2 + (1-2)5 \\ & = 2 - 5 = -3 \end{cases} \Rightarrow (t_1, \mathbf{y}_1) = (5,(11,-3))$$

Next, $t_2 = t_1 + \Delta t = 5 + 5 = 10$ and

$$\begin{cases} x_2 & = x_1 + \phi_1(0,(1,2))\Delta t \\ & = 11 + (-3+2(5))5 \\ & = 11 + 35 = 46 \\ y_2 & = y_1 + \phi_2(0,(1,2))\Delta t \\ & = -3 + (11-(-3))5 \\ & = -3 + 70 = 67 \end{cases} \Rightarrow (t_2, \mathbf{y}_2) = (10,(46,67)).$$

3. a) The system has **one** stable equilibrium (at $(-7,5)$).

   b) The system has **two** unstable equilibria (at $(8,5)$ and $(0,-6)$).

   c) The equilibria are constant solutions, so any of these three answers are valid:

$$\begin{cases} x = -7 \\ y = 5 \end{cases} \qquad \begin{cases} x = 8 \\ y = 5 \end{cases} \qquad \begin{cases} x = 0 \\ y = -6 \end{cases}$$

   (There are other ways to write this; for example, $\mathbf{y} = (-7,5)$, etc.)

d) Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution to this system satisfying $\mathbf{y}(0) = (4, -1)$.

   i. From the phase plane, $x(t)$ decreases then increases. The answer is **D**.

   ii. From the phase plane, $y(t)$ is always decreasing. The answer is **B**.

   iii. From the phase plane, $\lim\limits_{t\to\infty} x(t) = 8$.

   iv. From the phase plane, $\lim\limits_{t\to-\infty} y(t) = 5$.

4. Thinking of the system as $\mathbf{y}' = \Phi(\mathbf{y})$, we set $\Phi(\mathbf{y}) = \mathbf{0}$ and solve for x and y:

$$\begin{cases} 0 = y - 2x \\ 0 = y^2 - 2y - 80 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}y \\ 0 = (y-10)(y+8) \end{cases}$$

From the second equation, $y = 10$ or $y = -8$. From the first equation, the corresponding x-values are 5 and -4, so the two equilibria are $(5, 10)$ and $(-4, -8)$.

To classify the equilibria, compute the total derivative:

$$D\Phi = \begin{pmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_1}{\partial y} \\ \frac{\partial \phi_2}{\partial x} & \frac{\partial \phi_2}{\partial y} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 2y-2 \end{pmatrix}$$

Since this matrix is upper triangular, its eigenvalues are -2 and $2y - 2$. That means that for the equilibrium $(5, 10)$, the eigenvalues of $D\Phi(5, 10)$ are -2 and 18. Since there is one positive and one negative eigenvalue, $(5, 10)$ is an **unstable saddle**.

For the equilibrium $(-4, -8)$, the eigenvalues of $D\Phi(-4, -8)$ are -2 and -18. Since both eigenvalues are negative and real, $(-4, -8)$ is a **stable node**.

5. We start by finding the equilibria of the system in terms of r: set $\phi(y; r) = 0$ and solve for y to get

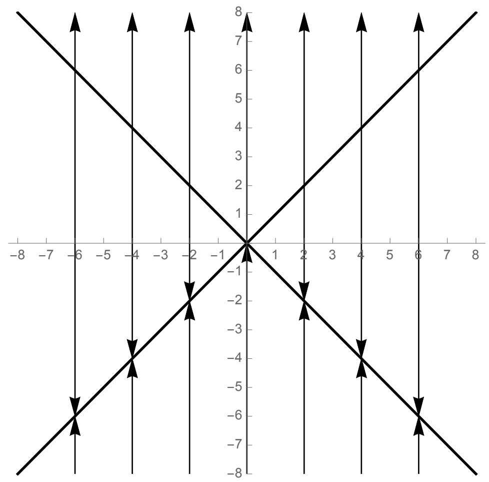
$$0 = y^2 - r^2 \Rightarrow 0 = (y - r)(y + r) \Rightarrow y = r, y = -r.$$

Next, classify these equilibria using the sign of ϕ'. $\phi'(y) = 2y$, so we conclude

$$\phi'(r) = 2r \Rightarrow y = r \text{ is } \begin{cases} \text{stable if } r < 0 \\ \text{semistable if } r = 0 \\ \text{unstable if } r > 0 \end{cases}$$

$$\phi'(-r) = -2r \Rightarrow y = -r \text{ is } \begin{cases} \text{unstable if } r < 0 \\ \text{semistable if } r = 0 \\ \text{stable if } r > 0 \end{cases}$$

We can now sketch the bifurcation diagram:



That means there is a **transcritical bifurcation** at $r = 0$ because the two equilibria cross and change behavior.

6. a) Separate the variables and integrate both sides:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{y+1}{t+1} \\
\frac{1}{y+1}\,dy &= \frac{1}{t+1}\,dt \\
\int \frac{1}{y+1}\,dy &= \int \frac{1}{t+1}\,dt \\
\ln(y+1) &= \ln(t+1) + C \\
y+1 &= e^{\ln(t+1)+C} = C(t+1) \\
y &= C(t+1) - 1
\end{aligned}$$

Now plug in the initial condition $y(2) = 3$ to get $3 = C(2+1) - 1$ and solve for C to get $C = \frac{4}{3}$. Thus the particular solution is $y = \frac{4}{3}(t+1) - 1$, i.e.

$$y = \frac{4}{3}t + \frac{1}{3}.$$

b) Separate the variables and integrate both sides:

$$\begin{aligned}
\frac{dy}{dt} &= \sqrt{ty} \\
\frac{1}{\sqrt{y}}\,dy &= \sqrt{t}\,dt \\
\int \frac{1}{\sqrt{y}}\,dy &= \int \sqrt{t}\,dt \\
2\sqrt{y} &= \frac{2}{3}t^{3/2} + C.
\end{aligned}$$

(If you solved for y, you'd get $y = (\frac{1}{3}t^{3/2} + C)^2 = \frac{1}{9}t^3 + \frac{2}{3}Ct\sqrt{t} + C^2$.)

7. a) First, divide through by t to write the equation as $y' + \frac{2}{t}y = 4t$. Then, compute the integrating factor:

$$\mu(t) = e^{\int_0^t p_0(s)\,ds} = e^{\int_0^t \frac{2}{s}\,ds} = e^{2\ln t} = t^2.$$

Multiply through by μ to obtain the equation

$$\begin{aligned}
\frac{d}{dt}(y\mu) &= 4t(t^2) \\
\frac{d}{dt}\left(yt^2\right) &= 4t^3 \\
yt^2 &= t^4 + C \\
y &= t^2 + Ct^{-2}.
\end{aligned}$$

b) The characteristic equation is $0 = \lambda^2 - 9\lambda - 22 = (\lambda - 11)(\lambda + 2)$ which has roots $\lambda = 11$ and $\lambda = -2$. Therefore the general solution is $y = C_1e^{11t} + C_2e^{-2t}$. To find the particular solution, differentiate to get $y' = 11C_1e^{11t} - 2C_2e^{-2t}$ and plug in the initial values to get

$$\begin{cases} 9 = C_1 + C_2 \\ 8 = 11C_1 - 2C_2 \end{cases} \Rightarrow C_1 = 2, C_2 = 7$$

Therefore the particular solution is $y = 2e^{11t} + 7e^{-2t}$.

8. a) The characteristic equation is $\lambda^3 + \lambda^2 - 20\lambda = \lambda(\lambda + 5)(\lambda - 4)$ which has roots $\lambda = 0$, $\lambda = 4$ and $\lambda = -5$. Therefore the general solution of the homogeneous is $y = C_1 + C_2e^{4t} + C_3e^{-5t}$. Since $q = 56e^{2t}$, guess $y_p = Ae^{2t}$ and plug in the original equation to get

$$8Ae^{2t} + 4Ae^{2t} - 40Ae^{2t} = 56e^{2t} \Rightarrow -28A = 56 \Rightarrow A = -2.$$

Therefore $y_p = -2e^{2t}$ so the general solution is $y = y_p + y_h$, i.e.

$$y = -2e^{2t} + C_1 + C_2e^{4t} + C_3e^{-5t}.$$

b) Thinking of the system as $\mathbf{y}' = A\mathbf{y}$, start with the eigenvalues of A:

$\det(A-\lambda I) = (0-\lambda)(-1-\lambda)-2(3) = \lambda^2+\lambda-6 = (\lambda+3)(\lambda-2) \Rightarrow \lambda = -3, \lambda = 2$

Next, eigenvectors. Let $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ for each eigenvalue:

$$\lambda = -3 : \begin{cases} 3y = -3x \\ 2x - y = -3y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1)$$

$$\lambda = 2 : \begin{cases} 3y = 2x \\ 2x - y = 2y \end{cases} \Rightarrow 3y = 2x \Rightarrow \mathbf{v} = (3, 2)$$

Thus the general solution is

$$\mathbf{y} = C_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Written coordinate-wise (not required), this is

$$\begin{cases} x(t) = C_1 e^{-3t} + 3C_2 e^{2t} \\ y(t) = -C_1 e^{-3t} + 2C_2 e^{2t} \end{cases}$$

9. Thinking of the system as $\mathbf{y}' = A\mathbf{y}$, start by finding eigenvalues of A:

$$\det(A - \lambda I) = (6 - \lambda)(4 - \lambda) + 1 = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2$$

so the only eigenvalue of A is $\lambda = 5$. Next, find eigenvector(s): let $\mathbf{v} = (x, y)$ and set $A\mathbf{v} = \lambda\mathbf{v}$ to get

$$\begin{cases} 6x - y = 5x \\ x + 4y = 5y \end{cases} \Rightarrow x = y \Rightarrow \mathbf{v} = (1, 1)$$

Now, find a generalized eigenvector $\mathbf{w} = (x, y)$ by solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$:

$$\begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \Rightarrow \mathbf{w} = (1, 0)$$

Now, our theorem on repeated eigenvalues tells us that the general solution is

$$\mathbf{y} = C_1 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left[e^{5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right].$$

Coordinate-wise, this is

$$\begin{cases} x = (C_1 + C_2)e^{5t} + C_2 te^{5t} \\ y = C_1 e^{5t} + C_2 te^{5t} \end{cases}$$

To find the particular solution, plug in the initial condition to get $3 = C_1 + C_2$ and $-1 = C_1$. Therefore $C_2 = 4$ so the particular solution is

$$\begin{cases} x = 3e^{5t} + 4te^{5t} \\ y = -e^{5t} + 4te^{5t} \end{cases}.$$

10. This is a second-order, non-linear equation with no t in it. Think of y as the independent variable and let $v = y' = \frac{dy}{dt}$. Then $y'' = v\frac{dv}{dy}$ so the equation becomes
$$v\frac{dv}{dy} = e^{-y}v.$$
This equation can be solved by separating variables and integrating both sides:
$$dv = e^{-y}\,dy \Rightarrow v = -e^{-y} + C.$$
Now back-substitute for v to get the equation
$$\frac{dy}{dt} = -e^{-y} + C = \frac{-1}{e^y} + C = \frac{Ce^y - 1}{e^y}.$$
This equation is separable and can be rewritten as
$$\frac{e^y}{Ce^y - 1}\,dy = dt;$$
integrate both sides (you need the u-substitution $u = Ce^y - 1$ on the left-hand side) to get
$$\frac{1}{C}\ln(Ce^y - 1) = t + D.$$
Solve for y to get
$$y = \ln\left[\frac{1}{C}(e^{C(t+D)} + 1)\right], \text{ i.e. } y = \ln(e^{Ct+D} + 1) - \ln C.$$

11. Think of the system $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ and start by finding the eigenvalues of A:
$$\det(A-\lambda I) = (5-\lambda)(-1-\lambda)+10 = \lambda^2-4\lambda+5 \Rightarrow \lambda = \frac{4 \pm \sqrt{4^2 - 4(5)}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$
Solve for the eigenvector $\mathbf{v} = (x, y)$ corresponding to $2 + i$:
$$\left\{ \begin{array}{l} 5x + 2y = (2+i)x \\ -5x - y = (2+i)y \end{array} \right. \Rightarrow 2y = (-3+i)x \Rightarrow \mathbf{v} = \begin{pmatrix} 2 \\ -3+i \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
We have $\alpha = 2$, $\beta = 1$, $\mathbf{a} = (2, -3)$ and $\mathbf{b} = (0, 1)$, so by the theorem governing solutions with complex eigenvalues we have
$$\mathbf{y} = C_1\left[e^{2t}\cos t\begin{pmatrix} 2 \\ -3 \end{pmatrix} - e^{2t}\sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + C_2\left[e^{2t}\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{2t}\sin t\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right].$$
To find the particular solution, plug in the initial condition to get
$$\left\{ \begin{array}{l} 1 = 2C_1 \\ 1 = -3C_1 + C_2 \end{array} \right. \Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{5}{2}$$

Therefore the particular solution, written coordinate-wise, is

$$\begin{cases} x(t) = e^{2t}\cos t + 5e^{2t}\sin t \\ y(t) = e^{2t}\cos t - 8e^{2t}\sin t \end{cases}.$$

12. Let $y(t)$ be the amount of salt in the tank at time t. The rate at which salt enters the tank is $.02$ kg/L $\times 4$ L/min $= .08 = \frac{2}{25}$ kg/min, and the rate at which salt leaves the tank is $y/40$ kg/L $\times 4$ L/min = $\frac{1}{10}y$ kg/min. This leads to the initial value problem

$$\begin{cases} y' = \frac{2}{25} - \frac{1}{10}y \\ y(0) = 0 \end{cases}$$

($y(0) = 0$ because the water is initially fresh.) To solve the ODE, one can use integrating factors or undetermined coefficients. Using undetermined coefficients, the corresponding homogeneous equation is $y' = \frac{-1}{10}y$ which has solution $y_h = Ce^{-t/10}$. Since $q = \frac{2}{25}$, guess $y_p = A$ and plug in to obtain $0 = \frac{2}{25} - \frac{A}{10}$. Solve for A to get $A = \frac{4}{5}$, so $y_p = \frac{4}{5}$ and the general solution is therefore

$$y = y_p + y_h = Ce^{-t/10} + \frac{4}{5}.$$

Solve for C using the initial condition $y(0) = 0$ to get the particular solution $y = -\frac{4}{5}e^{-t/10} + \frac{4}{5}$. Therefore at time 3, the amount of salt in the tank is

$$y(3) = -\frac{4}{5}e^{-3/10} + \frac{4}{5}.$$

13. The RLC series circuit equation is

$$Lq''(t) + Rq'(t) + \frac{1}{C}q(t) = E_S(t)$$

where q is the charge. In this problem, the above equation leads to the initial value problem

$$\begin{cases} 4q'' + 12q' + 25q = 102\cos\frac{t}{2} \\ q(0) = 13 \\ q'(0) = 3 \end{cases}$$

To solve the IVP, start with the characteristic equation $0 = 4\lambda^2 + 12\lambda + 25 = 4(\lambda^2 + 3\lambda + \frac{25}{4})$ which has roots

$$\lambda = \frac{-3 \pm \sqrt{9 - 25}}{2} = \frac{-3}{2} \pm 2i$$

Therefore the general solution of the homogeneous is

$$q_h = C_1e^{-3t/2}\cos 2t + C_2e^{-3t/2}\sin 2t.$$

To find q_p, guess $q_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}$. Then $q_p' = -\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}$ and $q_p'' = \frac{-A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}$. Plugging in the original equation, we get

$$4\left(\frac{-A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}\right) + 12\left(\frac{-A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}\right) + 25\left(A\cos\frac{t}{2} + B\sin\frac{t}{2}\right) = 102\cos\frac{t}{2}$$
$$(24A + 6B)\cos\frac{t}{2} + (24B - 6A)\sin\frac{t}{2} = 102\cos\frac{t}{2}$$

Therefore $24A + 6B = 102$ and $24B - 6A = 0$; solving for A and B we get $B = 1$ and $A = 4$. Therefore $y_p = 4\cos\frac{t}{2} + \sin\frac{t}{2}$ so the general solution is

$$q = q_p + q_h = 4\cos\frac{t}{2} + \sin\frac{t}{2} + C_1e^{-3t/2}\cos 2t + C_2e^{-3t/2}\sin 2t.$$

To find C_1 and C_2, differentiate to get

$$q' = -2\sin\frac{t}{2} + \frac{1}{2}\cos\frac{t}{2} - \frac{3}{2}C_1e^{-3t/2}\cos 2t - 2C_1e^{-3t/2}\sin 2t - \frac{3}{2}C_2e^{-3t/2}\sin 2t + 2C_2e^{-3t/2}\cos 2t;$$

then plug in the initial conditions to get

$$\begin{cases} 13 = 4 + C_1 \\ 3 = \frac{1}{2} - \frac{3}{2}C_1 + 2C_2 \end{cases} \Rightarrow C_1 = 9, C_2 = 8$$

Therefore the particular solution is $q(t) = q_p + q_h$, i.e.

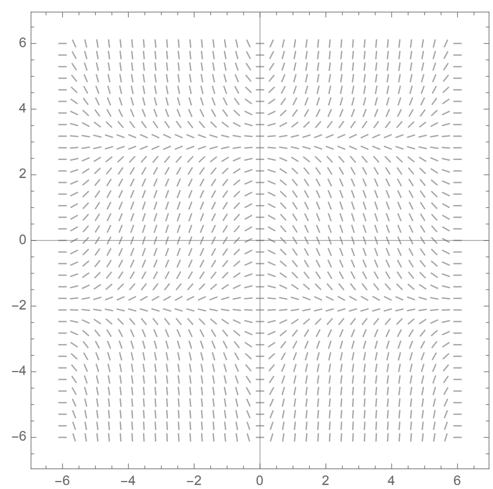
$$q(t) = 4\cos\frac{t}{2} + \sin\frac{t}{2} + 9e^{-3t/2}\cos 2t + 8e^{-3t/2}\sin 2t.$$

Chapter 3

Exams from Fall 2017

3.1 Fall 2017 Exam 1

1. (1.1) Briefly explain the difference between the terms "general solution" and "particular solution" in the context of ordinary differential equations.
2. (1.7) Sketch the phase line for the autonomous ODE $y' = 5y^2 - y^3$.
3. Here is the picture of the slope field associated to some first-order ODE $y' = \phi(t, y)$:



   a) (1.4) Write the equation of any one solution of this ODE.
   b) (1.4) Suppose $y(0) = -5$. Find $\lim_{t\to\infty} y(t)$.
   c) (1.4) Suppose $y(-1) = 1$. Estimate $y(2)$.
   d) (1.4) On the picture above, sketch the graph of the solution satisfying the initial condition $y(3) = -4$.

e) (1.5) Let $y = h(t)$ be the solution of the initial value problem

$$\begin{cases} y' = \phi(t, y) \\ y(0) = 2 \end{cases}$$

Suppose you used Euler's method to estimate $h(12)$ using 3 steps. What are the coordinates of the point you would you obtain as (t_1, y_1)?

4. (2.2 or 2.3) Find the general solution of the following ODE:

$$\frac{dy}{dt} = 5y + 20e^{5t}$$

Write your answer as a function $y = f(t)$.

5. (2.4) Find the particular solution of the following initial value problem:

$$\begin{cases} \frac{dy}{dt} = \frac{ty}{t^2+1} \\ y(0) = 4 \end{cases}$$

Write your answer as a function $y = f(t)$.

6. (2.2) Find the particular solution of the following initial value problem:

$$\begin{cases} y' = \frac{y}{t} - te^{-t} \\ y(-1) = 0 \end{cases}$$

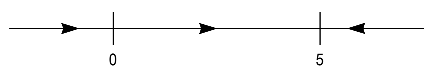
7. (no longer in MATH 330 as of Fall 2023) Find the general solution of the following ODE:

$$y' = \frac{\cos t - y - \cos y}{t - t \sin y}$$

Solutions

1. Given an ODE, the set of all solutions of that ODE is called the **general solution** of the ODE. This solution will have one or more arbitrary constants in it. If you are given an initial value, you can plug that initial value into the general solution, solving for the constant in the general solution. This produces a solution of the ODE with no arbitrary constants, which is called a **particular solution** of the ODE.

2. $\phi(y) = 5y^2 - y^3$; set $\phi(y) = 0$ and factor to solve for y: this gives $0 = y^2(5-y)$ so the two equilibria are $y = 0$ and $y = 5$. To classify them, compute derivatives of ϕ and plug in the equilibria. When $y = 5$, $\phi'(5) = 50 - 75 < 0$ so 5 is stable; when $y = 0$; $\phi'(0) = 0$ but $\phi''(0) = 10 \neq 0$ so 0 is semistable. Therefore the phase line looks like this:



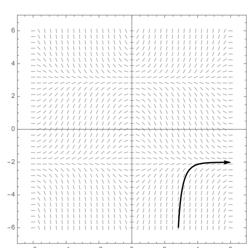
3. Here is the picture of the slope field associated to some first-order ODE $y' = \phi(t, y)$:

   a) $y = 3$ and $y = -2$ are solutions.

   b) If $y(0) = -5$, then $\lim\limits_{t\to\infty} y(t) = -2$.

   c) If $y(-1) = 1$, then $y(2) \approx -1$.

   d) Here is the graph:



   e) Notice that $\phi(0, 2)$ is the slope of the vector field at the point $(0, 2)$, which is 0. So if you used Euler's method with 3 steps, $\Delta t = \frac{t_n - t_0}{n} = \frac{12-0}{3} = 4$ so

$$\begin{cases} t_1 = t_0 + \Delta t = 0 + 4 = 3 \\ y_1 = y_0 + \phi(t_0, y_0)\Delta t = 2 + \phi(0,2) \cdot 4 = 2 + 0 \cdot 4 = 2 \end{cases}$$

   and therefore $(t_1, y_1) = (4, 2)$.

4. *Solution # 1:* Rewrite in standard form as $y' - 5y = 20e^{5t}$; then the integrating factor is

$$\mu = \exp\left(\int -5\, dt\right) = e^{-5t}.$$

After multiplying through by the integrating factor, the equation becomes

$$\frac{d}{dt}\left(ye^{-5t}\right) = 20e^{5t}e^{-5t} = 20.$$

Integrate both sides to get

$$ye^{-5t} = 20t + C;$$

solve for y to get $y = 20te^{5t} + Ce^{5t}$.

Solution # 2: Rewrite in standard form as $y' - 5y = 20e^{5t}$ and use undetermined coefficients. The corresponding homogeneous equation is $y' - 5y = 0$ which has solution $y_h = e^{5t}$. Since the right-hand side of the ODE is $20e^{5t}$, a normal guess for the particular solution would be $y_p = Ae^{5t}$ but since this is the same as y_h up to a constant, you need to multiply the guess by t, i.e. $y_p = Ate^{5t}$. Plugging in the equation, we get

$$\begin{aligned} y' - 5y &= 20e^{5t} \\ (Ate^{5t})' - 5(Ate^{5t}) &= 20e^{5t} \\ Ae^{5t} + 5Ate^{5t} - 5Ate^{5t} &= 20e^{5t} \\ Ae^{5t} &= 20e^{5t} \\ A &= 20 \end{aligned}$$

Therefore $y_p = 20te^{5t}$, so the solution is $y = y_p + Cy_h = 20te^{5t} + Ce^{5t}$.

5. This equation is separable; divide both sides by y and multiply both sides by dt to obtain

$$\frac{1}{y}\,dy = \frac{t}{t^2+1}\,dt.$$

Then integrate both sides (you need the u-sub $u = t^2 + 1$ on the right) to get

$$\ln y = \frac{1}{2}\ln(t^2+1) + C;$$

solving for y by exponentiating both sides gives

$$y = e^{\frac{1}{2}\ln(t^2+1)+C} = C\sqrt{t^2+1}.$$

Now, plug in the initial condition $t = 0, y = 4$ to get $4 = C\sqrt{0^2+1}$, i.e. $C = 4$. Thus the particular solution is $y = 4\sqrt{t^2+1}$.

6. This equation is first-order linear; rewrite it as

$$y' - \frac{1}{t}y = -te^{-t}.$$

From this point there are two methods of solution:

Solution # 1: The integrating factor is

$$\mu = \exp\left(\int \frac{-1}{t}\,dt\right) = \exp(-\ln t) = t^{-1} = \frac{1}{t};$$

after multiplying through the equation by the integrating factor we get

$$\frac{d}{dt}\left(\frac{y}{t}\right) = -e^{-t}.$$

Integrate both sides to get

$$\frac{y}{t} = e^{-t} + C.$$

Now plug in the initial condition $t = -1, y = 0$ to get $0 = e + C$, i.e. $C = -e$. Thus the particular solution is

$$\frac{y}{t} = e^{-t} - e.$$

If you solved for y (not required), this can be rewritten as $y = te^{-t} - et$.

Solution # 2: The corresponding homogeneous equation is $y' - \frac{1}{t}y = 0$ which has solution

$$y_h = \exp\left(\int \frac{1}{t}\,dt\right) = e^{\ln t} = t.$$

For the particular solution, guess $y_p = Ate^{-t} + Be^{-t}$; plugging in the equation we get

$$\begin{aligned} y' - \frac{1}{t}y &= -e^{-t} \\ (Ate^{-t} + Be^{-t})' - \frac{1}{t}(Ate^{-t} + Be^{-t}) &= -e^{-t} \\ Ae^{-t} - Ate^{-t} - Be^{-t} - Ae^{-t} + B\frac{1}{t}e^{-t} &= -e^{-t} \\ -Ate^{-t} - Be^{-t} + \frac{B}{t}e^{-t} &= -e^{-t} \end{aligned}$$

Therefore $-A = -1$ so $A = 1$ and $B = 0$, so $y_p = te^{-t}$. That makes the general solution $y = y_p + Cy_h = te^{-t} + Ct$. Plugging in the initial condition as in Solution # 1 gives the same particular solution: $C = -e$ and therefore $y = te^{-t} - et$.

7. Rewrite the equation as

$$y + \cos y - \cos t + (t - t \sin y)y' = 0.$$

Now let $M = y + \cos y - \cos t$ and let $N = t - t \sin y$. We see that $M_y = N_t = 1 - \sin y$ so the equation is exact. Now find ψ by integrating:

$$\psi = \int M\, dt = \int (y + \cos y - \cos t)\, dt = yt + t \cos y - \sin t + A(y)$$
$$\psi = \int N\, dy = \int (t - t \sin y)\, dy = ty + t \cos y + B(t)$$

To reconcile these answers, set $B(t) = -\sin t$ and $A(y) = 0$ so that $\psi(t, y) = ty + t \cos y - \sin t$. The general solution is therefore $\psi(t, y) = C$, i.e. $ty + t \cos y - \sin t = C$.

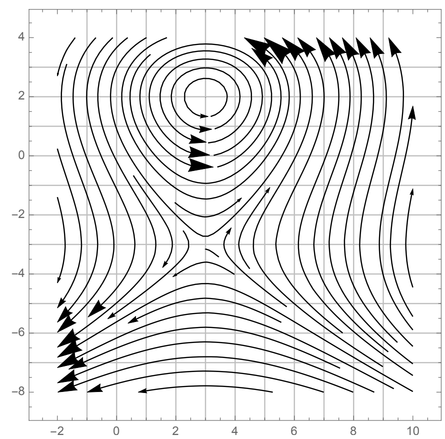
3.2 Fall 2017 Exam 2

1. (3.2) Let $\mathbf{y}(t) = (x(t), y(t))$ be the solution of the initial value problem

$$\begin{cases} \mathbf{y}' = (x - 2y + t, y - 3x) \\ \mathbf{y}(0) = (2, 1) \end{cases}.$$

   Suppose you wanted to estimate $\mathbf{y}(80)$ using Euler's method with 40 steps. Compute the points $\mathbf{y}_1$ and $\mathbf{y}_2$ that would be obtained by this method.

2. Here is the phase plane of a first-order, autonomous 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$:



   Use this picture to answer the following questions:

   a) (4.7) Find all saddles of this system (if there are no saddles, say so).

   b) (4.7) Find all nodes of this system (if there are no nodes, say so).

   c) (4.7) Find all centers of this system (if there are no centers, say so).

   d) (4.8) Is the trace of $D\Phi(3, 2)$ positive, negative or zero?

   e) (4.8) Is the determinant of $D\Phi(3, 2)$ positive, negative or zero?

   f) (3.7) Suppose $\mathbf{y}(0) = (5, -6)$. In this situation, which statement best describes the behavior of $x(t)$?

      A. $x(t)$ increases for all t
      B. $x(t)$ decreases for all t
      C. initially, $x(t)$ is increasing, but then it becomes decreasing
      D. initially, $x(t)$ is decreasing, but then it becomes increasing

   g) (3.7) Suppose $\mathbf{y}(0) = (5, -6)$. In this situation, which statement best describes the behavior of $y(t)$?

      A. $y(t)$ increases for all t

B. $y(t)$ decreases for all t
C. initially, $y(t)$ is increasing, but then it becomes decreasing
D. initially, $y(t)$ is decreasing, but then it becomes increasing

3. (4.5) Find the particular solution of the following initial value problem:

$$\begin{cases} x' = -7x + 5y \\ y' = x - 3y \end{cases} \qquad \begin{cases} x(0) = 2 \\ y(0) = 4 \end{cases}$$

4. (4.6) Find the general solution of the following system of ODEs:

$$\begin{cases} x' &= 5x - 4y + 3e^{-t} \\ y' &= 4x - 5y + 4e^{-t} \end{cases}$$

5. (4.5) Find the particular solution of the following initial value problem:

$$\begin{cases} \mathbf{y}' = (-7x + 2y, -25x + 3y) \\ \mathbf{y}(0) = (1, -5) \end{cases}$$

Write your final answer coordinate-wise.

Solutions

1. First, $\Delta t = \frac{t_n - t_0}{n} = \frac{80-0}{40} = 2$. Now, $(t_0, \mathbf{y}_0) = (0, (2,1))$ and $\Phi(\mathbf{y}_0) = (2 - 2(1) + 0,$ $1 - 3(2)) = (0, -5)$ so by the Euler's method formula,

$$\begin{cases} t_1 = t_0 + \Delta t = 0 + 2 = 2 \\ \mathbf{y}_1 = \mathbf{y}_0 + \Phi(t_0, \mathbf{y}_0)\Delta t = (2,1) + (0,-5)2 = (2,-9)\,. \end{cases}$$

Now, $\Phi(t_1, \mathbf{y}_1) = (2 - 2(-9) + 2, -9 - 3(2)) = (22, -15)$ so

$$\begin{cases} t_2 = t_1 + \Delta t = 2 + 2 = 4 \\ \mathbf{y}_2 = \mathbf{y}_1 + \Phi(t_1, \mathbf{y}_1)\Delta t = (2,-9) + (22,-15)2 = (46,-39)\,. \end{cases}$$

2. a) The only saddle is at $(3, -3)$.
   b) This system has no nodes.
   c) The only center is at $(3, 2)$.
   d) Since $(3, 2)$ is a center, $tr(D\Phi(3, 2)) = 0$.
   e) Since $(3, 2)$ is a center, $\det(D\Phi(3, 2))$ is positive.
   f) **B.** Since the graph of the solution always moves to the left (in the direction of increasing t), $x(t)$ decreases for all t.
   g) **C.** In the direction of increasing t, the graph of the solution initially goes upwards, then downwards.

3. Let $A = \begin{pmatrix} -7 & 5 \\ 1 & -3 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det \begin{pmatrix} -7-\lambda & 5 \\ 1 & -3-\lambda \end{pmatrix} &= (-7-\lambda)(-3-\lambda) - 5 \\ &= \lambda^2 + 10\lambda + 16 = (\lambda+8)(\lambda+2) \end{aligned}$$

so the eigenvalues are $\lambda = -8$ and $\lambda = -2$. Now for the eigenvectors; let $\mathbf{v} = (x, y)$:

$$\lambda = -8 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -7x + 5y = -8x \\ x - 3y = -8y \end{cases} \Rightarrow -x = 5y \Rightarrow (5, -1)$$

$$\lambda = -2 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} -7x + 5y = -2x \\ x - 3y = -2y \end{cases} \Rightarrow x = y \Rightarrow (1, 1)$$

Therefore the general solution is

$$\mathbf{y} = C_1 e^{-8t} \begin{pmatrix} 5 \\ -1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To find the particular solution, plug in the initial value $\mathbf{y}(0) = (2, 4)$ to get

$$\begin{cases} 2 = 5C_1 + C_2 \\ 4 = -C_1 + C_2 \end{cases} \Rightarrow 6C_1 = -2 \Rightarrow C_1 = \frac{-1}{3}, C_2 = \frac{11}{3}.$$

Therefore the particular solution is

$$\begin{aligned} \mathbf{y} &= \frac{-1}{3}e^{-8t}\begin{pmatrix} 5 \\ -1 \end{pmatrix} + \frac{11}{3}e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-5}{3}e^{-8t} + \frac{11}{3}e^{-2t} \\ \frac{1}{3}e^{-8t} + \frac{11}{3}e^{-2t} \end{pmatrix}. \end{aligned}$$

4. Let $A = \begin{pmatrix} 5 & -4 \\ 4 & -5 \end{pmatrix}$ so that the corresponding homogeneous system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned} \det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -4 \\ 4 & -5-\lambda \end{pmatrix} &= (5-\lambda)(-5-\lambda) + 16 \\ &= \lambda^2 - 9 = (\lambda+3)(\lambda-3) \end{aligned}$$

so the eigenvalues are $\lambda = -3$ and $\lambda = 3$. Now for the eigenvectors; let $\mathbf{v} = (x, y)$:

$$\lambda = -3 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 5x - 4y = -3x \\ 4x - 5y = -3y \end{cases} \Rightarrow y = 2x \Rightarrow (1, 2)$$

$$\lambda = 3 : A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{cases} 5x - 4y = 3x \\ 4x - 5y = 3y \end{cases} \Rightarrow x = 2y \Rightarrow (2, 1)$$

Therefore the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Now, we find a particular solution $\mathbf{y}_p$ using undetermined coefficients. Since $\mathbf{q} = (3e^{-t}, 4e^{-t})$, we guess $\mathbf{y}_p = (Ae^{-t}, Be^{-t})$. Plugging this into the system, we get

$$\begin{cases} -Ae^{-t} &= 5Ae^{-t} - 4Be^{-t} + 3e^{-t} \\ -Be^{-t} &= 4Ae^{-t} - 5Be^{-t} + 4e^{-t} \end{cases} \Rightarrow \begin{cases} -A &= 5A - 4B + 3 \\ -B &= 4A - 5B + 4 \end{cases}$$

From the second equation $4B = 4A + 4$, i.e. $B = A + 1$. Plugging this into the first equation gives $-A = 5A - 4(A+1) + 3$, i.e. $-A = A - 1$ so $A = \frac{1}{2}$ and $B = \frac{3}{2}$. Therefore

$$\mathbf{y}_p = \begin{pmatrix} Ae^{-t} \\ Be^{-t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-t} \\ \frac{3}{2}e^{-t} \end{pmatrix}.$$

Last, the solution is

$$\begin{aligned}\mathbf{y} &= \mathbf{y}_p + \mathbf{y}_h \\ &= \begin{pmatrix} \frac{-7}{10}e^{-t} \\ \frac{3}{10}e^{-t} \end{pmatrix} + C_1 e^{-3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}e^{-t} + C_1 e^{-3t} + 2C_2 e^{3t} \\ \frac{3}{2}e^{-t} + 2C_1 e^{-3t} + C_2 e^{3t} \end{pmatrix}.\end{aligned}$$

5. Let $A = \begin{pmatrix} -7 & 2 \\ -25 & 3 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$. First, find the eigenvalues of A:

$$\begin{aligned}\det(A - \lambda I) = \det\begin{pmatrix} -7-\lambda & 2 \\ -25 & 3-\lambda \end{pmatrix} &= (-7-\lambda)(3-\lambda) + 50 \\ &= \lambda^2 + 4\lambda + 29.\end{aligned}$$

By the quadratic formula, the eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{16 - 4(29)}}{2} = \frac{-4 \pm \sqrt{-100}}{2} = \frac{-4 \pm 10i}{2} = -2 \pm 5i$$

so $\alpha = -2$ and $\beta = 5$. Next, find the eigenvectors; let $\mathbf{v} = (x, y)$:

$$\begin{aligned}\lambda = -2 + 5i: \ A\mathbf{v} = \lambda\mathbf{v} &\Rightarrow \begin{cases} -7x + 2y = (-2+5i)x \\ 25x + 3y = (-2+5i)y \end{cases} \\ &\Rightarrow 2y = (5+5i)x \\ &\Rightarrow (2, 5+5i) = (2,5) + i(0,5).\end{aligned}$$

Therefore $\mathbf{a} = (2,5)$ and $\mathbf{b} = (0,5)$. By the theorem from the lecture notes, the general solution is therefore

$$\begin{aligned}\mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{-2t}\cos 5t\begin{pmatrix} 2 \\ 5 \end{pmatrix} - e^{-2t}\sin 5t\begin{pmatrix} 0 \\ 5 \end{pmatrix}\right] + C_2\left[e^{-2t}\cos 5t\begin{pmatrix} 0 \\ 5 \end{pmatrix} + e^{-2t}\sin 5t\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right] \\ &= \begin{pmatrix} 2C_1 e^{-2t}\cos 5t + 2C_2 e^{-2t}\sin 5t \\ (5C_1 + 5C_2)e^{-2t}\cos 5t + (5C_2 - 5C_1)e^{-2t}\sin 5t \end{pmatrix}.\end{aligned}$$

Now for the particular solution. Plugging in $t = 0$, $\mathbf{y} = (1, -5)$, we get

$$\begin{cases} 1 = 2C_1 \\ -5 = 5C_1 + 5C_2 \end{cases} \Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{-3}{2}.$$

Therefore the particular solution is

$$\mathbf{y} = \begin{pmatrix} e^{-2t}\cos 5t - 3e^{-2t}\sin 5t \\ -5e^{-2t}\cos 5t - 10e^{-2t}\sin 5t \end{pmatrix}.$$

Written coordinate-wise, this is

$$\begin{cases} x(t) & = e^{-2t}\cos 5t - 3e^{-2t}\sin 5t \\ y(t) & = -5e^{-2t}\cos 5t - 10e^{-2t}\sin 5t \end{cases}.$$

3.3 Fall 2017 Final Exam

1. a) (4.8) Write down an example of a 2×2 matrix A such that the constant-coefficient system $\mathbf{y}' = A\mathbf{y}$ has a stable node at the origin.

   b) (5.1) Consider the third-order differential equation $y''' + y' - 5y = e^t$. Convert this equation to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$, clearly defining what $\mathbf{y}$, A and $\mathbf{q}$ are.

   c) (3.6) Let $y' = \phi(t, y)$ be a first-order, linear ODE and suppose that $y_1(t) = 2t + t^2$ and $y_2(t) = t + t^3$ are particular solutions of $y' = \phi(t, y)$. Find the general solution of the ODE $y' = \phi(t, y)$.

   d) (1.7) Find $\lim\limits_{t\to\infty} h(t)$, where $y = h(t)$ is the solution of the IVP

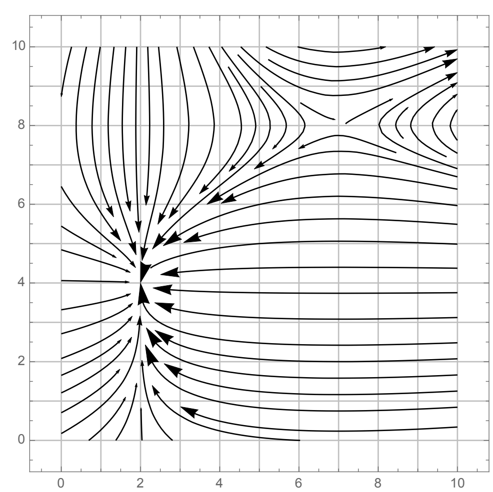
$$\begin{cases} y' = (y-2)(y-9) \\ y(0) = 5 \end{cases}.$$

2. (1.5) Consider the initial value problem

$$\begin{cases} \frac{dy}{dt} = \frac{t-1}{y+t} \\ y(1) = 2 \end{cases}.$$

   Estimate $y(10)$ by using Euler's method with 3 steps.

3. Here is a picture of the phase plane of some 2×2 first-order system $\mathbf{y}' = \Phi(\mathbf{y})$:



   Use this picture to answer the following questions:

   a) (4.7) Give the location of all equilibria of the system. Classify each equilibrium.

b) (3.7) Let $\mathbf{y} = (x(t), y(t))$ be the solution of this system satisfying $\mathbf{y}(0) = (6, 2)$.

i. Find $\lim_{t\to\infty} x(t)$.

ii. Find $\lim_{t\to\infty} y(t)$.

iii. Find $\lim_{t\to-\infty} x(t)$.

c) (3.7) Let $\mathbf{y} = (f(t), g(t))$ be the solution of this system satisfying $f(0) = 1$ and $g(0) = 6$.

i. Is $f'(0)$ positive, negative or zero?

ii. Is $g'(0)$ positive, negative or zero?

4. (4.7) Find and classify all equilibria of this system:

$$\begin{cases} x' = x^2 - y - 1 \\ y' = y^2 - 3y \end{cases}$$

5. (2.4) Find the particular solution of this initial value problem:

$$\begin{cases} y' = \frac{10}{6y^2+1} \\ y(4) = 3 \end{cases}$$

6. (2.2) Find the general solution of this ODE writing your answer as a function $y = f(t)$.

$$\frac{dy}{dt} + 6ty = 12t$$

7. (5.2) Find the particular solution of this initial value problem:

$$\begin{cases} y'' = -16y \\ y(0) = 3 \\ y'(0) = 5 \end{cases}$$

8. (5.2) Find the general solution of this ODE:

$$y'' - 7y' - 18y = -54t + 15$$

9. (4.5) Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (-3x + 4y, -x - 7y) \\ \mathbf{y}(0) = (7, -2) \end{cases}$$

10. (4.5) Find the general solution of this system, and write your answer coordinate-wise:

$$\begin{cases} x' = -7x + 25y \\ y' = -5x + 13y \end{cases}$$

11. (5.3) A 4 kg mass is attached to the end of a spring with spring constant 2 N/cm. Assume that the damping coefficient is 6 N sec/cm and the entire system is subject at time t to an external force of $2\cos\frac{t}{2} + 6\sin\frac{t}{2}$. If at time $t = 0$, the mass is moving with initial velocity -4 cm/sec and has position 2, find the position of the mass at time $t = \pi$.

12. (4.9) Two large tanks each hold some volume of liquid. Tank X holds 100 L of sulfuric acid solution which is initially 4% hydrochloric acid; tank Y holds 50 L of solution which is initially 12% hydrochloric acid. Pure water flows into tank Y at a rate of 3 L/min and pure water flows into tank X at a rate of 1 L/min. Tank Y drains into tank X at a rate of 2 L/min, and drains out of the system at a rate of 1 L/min. Tank X drains out of the system at a rate of 3 L/min, and does not drain into tank Y.

    Assuming that at all times the solution in each tank is kept mixed, find the concentration of hydrochloric acid in tank X at time 25.

Solutions

1. a) Any 2×2 matrix with two negative, real eigenvalues works: the easiest thing is to write down any triangular (or diagonal) matrix with negative numbers along the diagonal, like (for example) $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

   b) Rewrite the equation as $y''' = -y' + 5y + e^t$. Then, let $\mathbf{y} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$, let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -1 & 0 \end{pmatrix}$ and let $\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}$. The equation becomes $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ as desired.

   c) From the theory of first-order linear equations, we know that the difference of any two solutions of a linear ODE solves the corresponding homogeneous equation. So $y_1(t) - y_2(t) = t + t^2 - t^3$ is a solution of the corresponding homogeneous equation. Furthermore, we know that for a homogeneous first-order linear ODE, the solution set is the span (i.e. set of multiples) of any one nonzero solution, so the homogeneous equation has as its solution set $y_h = C(t + t^2 - t^3)$. The solution set of the original equation is therefore any one particular solution of the equation (like either y_1 or y_2) plus the solution set of the homogeneous, i.e. the general solution is

   $$y = 2t + t^2 + C(t + t^2 - t^3).$$

   d) Let $\phi(y) = (y-2)(y-9)$. Clearly $\phi(y) = 0$ when $y = 2$ or $y = 9$. Then $\phi'(y) = 2y - 11$, so $\phi'(2) = -7 < 0$ and $\phi'(9) = 7 > 0$. Thus 2 is a stable equilibrium and 9 is an unstable equilibrium of this equation. As $t \to \infty$, $h(t)$ will approach the stable equilibrium, so $\lim\limits_{t\to\infty} h(t) = 2$.

2. Let $\phi(t, y) = \frac{t-1}{y+t}$ so that the equation becomes $y' = \phi(t, y)$. Next, $\Delta t = \frac{t_n - t_0}{n} = \frac{10-1}{3} = 3$. We are given $(t_0, y_0) = (1, 2)$. Therefore $\phi(3, 10) = \frac{1-1}{2+1} = 0$ so

   $$\begin{cases} t_1 = t_0 + \Delta t = 1 + 3 = 4 \\ y_1 = y_0 + \phi(t_0, y_0)\Delta t = 2 + 0(3) = 2 \end{cases} \Rightarrow (t_1, y_1) = (4, 2).$$

   Next, $\phi(t_1, y_1) = \frac{4-1}{4+2} = \frac{1}{2}$ so

   $$\begin{cases} t_2 = t_1 + \Delta t = 4 + 3 = 7 \\ y_2 = y_1 + \phi(t_1, y_1)\Delta t = 2 + \frac{1}{2}(3) = \frac{7}{2} \end{cases} \Rightarrow (t_2, y_2) = (7, \frac{7}{2}).$$

Last, $\phi(t_2, y_2) = \frac{7-1}{\frac{7}{2}+7} = \frac{6}{\frac{21}{2}} = \frac{4}{7}$ so

$$\begin{cases} t_3 = t_2 + \Delta t = 7 + 3 = 10 \\ y_3 = y_2 + \phi(t_2, y_2)\Delta t = \frac{7}{2} + \frac{4}{7}(3) = \frac{7}{2} + \frac{12}{7} = \frac{73}{14} \end{cases} \Rightarrow (t_3, y_3) = (10, \frac{73}{14}).$$

So $y(10) \approx \frac{73}{14}$.

3. a) $(2,4)$ is a stable node; $(7,8)$ is a saddle.
   b) i. $\lim_{t\to\infty} x(t) = 2$.
      ii. $\lim_{t\to\infty} y(t) = 4$.
      iii. $\lim_{t\to-\infty} x(t) = \infty$.
   c) i. $f'(0) = \frac{dx}{dt}\Big|_{x=1,y=6} > 0$ since the graph of $\mathbf{y}$ is moving to the right at $t = 0$.
      ii. $g'(0) = \frac{dy}{dt}\Big|_{x=1,y=6} < 0$ since the graph of $\mathbf{y}$ is moving downward at $t = 0$.

4. Let $\Phi(x,y) = (x^2 - y - 1, y^2 - 3y)$. Setting $\Phi(x,y) = \mathbf{0}$, we obtain

$$\begin{cases} 0 = x^2 - y - 1 \\ 0 = y^2 - 3y \end{cases}$$

From the second equation (which factors as $0 = y(y-3)$), either $y = 0$ or $y = 3$. When $y = 0$, from the first equation $x^2 = 1$, so $x = \pm 1$. When $y = 3$, from the first equation $x^2 = 4$, so $x = \pm 2$. We therefore have four equilibria, which we classify by looking at the eigenvalues of the total derivative $D\Phi(x,y) = \begin{pmatrix} 2x & -1 \\ 0 & 2y-3 \end{pmatrix}$:

- $(1,0)$: $D\Phi(1,0) = \begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix}$ has eigenvalues 2 and -3, making $(1,0)$ a **saddle**.
- $(-1,0)$: $D\Phi(-1,0) = \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$ has eigenvalues -2 and -3, making $(-1,0)$ a **stable node**.
- $(2,3)$: $D\Phi(2,3) = \begin{pmatrix} 4 & -1 \\ 0 & 3 \end{pmatrix}$ has eigenvalues 4 and 3, making $(2,3)$ an **unstable node**.
- $(-2,3)$: $D\Phi(2,3) = \begin{pmatrix} -4 & -1 \\ 0 & 3 \end{pmatrix}$ has eigenvalues -4 and 3, making $(-2,3)$ a **saddle**.

5. This equation is separable: rewrite it as $(6y^2 + 1)\,dy = 10\,dt$. Then integrate both sides to get $2y^3 + y = 10t + C$. Plug in the initial condition $(4, 3)$ to get $2(27) + 3 = 10(4) + C$, i.e. $C = 17$. Thus the particular solution is $2y^3 + y = 10t + 17$.

6. Multiply through by the integrating factor $\mu = \exp\left(\int 6t\,dt\right) = \exp(3t^2)$ to obtain the equation $\frac{d}{dt}\left(ye^{3t^2}\right) = 12te^{3t^2}$. Now integrate both sides (you need the u-substitution $u = 3t^2, du = 6t\,dt$ on the right-hand side) to get $ye^{3t^2} = 2e^{3t^2} + C$. Solving for y, we obtain $y = 2 + Ce^{-3t^2}$.

7. Rewrite the equation as $y'' + 16y = 0$ and solve the characteristic equation $\lambda^2 + 16 = 0$ to obtain $\lambda = \pm 4i$. Thus the general solution is $y = C_1 \cos 4t + C_2 \sin 4t$. To find the particular solution, differentiate to obtain $y' = -4C_1 \sin 4t + 4C_2 \cos 4t$. Plugging in $y(0) = 3$ and $y'(0) = 5$ yields

$$\begin{cases} 3 = C_1 \\ 5 = 4C_2 \end{cases} \Rightarrow C_1 = 3, C_2 = \frac{5}{4}.$$

Thus the particular solution is $y = 3\cos 4t + \frac{5}{4}\sin 4t$.

8. First, solve the homogeneous equation $y'' - 7y' - 18y = 0$ by considering the characteristic equation $0 = \lambda^2 - 7\lambda - 18 = (\lambda - 9)(\lambda + 2)$. This equation has solutions $\lambda = 9, \lambda = -2$ so the solution of the homogeneous is $y_h = C_1e^{9t} + C_2e^{-2t}$.

   Now find a particular solution using undetermined coefficients; guess $y_p = At + B$ and plug into the equation to obtain

$$0 - 7(A) - 18(At + B) = -54t + 15,$$

   i.e. $-18At + (-7A - 18B) = -54t + 15$. Thus $-18A = -54$, so $A = 3$. Then $-7(3) - 18B = 15$ so $-18B = 36$ so $B = -2$. Thus $y_p = 3t - 2$ so the general solution of the equation is

$$y = y_p + y_h = 3t - 2 + C_1e^{9t} + C_2e^{-2t}.$$

9. First, find the eigenvalue(s):

$$\det\begin{pmatrix} -3-\lambda & 4 \\ -1 & -7-\lambda \end{pmatrix} = (-3-\lambda)(-7-\lambda) + 4 = \lambda^2 + 10\lambda + 25 = (\lambda+5)^2$$

   so the only eigenvalue is $\lambda = -5$ (repeated twice). Next, the eigenvector(s):

$$\begin{cases} -3x + 4y = -5x \\ -x - 7y = -5y \end{cases} \Rightarrow x = -2y \Rightarrow \mathbf{v} = (2, -1).$$

Since there is only one linearly independent eigenvector, we find a generalized eigenvector $\mathbf{w} = (x, y)$:

$$\begin{pmatrix} -3-(-5) & 4 \\ -1 & -7-(-5) \end{pmatrix} \mathbf{w} = \mathbf{v} \Rightarrow \begin{cases} 2x + 4y = 2 \\ -x - 2y = -1 \end{cases} \Rightarrow \mathbf{w} = (1, 0).$$

Now by the formula from the lecture notes, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 e^{-5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 \left[e^{-5t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} (2C_1 + C_2)e^{-5t} + 2C_2 te^{-5t} \\ -C_1 e^{-5t} - C_2 te^{-5t} \end{pmatrix}. \end{aligned}$$

Now for the particular solution. Plugging in the initial condition $\mathbf{y}(0) = (7, -2)$, we get

$$\begin{cases} 7 = 2C_1 + C_2 \\ -2 = -C_1 \end{cases}$$

which leads to $C_1 = 2$ and $C_2 = 3$. Thus the particular solution is

$$\mathbf{y} = \begin{pmatrix} 7e^{-5t} + 6te^{-5t} \\ -2e^{-5t} - 3te^{-5t} \end{pmatrix}.$$

10. First, find the eigenvalue(s):

$$\det \begin{pmatrix} -7-\lambda & 25 \\ -5 & 13-\lambda \end{pmatrix} = (-7-\lambda)(13-\lambda) + 125 = \lambda^2 - 6\lambda + 34$$

so by the quadratic formula, the eigenvalues are

$$\lambda = \frac{6 \pm \sqrt{(-6)^2 - 4(34)}}{2} = 3 \pm \frac{1}{2}\sqrt{-100} = 3 \pm 5i.$$

Next, the eigenvector for $3 + 5i$:

$$\begin{cases} -7x + 25y = (3+5i)x \\ -5x + 13y = (3+5i)y \end{cases} \Rightarrow 25y = (10+5i)x \Rightarrow 5y = (2+i)x \Rightarrow \mathbf{v} = (5, 2+i) = (5, 2) + i(0, 1).$$

Now by the formula in the lecture notes, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 \left[e^{3t} \cos 5t \begin{pmatrix} 5 \\ 2 \end{pmatrix} - e^{3t} \sin 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + C_2 \left[e^{3t} \cos 5t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + e^{3t} \sin 5t \begin{pmatrix} 5 \\ 2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 5C_1 e^{3t} \cos 5t + 5C_2 e^{3t} \sin 5t \\ (2C_1 + C_2)e^{3t} \cos 5t + (-C_1 + 2C_2)e^{3t} \sin 5t \end{pmatrix}. \end{aligned}$$

Written coordinate-wise, this is

$$\begin{cases} x(t) = 5C_1 e^{3t} \cos 5t + 5C_2 e^{3t} \sin 5t \\ y(t) = (2C_1 + C_2)e^{3t} \cos 5t + (-C_1 + 2C_2)e^{3t} \sin 5t \end{cases}.$$

11. From the spring equation $mx''(t) + bx'(t) + kx(t) = F_{ext}(t)$, we obtain, by plugging in the constants given in the problem, the following IVP:

$$\begin{cases} 4x'' + 6x' + 2x = 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ x(0) = 2 \\ x'(0) = -4 \end{cases}$$

To solve this, start with the corresponding homogeneous equation $4x''+6x'+2x = 0$ which has characteristic equation $4\lambda^2 + 6\lambda + 2 = 0$. This equation factors as $2(2\lambda+1)(\lambda+1) = 0$ so it has solutions $\lambda = -1$ and $\lambda = \frac{-1}{2}$. Thus the general solution of the homogeneous is $y_h = C_1e^{-t} + C_2e^{-t/2}$.

Next, find a particular solution with undetermined coefficients. Guess $x_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}$; then $x_p' = -\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}$ and $x_p'' = -\frac{A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}$. Plugging in the original equation, we get

$$\begin{aligned} 4(-\frac{A}{4}\cos\frac{t}{2} - \frac{B}{4}\sin\frac{t}{2}) + 6(-\frac{A}{2}\sin\frac{t}{2} + \frac{B}{2}\cos\frac{t}{2}) + 2(A\cos\frac{t}{2} + B\sin\frac{t}{2}) &= 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ \Rightarrow (A+3B)\cos\frac{t}{2} + (-3A+B)\sin\frac{t}{2} &= 2\cos\frac{t}{2} + 6\sin\frac{t}{2} \\ \Rightarrow \begin{cases} A+3B = 2 \\ -3A+B = 6 \end{cases} &\Rightarrow A = -\frac{8}{5}, B = \frac{6}{5}. \end{aligned}$$

Therefore $x_p = -\frac{8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2}$ so

$$x = x_p + x_h = -\frac{8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2} + C_1e^{-t} + C_2e^{-t/2}.$$

Now find C_1 and C_2 by plugging in the initial conditions: plugging in $x(0) = 2$, we get $2 = -\frac{8}{5} + C_1 + C_2$ so $C_1 + C_2 = \frac{18}{5}$. Plugging in $x'(0) = -4$, we get $-4 = \frac{3}{5} - C_1 - \frac{1}{2}C_2$, i.e. $-C_1 - \frac{1}{2}C_2 = \frac{-23}{5}$; solving the two equations together gives $C_2 = -2, C_1 = \frac{28}{5}$ so the particular solution of the spring equation is

$$x(t) = \frac{-8}{5}\cos\frac{t}{2} + \frac{6}{5}\sin\frac{t}{2} + \frac{28}{5}e^{-t} - 2e^{-t/2}.$$

Finally, answer the question by plugging in $x = \pi$ to obtain

$$\begin{aligned} x(\pi) &= \frac{-8}{5}\cos\frac{\pi}{2} + \frac{6}{5}\sin\frac{\pi}{2} + \frac{28}{5}e^{-\pi} - 2e^{-\pi/2} \\ &= \frac{6}{5} + \frac{28}{5}e^{-\pi} - 2e^{-\pi/2}. \end{aligned}$$

12. Let $x(t)$ and $y(t)$ be the amount of hydrochloric acid in tanks X and Y, respectively, at time t. Let $\mathbf{y} = (x(t), y(t))$; from the given information, we obtain the following IVP:

$$\begin{cases} \mathbf{y}' = (2y/50 - 3x/100, -3y/50) \\ \mathbf{y}(0) = (4, 6) \end{cases}$$

Written in matrix form, this equation is $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{pmatrix} \frac{-3}{100} & \frac{1}{25} \\ 0 & \frac{-3}{50} \end{pmatrix}.$$

Since this matrix is triangular, its eigenvalues are its diagonal entries: $\lambda = \frac{-3}{100}, \lambda = \frac{-3}{50}$. Now find eigenvectors:

- $\lambda = \frac{-3}{100}$: $\begin{cases} \frac{-3}{100}x + \frac{1}{25}y = \frac{-3}{100}x \\ \frac{-3}{50}y = \frac{-3}{100}y \end{cases} \Rightarrow y = 0 \Rightarrow (1, 0)$
- $\lambda = \frac{-3}{50}$: $\begin{cases} \frac{-3}{100}x + \frac{1}{25}y = \frac{-3}{50}x \\ \frac{-3}{50}y = \frac{-3}{50}y \end{cases} \Rightarrow \frac{1}{25}y = \frac{-3}{100}x \Rightarrow 4y = -3x \Rightarrow (4, -3)$

Therefore the general solution of this system is

$$\mathbf{y} = C_1 e^{-3t/100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-3t/50} \begin{pmatrix} 4 \\ -3 \end{pmatrix}.$$

Plugging in the initial conditions, we get

$$\begin{cases} 4 = C_1 + 4C_2 \\ 6 = -3C_2 \end{cases} \Rightarrow C_1 = 12, C_2 = -2.$$

Thus the particular solution is

$$\mathbf{y} = 12e^{-3t/100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2e^{-3t/50} \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

which coordinate-wise is

$$\begin{cases} x(t) = 12e^{-3t/100} - 8e^{-3t/50} \\ y(t) = 6e^{-3t/50} \end{cases}.$$

So the amount of hydrochloric acid in tank X at time 25 is $x(25) = 12e^{-3/4} - 8e^{-3/2}$; the concentration is given by this amount divided by the volume of tank X, which gives

$$\frac{12e^{-3/4} - 8e^{-3/2}}{100} = \frac{7e^{-3/4} - 2e^{-3/2}}{25}.$$

Chapter 4

Exams from Fall 2019

4.1 Fall 2019 Exam 1

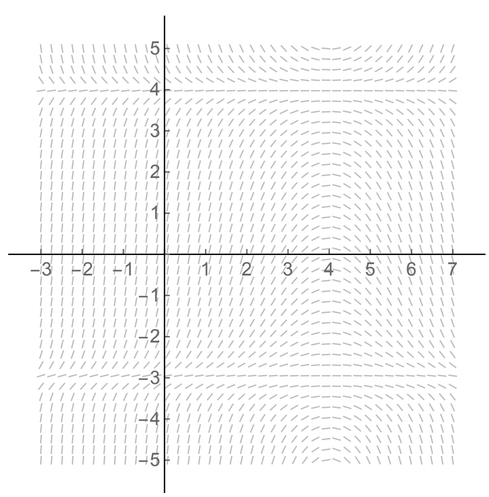
1. (1.3) Write down an initial value problem for which the function $y = 7e^{3t}$ is the only solution. There is a catch: your differential equation may not contain the independent variable t anywhere in it.

2. (1.5) Let $y = f(t)$ be the solution of the initial value problem

$$\begin{cases} y' = y^2 + t \\ y(1) = -2 \end{cases} .$$

   Estimate $f(5)$ by performing Euler's method with two steps.

3. (1.4) Here is the picture of the slope field associated to a first-order ODE $y' = \phi(t, y)$:



a) Write the explicit equation of one solution of this differential equation.

b) Let $y = f(t)$ be the solution to this differential equation satisfying $f(3) = 0$.

   i. Estimate $f(2)$.

   ii. When $t = 5$, is f increasing or decreasing?

   iii. What is the maximum value obtained by f?

   iv. Find $\lim_{t\to\infty} f(t)$.

4. (no longer in MATH 330 as of Fall 2023) Find the general solution of the following ODE:
$$y' = \frac{1 - 6ty}{3t^2 + 2}$$

5. (2.2 or 2.3) Find the general solution of the following ODE:
$$\frac{dy}{dt} - 3y = 5e^{2t}$$

6. (2.4) Find the particular solution of the following initial value problem:
$$\begin{cases} y' = e^y \\ y(0) = 0 \end{cases}$$
Write your answer as a function $y = f(t)$.

7. (2.2) Find the particular solution of the following initial value problem:
$$\begin{cases} t\frac{dy}{dt} = y + t^3 \\ y(2) = 10 \end{cases}$$
Write your answer as a function $y = f(t)$.

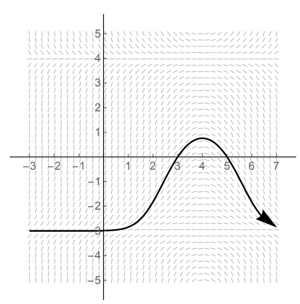
Solutions

1. This is an exponential growth model with $y_0 = 7$ and $r = 3$; an appropriate IVP is
$$\begin{cases} y' = 3y \\ y(0) = 7 \end{cases}.$$

2. First, $\Delta t = \frac{1}{n}(t_n - t_0) = \frac{1}{2}(5-1) = 2$. Next, we are given $(t_0, y_0) = (1, -2)$. Now $\phi(t_0, y_0) = (-2)^2 + 1 = 5$ so
$$\begin{aligned} t_1 &= t_0 + \Delta t = 1 + 2 = 3 \\ y_1 &= y_0 + \phi(t_0, y_0)\Delta t = -2 + 5(2) = 8. \end{aligned}$$
Therefore $(t_1, y_1) = (3, 8)$. Now $\phi(t_1, y_1) = 8^2 + 3 = 67$ so
$$\begin{aligned} t_2 &= t_1 + \Delta t = 3 + 2 = 5 \\ y_2 &= y_1 + \phi(t_1, y_1)\Delta t = 8 + 67(2) = 142. \end{aligned}$$
Therefore $(t_2, y_2) = (5, 142)$, so $f(5) \approx 142$.

3. a) $y = 4$ and $y = -3$ are both solutions.

   b) Here is a graph of f, obtained by starting at $(3, 0)$ and following the vector field:



   From the graph, we see:

   i. $f(2) \approx -2$.

   ii. f is decreasing when $t = 5$.

   iii. The maximum value obtained by f is about $\frac{3}{4}$ (any answer between $\frac{1}{2}$ and 1 is fine).

   iv. $\lim\limits_{t\to\infty} f(t) = -3$.

4. Rewrite this equation as $(6ty - 1) + (3t^2 + 2)\frac{dy}{dt}$. Letting $M = 6ty - 1$ and $N = 3t^2 + 2$, we see that
$$M_y = 6t = N_t$$

so the equation is exact. Now

$$\begin{aligned}\psi(t,y) &= \int M\,dt = \int (6ty-1)\,dt = 3t^2y - t + A(y)\\ &= \int N\,dy = \int (3t^2+2)\,dy = 3t^2y + 2y + B(t).\end{aligned}$$

By setting $B(t) = -t$ and $A(y) = 2y0$, we reconcile these integrals to obtain $\psi(t,y) = 3t^2y - t + 2y$. Thus the general solution is $\psi(t,y) = C$, i.e. $3t^2y - t + 2y = C$.

5. *Method 1 (integrating factors):* The integrating factor is $\mu(t) = \exp\left[\int(-3)\,dt\right] = e^{-3t}$. After multiplying through by $\mu(t)$, the equation becomes

$$\begin{aligned}\frac{dy}{dt}e^{-3t} - 3e^{-3t}y &= 5e^{2t}(e^{-3t})\\ \frac{d}{dt}\left(ye^{-3t}\right) &= 5e^{-t}\\ ye^{-3t} &= \int 5e^{-t}\,dt\\ ye^{-3t} &= -5e^{-t} + C\\ y &= e^{3t}\left(-5e^{-t} + C\right)\\ y &= -5e^{2t} + Ce^{3t}.\end{aligned}$$

Method 2 (undetermined coefficients): The corresponding homogeneous equation is $\frac{dy}{dt} - 3y = 0$ which has solution $y_h = e^{3t}$ (exponential growth model).

Now, guess $y_p = Ae^{2t}$ and plug into the left-hand side of the equation to get $2Ae^{2t} - 3Ae^{2t} = 5e^{2t}$. That means $2A - 3A = 5$, i.e. $A = -5$. Therefore $y_p = -5e^{2t}$ so $y = y_p + Cy_h$, i.e.

$$y = -5e^{2t} + Ce^{3t}.$$

6. Start with the ODE, which is separable:

$$\begin{aligned}\frac{dy}{dt} &= e^y\\ e^{-y}\,dy &= dt\\ \int e^{-y}\,dy &= \int dt\\ -e^{-y} &= t + C\end{aligned}$$

Next, solve for C using the initial condition (you could have solved for y first):

$$-e^{-0} = 0 + C \Rightarrow -1 = C$$

Therefore the particular solution is

$$-e^{-y} = t - 1$$

Solve for y to get

$$y = -\ln(1-t).$$

7. First, move the y term to the left-hand side and then divide through by t to write this linear equation in its standard form:

$$\frac{dy}{dt} - \frac{1}{t}y = t^2.$$

Since this equation is linear but not constant coefficient, it is best to use integrating factors. We have $\mu(t) = \exp\left[\int -\frac{1}{t}\,dt\right] = e^{-\ln t} = \frac{1}{t}$. After multiplying through by $\mu(t)$, the equation becomes

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{t}y\right) &= t \\
\frac{y}{t} &= \int t\,dt \\
\frac{y}{t} &= \frac{1}{2}t^2 + C \\
y &= t\left(\frac{1}{2}t^2 + C\right) \\
y &= \frac{1}{2}t^3 + Ct.
\end{aligned}$$

Last, plugging in the initial condition $(2, 10)$ gives $10 = \frac{1}{2}(8) + C(2)$ so $C = 3$. Thus the particular solution is $y = \frac{1}{2}t^3 + 3t$.

4.2 Fall 2019 Exam 2

1. a) (3.2) Consider the 2×2 initial value problem

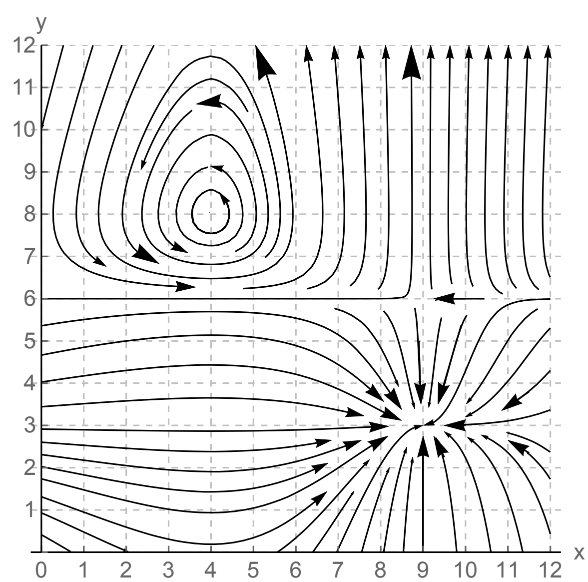
$$\begin{cases} x'(t) = y \\ y'(t) = x \end{cases} \qquad \begin{cases} x(0) = 4 \\ y(0) = -1 \end{cases}.$$

   Use Euler's method with one step to estimate (x, y) when $t = 2$.

   b) (4.7) Find all equilibria of the autonomous 2×2 system of ODEs

$$\begin{cases} x'(t) = x + y \\ y'(t) = y^2 + x - 6 \end{cases}$$

2. Here is the phase plane of a first-order, autonomous 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$:



Use this picture to answer the following questions:

   a) (4.7) Find the coordinates of any stable equilibrium of this system.

   b) (4.7) Find the coordinates of any unstable equilibrium of this system.

   c) (4.8) Which of these statements is true about the eigenvalues of the matrix $D\Phi(4, 8)$?

      A. they are both positive real numbers
      B. they are both negative real numbers
      C. one is a positive real number; the other is negative
      D. they are not real numbers

d) (4.8) Which of these statements is true about the eigenvalues of the matrix $D\Phi(9,3)$?

A. they are both positive real numbers
B. they are both negative real numbers
C. one is a positive real number; the other is negative
D. they are not real numbers

e) (3.7) Suppose $\mathbf{y}(0) = (1,1)$. In this situation, which statement best describes the behavior of $x(t)$?

A. $x(t)$ increases for all t
B. $x(t)$ decreases for all t
C. initially, $x(t)$ is increasing, but then it becomes decreasing
D. initially, $x(t)$ is decreasing, but then it becomes increasing

3. (4.5) Find the particular solution of the following initial value problem:

$$\begin{cases} x' = 5x + 4y \\ y' = -2x + y \end{cases} \qquad \begin{cases} x(0) = 3 \\ y(0) = -2 \end{cases}$$

4. (4.5) Find the particular solution of the following initial value problem:

$$\begin{cases} \mathbf{y}' = (10x - 3y, -6x - 7y) \\ \mathbf{y}(0) = (10,3) \end{cases}$$

5. (4.5) Find the general solution of the following system of ODEs:

$$\begin{cases} x' &= 6x - 4y \\ y' &= x + 2y \end{cases}$$

Write your final answer coordinate-wise.

Solutions

1. a) We are given $\phi_1(x,y) = y$, $\phi_2(x,y) = x$, $x_0 = 4$, $y_0 = -1$, $t_0 = 0$, $t_n = 2$ and $n = 1$. We first figure $\Delta t = \frac{t_n - t_0}{n} = \frac{2-0}{1} = 2$. Then, by Euler's method, we have:
$$\begin{cases} x_1 = x_0 + \phi_1(x,y)\Delta t = 4 + (-1)2 = 2 \\ y_1 = y_0 + \phi_2(x,y)\Delta t = -1 + 4(2) = 7. \end{cases}$$
Therefore, when $t = 2$, $(x,y) \approx (2,7)$.

   b) Thinking of the system as $\mathbf{y}' = \Phi(\mathbf{y})$, we set $\Phi(\mathbf{y}) = \mathbf{0}$ to get
$$\begin{cases} 0 = x + y \\ 0 = y^2 + x - 6 \end{cases}.$$
From the first equation, $x = -y$. Substituting in the second equation, we get $0 = y^2 - y - 6 = (y-3)(y+2)$. Therefore $y = 3$ or $y = -2$. Since $x = -y$, we have the two equilibria $(-3,3)$ and $(2,-2)$.

2. a) The only stable equilibrium is the node at $(9,3)$.

   b) The only unstable equilibrium is the saddle at $(9,6)$.

   c) Since $(4,8)$ is a center, both of the eigenvalues of $D\Phi(4,8)$ are pure imaginary. This makes the answer **D**.

   d) Since $(9,3)$ is a stable node, both of the eigenvalues of $D\Phi(9,3)$ are negative real numbers. This makes the answer **B**.

   e) The curve through $(1,1)$ always moves to the right as t increases, making the answer **A**.

3. Let $A = \begin{pmatrix} 5 & 4 \\ -2 & 1 \end{pmatrix}$. To solve $\mathbf{y}' = A\mathbf{y}$, start with eigenvalues of A:
$$\det(A - \lambda I) = (5-\lambda)(1-\lambda) + 8 = \lambda^2 - 6\lambda + 13 = 0$$
has solution $\lambda = \frac{6 \pm \sqrt{36 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$. Therefore $\alpha = 3$ and $\beta = 2$. Next, we find the eigenvector $\mathbf{v} = (x,y)$ corresponding to $\lambda = 3 + 2i$:
$$A\mathbf{v} = \mathbf{v} \Rightarrow \begin{cases} 5x + 4y = (3+2i)x \\ -2x + y = (3+2i)y \end{cases} \Rightarrow x = (-1-i)y \Rightarrow \mathbf{v} = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$
Therefore $\mathbf{a} = (-1,1)$ and $\mathbf{b} = (-1,0)$. By the theorem from the lecture notes, the general solution is therefore
$$\begin{aligned} \mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{3t}\cos 2t\begin{pmatrix} -1 \\ 1 \end{pmatrix} - e^{3t}\sin 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + C_2\left[e^{3t}\cos 2t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + e^{3t}\sin 2t\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} (-C_1 - C_2)e^{3t}\cos 2t + (C_1 - C_2)e^{3t}\cos 2t \\ C_1 e^{3t}\cos 2t + C_2 e^{3t}\sin 2t \end{pmatrix}. \end{aligned}$$

Now, plug in the initial condition $x(0) = 3, y(0) = -2$ to obtain

$$\begin{cases} 3 = -C_1 - C_2 \\ -2 = C_1 \end{cases} \Rightarrow C_1 = -2, C_2 = -1.$$

Therefore the particular solution is

$$\mathbf{y} = \begin{pmatrix} 3e^{3t}\cos 2t - e^{3t}\cos 2t \\ -2e^{3t}\cos 2t - e^{3t}\sin 2t \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 10 & -3 \\ -6 & -7 \end{pmatrix}$ and start with eigenvalues of A:

$$\det(A - \lambda I) = (10-\lambda)(-7-\lambda) - 18 = \lambda^2 - 3\lambda - 88 = (\lambda - 11)(\lambda + 7) = 0$$

gives eigenvalues $\lambda = 11, \lambda = -8$. Now for the eigenvectors. Let $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ to obtain:

$$\lambda = 11 : \begin{cases} 10x - 3y = 11x \\ -6x - 7y = 11y \end{cases} \Rightarrow -3y = x \Rightarrow \mathbf{v} = (-3, 1).$$

$$\lambda = -8 : \begin{cases} 10x - 3y = -8x \\ -6x - 7y = -8y \end{cases} \Rightarrow y = 6x \Rightarrow \mathbf{v} = (1, 6).$$

Since there are two distinct real eigenvalues, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= C_1 e^{11t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + C_2 e^{-8t}\begin{pmatrix} 1 \\ 6 \end{pmatrix}. \end{aligned}$$

Now plug in the initial condition $\mathbf{y}(0) = (10, 3)$ to get

$$\begin{cases} 10 = -3C_1 + C_2 \\ 3 = C_1 + 6C_2 \end{cases} \Rightarrow C_1 = -3, C_2 = 1.$$

Thus the particular solution is

$$\mathbf{y} = -3e^{11t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + e^{-8t}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 9e^{11t} + e^{-8t} \\ -3e^{11t} + 6e^{-8t} \end{pmatrix}.$$

5. Start with the eigenvalues of $A = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix}$:

$$\det(A - \lambda I) = (6-\lambda)(2-\lambda) + 4 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Therefore $\lambda = 4$ is a repeated eigenvalue. Next, find an eigenvector $\mathbf{v} = (x, y)$ by solving $A\mathbf{v} = \lambda\mathbf{v}$ to obtain

$$\begin{cases} 6x - 4y = 4x \\ x + 2y = 4y \end{cases} \Rightarrow x = 2y \Rightarrow \mathbf{v} = (2, 1).$$

Next, find an generalized eigenvector $\mathbf{w} = (x, y)$ by solving $(A - 4I)\mathbf{w} = \mathbf{v}$ to get

$$\begin{cases} 2x - 4y = 2 \\ x - 2y = 1 \end{cases} \Rightarrow x = 2y + 1 \Rightarrow \mathbf{w} = (1, 0).$$

Now by the theorem from the lecture notes, the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y} &= C_1 e^{\lambda t}\mathbf{v} + C_2\left[e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}\right] \\ &= C_1 e^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2\left[e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] \end{aligned}$$

which when written coordinate-wise, is

$$\begin{cases} x(t) &= (2C_1 + C_2)e^{4t} + 2C_2 te^{4t} \\ y(t) &= C_1 e^{4t} + C_2 te^{4t} \end{cases}$$

4.3 Fall 2019 Final Exam

1. (3.6) David McClendon gives his students a 2×2 initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \mathbf{y}_0$ to work on in class. He then walks around holding a copy of his (correct) solution, checking the students' work as they make progress.

   a) Suppose Dr. McClendon observes a student who has obtained different eigenvalues of A than he did. Is this student necessarily wrong?

   b) Suppose Dr. McClendon sees a student with the same eigenvalues he got, but different eigenvectors. Is this student necessarily wrong?

   c) Suppose Dr. McClendon sees a students' general solution of the ODE, which doesn't look the same as his. Is this student necessarily wrong?

   d) Suppose Dr. McClendon sees that a student has obtained a different C_1 and C_2 than he did. Is this student necessarily wrong?

   e) Suppose that Dr. McClendon sees that a student has obtained a different-looking, simplified, particular solution of the initial value problem. Is this student necessarily wrong?

2. a) (4.3) State Euler's Formula.

   b) (4.3) Explain the importance of Euler's Formula, in the context of differential equations.

3. (4.8) In each part (a)-(f) of this problem, you are given a 2×2 constant-coefficient, linear homogeneous system of differential equations. Write the letter (A through I) of the picture below which gives the phase plane of that system.

   a) $\begin{cases} x' = 3x \\ y' = y \end{cases}$

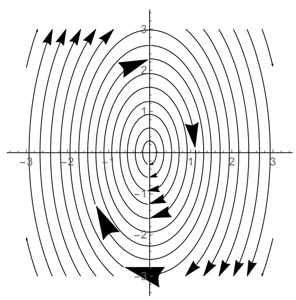
   b) $\begin{cases} x' = x \\ y' = 3y \end{cases}$

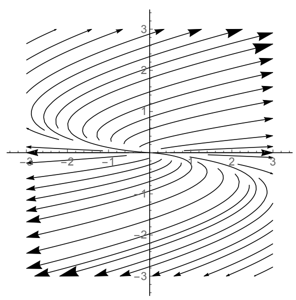
   c) $\begin{cases} x' = y \\ y' = 3x \end{cases}$

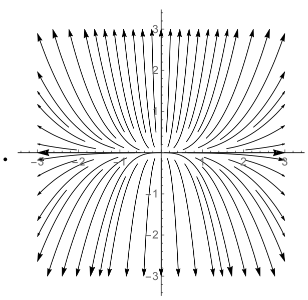
   d) $\begin{cases} x' = y \\ y' = -3x \end{cases}$

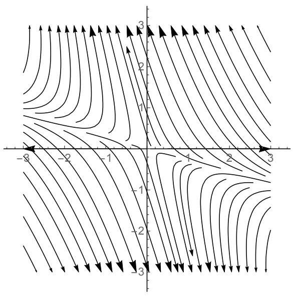
   e) $\begin{cases} x' = x + 3y \\ y' = y \end{cases}$

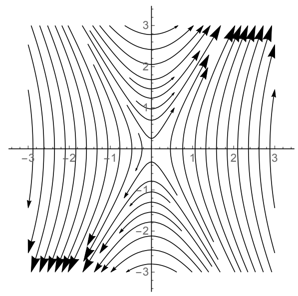
   f) $\begin{cases} x' = x - 3y \\ y' = x + y \end{cases}$

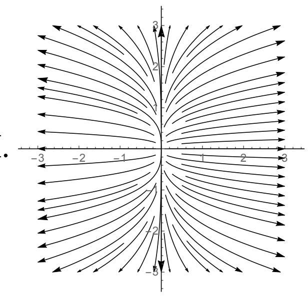
A.


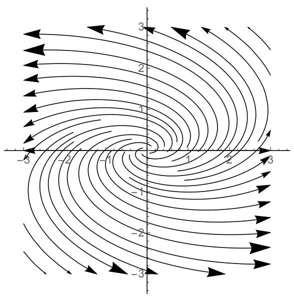
B.


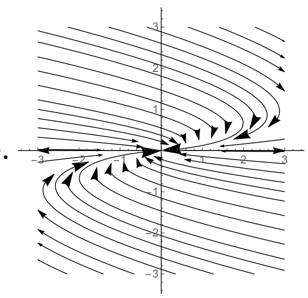
C.


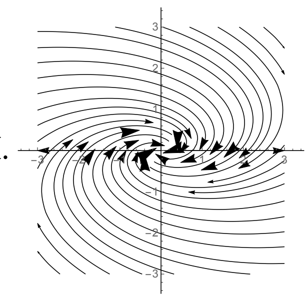
D. 

F. 

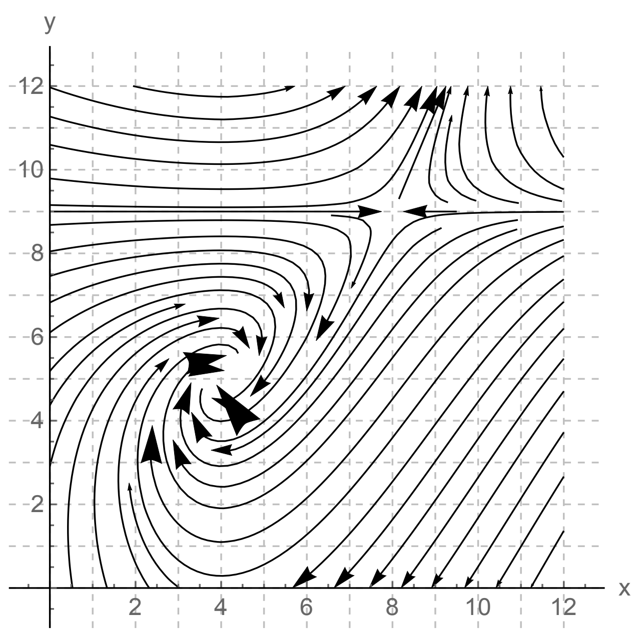
H. 

E. 

G. 

I. 

4. The phase plane of a first-order, autonomous 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$ is given below. Use the picture to answer the questions posed in this problem.



a) (4.7) Identify the location of any saddles of the system (if there are none, say so).

b) (4.7) Identify the location of any nodes of the system (if there are none, say so).

c) (3.7) Suppose $\mathbf{y}(t) = (x(t), y(t))$ is the solution of $\mathbf{y}' = \Phi(\mathbf{y})$ satisfying $\mathbf{y}(0) = (10, 4)$.

i. Find $\lim_{t\to\infty} x(t)$.
ii. Find $\lim_{t\to-\infty} y(t)$.
iii. Estimate $\left.\frac{dy}{dx}\right|_{t=0}$.
iv. Which one of these statements is true?
   A. At time 0, x is increasing but y is decreasing.
   B. At time 0, x and y are both increasing.
   C. At time 0, x is decreasing but y is increasing.
   D. At time 0, both x and y are decreasing.

5. (1.7) Sketch the phase line associated to this autonomous differential equation:
$$y' = y^2 - 8y + 15$$

6. (2.2 or 2.3) Find the general solution of this differential equation:
$$\frac{dy}{dt} = 8\sin 2t - 4y$$

7. (2.4) Find the particular solution of this initial value problem:
$$\begin{cases} \frac{dy}{dt} = te^{-2y} \\ y(2) = 0 \end{cases}$$
Write your answer as a function $y = f(t)$.

8. (5.2) Find the particular solution of this initial value problem:
$$\begin{cases} y''' - y' = 0 \\ y(0) = 2 \\ y'(0) = -1 \\ y''(0) = 0 \end{cases}$$

9. (5.2) Find the general solution of this ODE:
$$y''(t) - 3y'(t) - 10y(t) = 98e^{5t}$$

10. (4.6) Find the general solution of this system, and write your answer coordinate-wise:
$$\begin{cases} x' = \phantom{6x + {}} 2y + 7e^{2t} \\ y' = 6x + 4y + 3e^{2t} \end{cases}$$

11. (4.5) Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (5x + y, -2x + 3y) \\ \mathbf{y}(0) = (-2, 3) \end{cases}$$

Simplify your final answer by combining like terms.

12. (4.5) Find the particular solution of this initial value problem, and write your answer coordinate-wise:

$$\begin{cases} x' = 11x + 2y \\ y' = -8x + 3y \end{cases} \qquad \begin{cases} x(0) = 2 \\ y(0) = 1 \end{cases}$$

13. (2.5) Suppose that on an alien planet, the acceleration due to gravity is exactly 12 m/sec^2 (on Earth, the acceleration is 9.8 m/sec^2). An alien who lives on this world drops an object of mass 14 kg out the window of his flying spacecraft, and the object falls to the planet's surface. Assume that no forces act on the object other than gravity and air resistance, and that the drag coefficient of this object in the atmosphere of the alien world is 2 N sec/m.

    a) Write down an initial value problem that models the velocity of the object.
    b) Find the terminal velocity of the object.
    c) Find the exact velocity of the object one second after the alien drops it.

14. (5.3) A 1 Ω resistor, 3 H inductor and a 6 F capacitor are hooked up in series with a voltage source to form an RLC circuit. At all times, the voltage source supplies 2 V to the circuit. If at time 0, the charge across the resistor is 10 coulombs, but there is no current running through the resistor, find the charge in the circuit at time t.

Solutions

1. a) **YES.** The eigenvalues of a matrix are determined by the matrix.
   b) **NO.** The student might have chosen nonzero multiples of the ones Dr. McClendon did.
   c) **NO.** The student could have different eigenvectors, the solution could look different.
   d) **NO.** A different general solution would give rise to different C_1 and C_2.
   e) **YES.** By the existence-uniqueness theorem, there is one and only one solution.

2. a) For any complex number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.
   b) Euler's Formula is used in differential equations to derive the formula that allows us to write real solutions (involving cosines and sines) to systems or higher-order equations which have non-real eigenvalues.

3. a) The matrix $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ has two positive, real eigenvalues, so $(0,0)$ is an unstable node. Since the eigenvalue in the x-direction is greater, the solutions will move more rapidly horizontally than vertically. This gives picture **H**.
   b) The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ has two positive, real eigenvalues, so $(0,0)$ is an unstable node. Since the eigenvalue in the y-direction is greater, the solutions will move more rapidly vertically than horizontally. This gives picture **C**.
   c) The matrix $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ has determinant -3, so $(0,0)$ is a saddle, making the only possible answers D and F. Notice that $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (1,0)$, so at $(0,1)$, the vector field points to the right. Thus the correct answer is **F**.
   d) The matrix $\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$ has trace 0 and determinant 3, so $(0,0)$ is a center; the only center is **A**.
   e) The matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ has a repeated real eigenvalue $\lambda = 1$, so $(0,0)$ is an unstable node with one straight-line passing through it, which must be picture **B**.

f) The matrix $\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$ has trace $t = 2$ and determinant $d = 4$. Since $d > \frac{1}{4}t^2$, $(0, 0)$ is an unstable spiral, which must be **E**.

4. a) The only saddle is at $(8, 9)$.

   b) This system has no nodes.

   c) i. Find $\lim_{t\to\infty} x(t) = 4$, since the solution spirals inward towards $(4, 5)$.

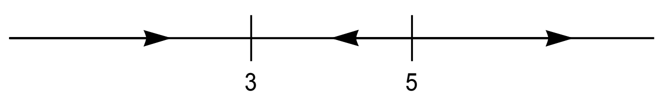
      ii. Find $\lim_{t\to-\infty} y(t) = 9$, since the solution comes from height $y = 9$.

      iii. Estimate $\frac{dy}{dx}\Big|_{t=0} \approx 1$, since the slope of the line tangent to the solution curve near $(10, 4)$ is about 1.

      iv. Since the curve moves down and to the left in the direction of increasing t, the answer is **D**.

5. Thinking of the equation as $y' = \phi(y)$, set $\phi(y) = 0$: this gives $(y-3)(y-5) = 0$, so $y = 3$ and $y = 5$ are the two equilibria.

   To classify these equilibria, differentiate ϕ to get $\phi'(y) = 2y - 8$. Then $\phi'(3) = -2 < 0$ so $y = 3$ is stable, but $\phi'(5) = 2 > 0$ so $y = 5$ is unstable. This gives the following phase line:



6. Rewrite the equation as $y' + 4y = 8\sin 2t$. The corresponding homogeneous equation is $y' + 4y = 0$ which has solution $y_h = e^{-4t}$. To find a particular solution, use undetermined coefficients. Guess $y_p = A\sin 2t + B\cos 2t$. Plugging in, this gives

$$\begin{aligned} (2A\cos 2t - 2B\sin 2t) + 4(A\sin 2t + B\cos 2t) &= 8\sin 2t \\ (2A + 4B)\cos 2t + (4A - 2B)\sin 2t &= 8\sin 2t \end{aligned}$$

$$\begin{cases} 2A + 4B &= 0 \\ 4A - 2B &= 8 \end{cases} \Rightarrow B = \frac{-4}{5}, A = \frac{8}{5}.$$

   Thus $y_p = \frac{8}{5}\sin 2t - \frac{4}{t}\cos 2t$. Finally, the general solution is

$$y = y_p + Cy_h = \frac{8}{5}\sin 2t - \frac{4}{5}\cos 2t + Ce^{-4t}.$$

7. This equation is separable; rewrite it as $e^{2y}\,dy = t\,dt$ and integrate both sides to get $\frac{1}{2}e^{2y} = \frac{1}{2}t^2 + C$. Plug in the initial condition $t = 2$, $y = 0$ to get $\frac{1}{2} = 2 + C$, so $C = -\frac{3}{2}$. Thus the particular solution is $\frac{1}{2}e^{2y} = \frac{1}{2}t^2 - \frac{3}{2}$. To solve for y,

multiply through by 2 to get $e^{2y} = t^2 - 3$; then take the natural log of both sides and last, divide through by 2 to get

$$y = \frac{1}{2}\ln(t^2 - 3).$$

8. Start by factoring the characteristic equation $\lambda^3 - \lambda = 0$ to get $\lambda(\lambda-1)(\lambda+1) = 0$. Thus the general solution is $y = C_1 + C_2e^t + C_3e^{-t}$. To find the particular solution, differentiate it and plug in the initial value:

$$\begin{cases} y = C_1 + C_2e^t + C_3e^{-t} \\ y' = C_2e^t - C_3e^{-t} \\ y'' = C_2e^t + C_3e^{-t} \end{cases} . \text{ So} \begin{cases} y(0) = 2 \\ y'(0) = -1 \\ y''(0) = 0 \end{cases} \text{ implies } \begin{cases} C_1 + C_2 + C_3 = 2 \\ C_2 - C_3 = -1 \\ C_2 + C_3 = 0 \end{cases}$$

From this, we get $C_1 = 2, C_2 = -\frac{1}{2}, C_3 = \frac{1}{2}$, so the particular solution is

$$y = 2 - \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$

9. Start with the characteristic equation $\lambda^2 - 3\lambda - 10 = 0$; factor to get $(\lambda-5)(\lambda+2)$ so the general solution of the homogeneous is $y_h = C_1e^{5t} + C_2e^{-2t}$. For the particular solution, since e^{5t} is already part of the solution of the homogeneous, guess $y_p = Ate^{5t}$. Plugging this in the original equation gives

$$\begin{aligned} y_p'' - 3y_p' - 10y_p &= 98e^{5t} \\ (10Ae^{5t} + 25Ate^{5t}) - 3(Ae^{5t} + 5Ate^{5t}) - 10Ate^{5t} &= 98e^{5t} \\ 7Ae^{5t} &= 98e^{5t} \\ 7A &= 98 \\ A &= 14 \end{aligned}$$

So $y_p = 14te^{5t}$, making the general solution of the original equation

$$y = y_p + y_h = 14te^{5t} + C_1e^{5t} + C_2e^{-2t}.$$

10. Write $A = \begin{pmatrix} 0 & 2 \\ 6 & 4 \end{pmatrix}$ and start by finding eigenvalues of A. $\det(A - \lambda I) = (-\lambda)(4-\lambda) - 12 = \lambda^2 - 4\lambda - 12 = (\lambda-6)(\lambda+2)$ so the eigenvalues are $\lambda = 6$ and $\lambda = -2$. Now for the eigenvectors: write $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ to get:

$$\lambda = 6 : \begin{cases} 2y &= 6x \\ 6x + 4y &= 6y \end{cases} \Rightarrow y = 3x \Rightarrow (1, 3)$$

$$\lambda = -2 : \begin{cases} 2y &= -2x \\ 6x + 4y &= -2y \end{cases} \Rightarrow y = -x \Rightarrow (1, -1)$$

So the general solution of the homogeneous is

$$\mathbf{y}_h = C_1 e^{6t}\begin{pmatrix}1\\3\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Now for the non-homogeneous part. Guess $\mathbf{y}_p = (Ae^{2t}, Be^{2t})$ and plug in to the original system to get

$$\begin{cases} 2Ae^{2t} & = 2Be^{2t} + 7e^{2t} \\ 2Be^{2t} & = 6Ae^{2t} + 4Be^{2t} + 3 \end{cases} \Rightarrow \begin{cases} 2A & = 2B + 7 \\ 2B & = 6A + 4B + 3 \end{cases} \Rightarrow A = \frac{1}{2}, B = -3.$$

So the particular solution is $\mathbf{y}_p = \left(\frac{1}{2}e^{2t}, -3e^{2t}\right)$, making the solution of the original equation

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h = \begin{pmatrix}\frac{1}{2}e^{2t}\\-3e^{2t}\end{pmatrix} + C_1 e^{6t}\begin{pmatrix}1\\3\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Written coordinate-wise, this is

$$\begin{cases} x(t) = \frac{1}{2}e^{2t} + C_1 e^{6t} + C_2 e^{-2t} \\ y(t) = -3e^{-2t} + 3C_1 e^{6t} - C_2 e^{-2t}. \end{cases}$$

11. Write $A = \begin{pmatrix}5 & 1\\-2 & 3\end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$, then start with eigenvalues of A. $\det(A - \lambda I) = (5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17 = 0$ when $\lambda = \frac{8\pm\sqrt{64-4(17)}}{2} = \frac{8\pm\sqrt{-4}}{2} = 4 \pm i$. Now for the eigenvector corresponding to $\lambda = 4+i$ (to find this, set $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$):

$$\lambda = 4+i : \begin{cases} 5x + y & = (4+i)x \\ -2x + 3y & = (4+i)y \end{cases} \Rightarrow y = (-1+i)x \Rightarrow \mathbf{v} = (1, -1+i) = (1,-1) + i(0,1)$$

So by setting $\mathbf{a} = (1,-1)$, $\mathbf{b} = (0,1)$, $\alpha = 4$ and $\beta = 1$ in the theorem from class, we have the general solution

$$\mathbf{y} = C_1 e^{4t}\left[\cos t\begin{pmatrix}1\\-1\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix}\right] + C_2 e^{4t}\left[\cos t\begin{pmatrix}0\\1\end{pmatrix} + \sin t\begin{pmatrix}1\\-1\end{pmatrix}\right].$$

Plugging in the initial condition $\mathbf{y}(0) = (-2, 3)$ gives

$$\begin{cases} -2 & = C_1 \\ 3 & = -C_1 + C_2 \end{cases} \Rightarrow C_1 = -2, C_2 = 1.$$

Thus the particular solution is

$$\begin{aligned}\mathbf{y} &= -2e^{4t}\left[\cos t\begin{pmatrix}1\\-1\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix}\right] + e^{4t}\left[\cos t\begin{pmatrix}0\\1\end{pmatrix} + \sin t\begin{pmatrix}1\\-1\end{pmatrix}\right] \\ &= \begin{pmatrix}-2e^{4t}\cos t + e^{4t}\sin t\\ 3e^{4t}\cos t + e^{4t}\sin t\end{pmatrix}.\end{aligned}$$

12. Write $A = \begin{pmatrix} 11 & 2 \\ -8 & 3 \end{pmatrix}$ so that the system is $\mathbf{y}' = A\mathbf{y}$, then start with eigenvalues of A. $\det(A - \lambda I) = (11-\lambda)(3-\lambda) + 16 = \lambda^2 - 14\lambda + 49 = (\lambda - 7)^2$ so $\lambda = 7$ is a repeated eigenvalue. Next, find an eigenvector:

$$\begin{cases} 11x + 2y &= 7x \\ -8x + 3y &= 7y \end{cases} \Rightarrow 2y = -4x \Rightarrow \mathbf{v} = (1, -2).$$

Now, for a generalized eigenvector. Set $(A - 7I)\mathbf{w} = \mathbf{v}$ and solve for $\mathbf{w}$ to get

$$\begin{cases} 4x + 2y &= 1 \\ -8x - 4y &= -2 \end{cases} \Rightarrow 4x + 2y = 1 \Rightarrow \mathbf{w} = \left(0, \frac{1}{2}\right).$$

Now, from the theorem in class, we have the general solution

$$\mathbf{y} = C_1 e^{7t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 \left[te^{7t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{7t} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right].$$

Now, plug in the initial condition $\mathbf{y}(0) = (2, 1)$ to get

$$\begin{cases} 2 &= C_1 \\ 1 &= -2C_1 + \frac{1}{2}C_2 \end{cases} \Rightarrow C_1 = 2, C_2 = 10$$

So the particular solution is

$$\begin{aligned} \mathbf{y} &= 2e^{7t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 10 \left[te^{7t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{7t} \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \right] \\ &= \begin{pmatrix} 2e^{7t} + 10te^{7t} \\ e^{7t} - 20te^{7t} \end{pmatrix}. \end{aligned}$$

Written coordinate-wise, this is

$$\begin{cases} x(t) &= 2e^{7t} + 10te^{7t} \\ y(t) &= e^{7t} - 20te^{7t} \end{cases}.$$

13. a) Let the velocity of the object at time t be $v(t)$. Since the object is dropped (as opposed to flung downward), we have the initial condition $v(0) = 0$. The forces acting on the object are gravity $mg = 14(12) = 168$ and drag $-\beta v = -2v$, so by Newton's Second Law we have

$$\begin{aligned} ma &= \sum F \\ mv' &= mg - \beta v \\ 14v' &= 14(12) - 2v \\ v' &= 12 - \frac{1}{7}v. \end{aligned}$$

This gives the IVP $\begin{cases} v' = 12 - \frac{1}{7}v \\ v(0) = 0 \end{cases}$.

b) The terminal velocity corresponds to the equilibrium of this system, which is when $12 - \frac{1}{7}v = 0$, i.e. $v = 84$ m/sec.

c) Let's start by solving the ODE $v' = 12 - \frac{1}{7}v$. Rewrite this as $v' + \frac{1}{7}v = 12$; the corresponding homogeneous equation $v' + \frac{1}{7}v = 0$ has solution $v_h = e^{-t/7}$. Guess the particular solution $v_p = A$; plugging in gives $\frac{1}{7}A = 12$ so $A = 84$. Putting this together, the general solution of the ODE is $y = y_p + Cy_h = 84 + Ce^{t/7}$.
Now, plug in the initial condition $v(0) = 0$ to get $0 = 84 + C(1)$, i.e. $C = -84$. This gives the particular solution $v(t) = 84 - 84e^{-t/7}$. Finally, the answer to the question is $v(1) = 84 - 84e^{-1/7}$ m/sec.

14. We are given $R = 1$, $L = 3$, $C = 6$ and $E_S(t) = 2$; the equation for RLC circuits in the notes gives

$$\begin{aligned} Lq'' + Rq' + \frac{1}{C}q &= E_S(t) \\ 3q'' + q' + \frac{1}{6}q &= 2 \end{aligned}$$

where $q = q(t)$ is the charge at time t. To solve this, first solve the characteristic equation $3\lambda^2 + \lambda + \frac{1}{6} = 0$. By the quadratic formula, this has solution

$$\lambda = \frac{-1 \pm \sqrt{1 - 4(3)(\frac{1}{6})}}{2(3)} = \frac{-1 \pm i}{6} = \frac{-1}{6} \pm i\frac{1}{6}.$$

So the general solution of the homogeneous is

$$q_h = C_1 e^{-t/6}\cos\frac{t}{6} + C_2 e^{-t/6}\sin\frac{t}{6}.$$

Next, guess the solution of the homogeneous as $q_p = A$. Plugging in, we get $\frac{1}{6}A = 2$ so $A = 12$. This makes the general solution of the equation $q = q_p + q_h = 12 + C_1 e^{-t/6}\cos\frac{t}{6} + C_2 e^{-t/6}\sin\frac{t}{6}$.

Now for the particular solution. We are given $q(0) = 10$ and $q'(0) = 0$. Differentiating our general solution, we get

$$\begin{cases} q(t) &= 12 + C_1 e^{-t/6}\cos\frac{t}{6} + C_2 e^{-t/6}\sin\frac{t}{6} \\ q'(t) &= \frac{-1}{6}C_1 e^{-t/6}\cos\frac{t}{6} - \frac{1}{6}C_1 e^{-t/6}\sin\frac{t}{6} - \frac{1}{6}C_2 e^{-t/6}\sin\frac{t}{6} + \frac{1}{6}C_2 e^{-t/6}\cos\frac{t}{6} \end{cases}$$

and plugging in the initial conditions, we get

$$\begin{cases} 10 &= 12 + C_1 \\ 0 &= \frac{-1}{6}C_1 + \frac{1}{6}C_2 \end{cases} \Rightarrow C_1 = -2, C_2 = -2.$$

Thus the charge in the circuit at time t is

$$q(t) = 12 - 2e^{-t/6}\cos\frac{t}{6} - 2e^{-t/6}\sin\frac{t}{6}.$$

Chapter 5

Exams from Fall 2023

5.1 Fall 2023 Exam 1

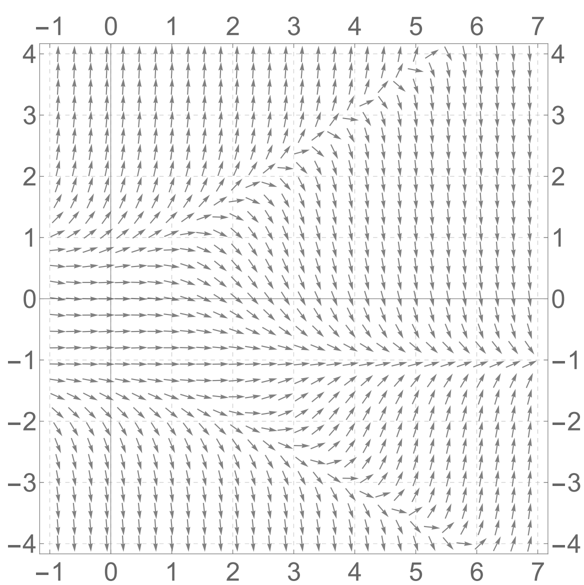
1. (2.1) Write down any example of a differential equation which is second-order, linear, and homogeneous, but not constant-coefficient.

2. (1.5) Let $y = f(t)$ be the solution of the initial value problem

$$\begin{cases} y' = 2y + t + 2 \\ y(-1) = -4 \end{cases}.$$

   Estimate $f(3)$ by performing Euler's method with two steps.

3. Here is the picture of the slope field associated to a first-order ODE $y' = \phi(t, y)$:



a) (1.4) Write the explicit equation of one solution of this differential equation.

b) (1.4) On the picture above, sketch the solution to this differential equation satisfying the initial value $y(4) = -2$.

c) Classify each statement as true or false:

   i. (1.7) This ODE is autonomous.

   ii. (1.4) If you know $y(0)$ is between 0 and 1, then you can predict $\lim_{t\to\infty} y(t)$ accurately.

   iii. (1.4) The solution to this ODE satisfying $y(4) = 1$ is decreasing for all t.

   iv. (1.4) If $y = f(t)$ is the solution to this ODE satisfying $f(2) = 1$, then $f(3) \approx 0$.

   v. (1.4) $\phi(4, 1) > 0$.

4. (2.4) Find the particular solution of the following initial value problem:

$$\begin{cases} y' = \dfrac{\sqrt{t}}{y^2+1} \\ y(1) = 0 \end{cases}$$

5. (2.2) Find the particular solution of the following initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{t^4 - y}{t} \\ y(1) = 2 \end{cases}$$

   Write your answer as a function $y = f(t)$.

6. (2.3) Find the general solution of $y' - 3y = 5\sin t$.

7. (2.4) Find the general solution of $\dfrac{dy}{dt} = y^2$. Write your answer in the form $y = f(t)$.

8. (2.5) Suppose you are studying a model for some positive quantity $q(t)$ which is described by the following compartmental model (here, r and s are positive constants):

$$\xrightarrow{\;r\,q\;} \boxed{\;q(t)\;} \xrightarrow{\;s\,q^2\;}$$

   In terms of r and/or s, determine $\lim_{t\to\infty} q(t)$.

9. **(Bonus)** Find the general solution of the following ODE:

$$y' = e^t(e^t - y)$$

Solutions

1. Answers may vary here, but one simple example would be $\boxed{ty'' + y' = 0}$.

2. First, $\Delta t = \dfrac{t_n - t_0}{n} = \dfrac{3-(-1)}{2} = 2$. Next, we are given $(t_0, y_0) = (-1, -4)$. Now $\phi(t_0, y_0) = 2(-4) + (-1) + 2 = -7$, so

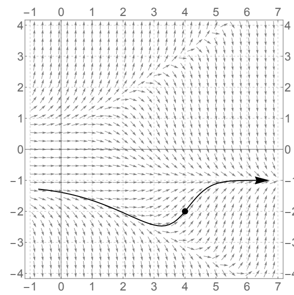
$$\begin{aligned} t_1 &= t_0 + \Delta t = -1 + 2 = 1 \\ y_1 &= y_0 + \phi(t_0, y_0)\Delta t = -4 + -7(2) = -4 - 14 = -18. \end{aligned}$$

   Therefore $(t_1, y_1) = (1, -18)$. Now $\phi(t_1, y_1) = 2(-18) + (1) + 2 = -33$, so

$$\begin{aligned} t_2 &= t_1 + \Delta t = 1 + 2 = 3 \\ y_2 &= y_1 + \phi(t_1, y_1)\Delta t = -18 + (-33)2 = -18 - 66 = -84. \end{aligned}$$

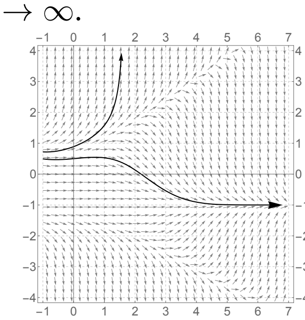
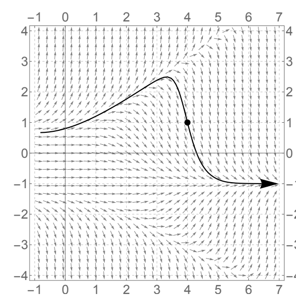
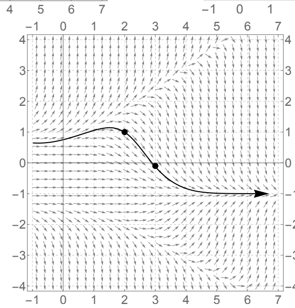
   Thus $\boxed{y(3) \approx -84}$.

3. a) $\boxed{y = -1}$ is a horizontal line which appears to be a solution of the ODE.

b) Start at $(4, -2)$ and flow with the vector field: 

c) i. **FALSE:**. If the ODE was autonomous (had the form $y' = \phi(y)$), then the slopes would have to be the same across every horizontal line in the slope field. Clearly that's not the case here

ii. **FALSE:**. Consider the two solutions shown below at left, which satisfy $y(0) = \frac{1}{2}$ and $y(0) = \frac{8}{9}$. One goes to -1 as $t \to \infty$; the other goes to ∞ as $t \to \infty$.

iii. **FALSE:** From the middle picture above, the solution through $(4, 1)$ is increasing when $t < 3$.

iv. **TRUE:**. See the picture above at right for the solution passing through $(2, 1)$.

v. **FALSE:**. At the point $(4, 1)$, the arrow points down to the right, indicating that the slope $y' = \phi(4, 1)$ is negative.

4. This equation is not linear, so we separate variables to get $(y^2 + 1)\,dy = \sqrt{t}\,dt$. Now, integrate both sides to get the general solution $\dfrac{1}{3}y^3 + y = \dfrac{2}{3}t^{3/2} + C$. Next, plug in the initial condition $(1, 0)$ to the general solution to get $0 + 0 = \dfrac{2}{3} + C$,

i.e. $C = -\frac{2}{3}$. All together, the particular solution is

$$\boxed{\frac{1}{3}y^3 + y = \frac{2}{3}t^{3/2} - \frac{2}{3}}.$$

5. This equation is linear, but not constant-coefficient, so we use integrating factors. First, split the fraction on the right to write the equation as $\dfrac{dy}{dt} = t^3 - \dfrac{1}{t}y$. Then, add $\dfrac{1}{t}y$ to both sides to get $\frac{dy}{dt} + \frac{1}{t}y = t^3$. Now, the integrating factor is $\mu(t) = \exp\left(\int \frac{1}{t}\,dt\right) = e^{\ln t} = t$. Multiply through by the integrating factor to make the equation $\dfrac{d}{dt}\left[y\,t\right] = t^4$. Integrate both sides to get $yt = \frac{1}{5}t^5 + C$, and divide through by t to get the general solution $y = \frac{1}{5}t^4 + Ct^{-1}$. Finally, the initial condition is $y(1) = 2$, which yields $2 = \frac{1}{5} + C$, i.e. $C = \frac{9}{5}$. All together, the particular solution is

$$\boxed{y = \frac{1}{5}t^4 + \frac{9}{5}t^{-1}}.$$

6. This equation is first-order linear with constant coefficients, so we use undetermined coefficients.

   The corresponding homogeneous equation is $y' - 3y = 0$, which has solution $y_h = e^{3t}$ (exponential growth model).

   To find y_p, we guess $y_p = A\sin t + B\cos t$. Plug this into the original differential equation to get

$$\begin{aligned} (A\sin t + B\cos t)' - 3(A\sin t + B\cos t) &= 5\sin t \\ A\cos t - B\sin t - 3A\sin t - 3B\cos t &= 5\sin t \\ (A - 3B)\cos t + (-B - 3A)\sin t &= 5\sin t + 0\cos t \end{aligned}$$

   Equating the coefficients on $\sin t$ and $\cos t$, this gives the system of equations

$$\begin{cases} A - 3B &= 0 \\ -B - 3A &= 5 \end{cases} \Rightarrow A = -\frac{3}{2}, B = -\frac{1}{2}.$$

   Therefore $y_p = -\frac{3}{2}\sin t - \frac{1}{2}\cos t$, making the general solution of the ODE $y = y_p + Cy_h$, i.e.

$$\boxed{y = -\frac{3}{2}\sin t - \frac{1}{2}\cos t + Ce^{3t}}.$$

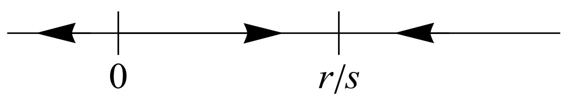
7. This equation is separable: first, separate the variables by dividing through by y^2 and multiplying through by dt to get $\frac{1}{y^2}dy = dt$. Then integrate both

sides to get $\dfrac{-1}{y} = t + C$. Finally, solve for y: take the reciprocal of both sides to get $-y = \dfrac{1}{t+C}$, then multiply through by (-1) to get the general solution, which is $\boxed{y = \dfrac{-1}{t+C}}$.

8. Using the concept that $\dfrac{dq}{dt} = (\text{rate in}) - (\text{rate out})$, we obtain the ODE $\dfrac{dq}{dt} = rq - sq^2$. This equation is autonomous, so we can determine $\lim\limits_{t\to\infty} q(t)$ by finding and classfying the equilibria.

   First, find the equilibria by setting $rq - sq^2 = 0$. Solve by factoring to get $q(r - sq) = 0$, which gives $q = 0$ and $q = \dfrac{r}{s}$ as the equilibria.

   Next, classify the equilibria: thinking of the equation as $\dfrac{dq}{dt} = \phi(q)$, we have $\phi'(q) = r - 2qs$. When $q = 0$, $\phi'(0) = r > 0$ so 0 is unstable. When $q = \dfrac{r}{s}$, $\phi'\left(\dfrac{r}{s}\right) = r - 2\left(\dfrac{r}{s}\right)s = -r < 0$, so $\frac{r}{s}$ is stable. This gives the phase line



   which tells us that (since we are told that q is positive) that $\lim\limits_{t\to\infty} q(t) = \boxed{\dfrac{r}{s}}$.

9. **(Bonus)** This is first-order linear but not constant-coefficient, so we solve with integrating factors. First, distribute the e^t on the right-hand side and then add $e^t y$ to both sides to get $y' + e^t y = e^{2t}$. Now, the integrating factor is $\mu(t) = \exp\left(\int e^t\,dt\right) = e^{e^t}$. Multiplying through by μ gives

$$\frac{d}{dt}\left[ye^{e^t}\right] = e^{e^t}e^{2t}.$$

   Now integrate both sides (integrating the right-hand side here is the hard part). For the right-hand side, write e^{2t} as $e^t e^t$ and then use the u-sub $u = e^t$, $du = e^t\,dt$ to get the right-hand side as

$$\int e^{e^t}e^{2t}\,dt = \int e^{e^t}e^t e^t\,dt = \int e^u u\,du.$$

   Now, you use parts on the integral $\int e^u u\,du$ with $r = u$ and $ds = e^u\,du$; eventually this leads to $ue^u - e^u + C$. Back-substitute for t to get $e^t e^{e^t} - e^{e^t} + C$ on the right-hand side, which means the general solution is

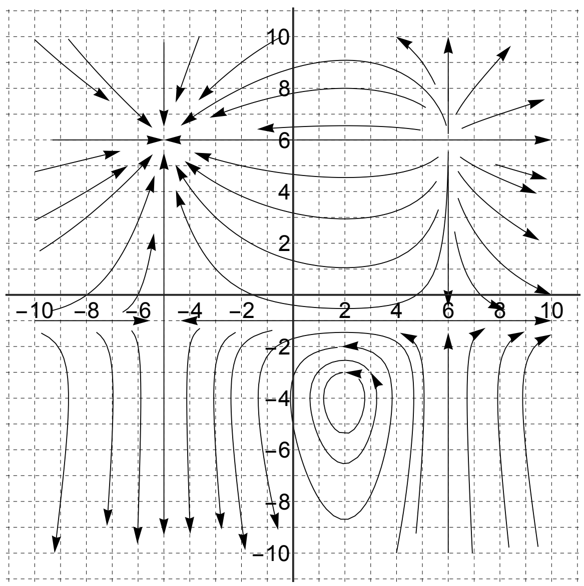
$$\boxed{ye^{e^t} = e^t e^{e^t} - e^{e^t} + C}, \text{ a.k.a. } \boxed{y = e^t - 1 - Ce^{-e^t}}.$$

5.2 Fall 2023 Exam 2

1. (4.7) Find and classify all equilibria of this system:

$$\begin{cases} x' &= x - 2y + 3 \\ y' &= y^2 - x^2 \end{cases}$$

2. The phase plane associated to some 2×2 system $\mathbf{y}' = \Phi(\mathbf{y})$ is shown below:



Use this picture to answer the following questions:

a) (4.7) Give the coordinates of any saddle of the system.

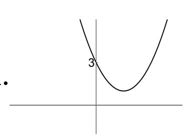
b) (4.7) Give the coordinates of any stable node of the system.

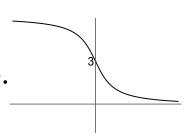
c) (4.7) Give the coordinates of any unstable node of the system.

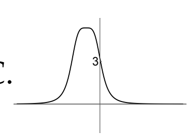
d) (4.7) Give the coordinates of any center of the system.

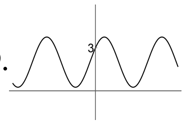
e) (3.7) Suppose $\mathbf{y}(0) = (4, 9)$. What is $\lim_{t \to \infty} x(t)$?

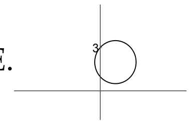
f) (3.7) Suppose $\mathbf{y}(0) = (3, -6)$. Which of these best represents the graph of $x(t)$?

A. 

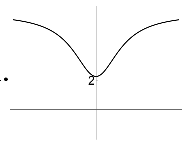
B. 

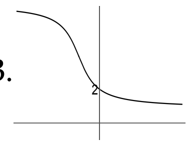
C. 

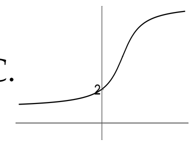
D. 

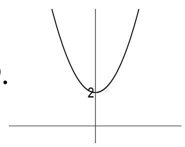
E. 

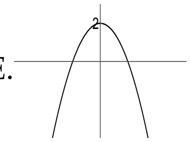
g) (3.7) Suppose $\mathbf{y}(0) = (1, 2)$. Which of these best represents the graph of $y(t)$?

A. 

B. 

C. 

D. 

E. 

3. (4.5) Find the particular solution of this initial value problem:

$$\begin{cases} \mathbf{y}' = (-8x + 3y, -6x + y) \\ \mathbf{y}(0) = (-3, 2) \end{cases}$$

4. (4.5) Find the general solution of this system of ODEs:

$$\begin{cases} x' &= -3x - 13y \\ y' &= x + 3y \end{cases}$$

5. (4.6) Find the general solution of this system of ODEs:

$$\begin{cases} x'(t) &= 5x(t) + y(t) - 13e^{-2t} \\ y'(t) &= -x(t) + 3y(t) + 7e^{-2t} \end{cases}$$

Solutions

1. Thinking of the system as $\mathbf{y}' = \Phi(\mathbf{y})$, we set $\Phi(\mathbf{y}) = \mathbf{0}$ and solve for $\mathbf{y} = (x, y)$:

$$\begin{cases} 0 &= x - 2y + 3 \\ 0 &= y^2 - x^2 = (y-x)(y+x) \end{cases}$$

From the second equation, $y = x$ or $y = -x$. If $y = x$, then the first equation is $0 = x - 2x + 3$, i.e. $x = 3$, leading to the equilibrium $(3, 3)$. But if $y = -x$, then the first equation is $0 = x - 2(-x) + 3$, i.e. $3x + 3 = 0$, i.e. $x = -1$, leading to the equilibrium $(-1, 1)$.

To classify the equilibria, we compute the total derivative $D\Phi = \begin{pmatrix} 1 & -2 \\ -2x & 2y \end{pmatrix}$. Now,

$$D\Phi(3,3) = \begin{pmatrix} 1 & -2 \\ -6 & 6 \end{pmatrix} \text{ has trace } 7 \text{ and determinant } -6;$$

since $d < 0$, $\boxed{(3,3) \text{ is a } \textbf{saddle}}$;

$$D\Phi(-1,1) = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix} \text{ has trace } 3 \text{ and determinant } 6;$$

since $t > 0$ and $d > \frac{1}{4}t^2$, $\boxed{(-1,1) \text{ is an } \textbf{unstable spiral}}$.

2. a) There are two saddles: one at $\boxed{(-5,-1)}$ and one at $\boxed{(6,-1)}$.

   b) The only stable node is at $\boxed{(-5,6)}$.

   c) The only unstable node is at $\boxed{(6,6)}$.

   d) The only center is at $\boxed{(2,-4)}$.

   e) The solution curve through $(4, 9)$ goes to the left and approaches $(-5, 6)$, so $\lim_{t\to\infty} x(t) = \boxed{-5}$.

   f) The solution curve through $(3, -6)$ is a loop, so $x(t)$ goes back and forth between its maximum and minimum values repeatedly as the curve goes around and around a loop. Thus the solution is a sinusoidal curve, which is $\boxed{\textbf{D}}$.

   g) The solution curve through $(1, 2)$ starts at $(6, 6)$, goes down and to the left, then up and to the left to $(-5, 6)$. Thus the y-coordinate starts near 6, goes down to about 2, then increases to 6. Of the provided graphs, the only curve that behaves like this is $\boxed{\textbf{A}}$.

3. Let $A = \begin{pmatrix} -8 & 3 \\ -6 & 1 \end{pmatrix}$ and start with eigenvalues of A:

$$\det(A - \lambda I) = (-8-\lambda)(1-\lambda) + 18 = \lambda^2 + 7\lambda + 10 = (\lambda+2)(\lambda+5) = 0$$

gives eigenvalues $\lambda = -2, \lambda = -5$. Now for the eigenvectors. Let $\mathbf{v} = (x, y)$ and solve $A\mathbf{v} = \lambda\mathbf{v}$ to obtain:

$$\lambda = -2 : \begin{cases} -8x - 3y = -2x \\ -6x + y = -2y \end{cases} \Rightarrow y = 2x \Rightarrow \mathbf{v} = (1, 2).$$

$$\lambda = -8 : \begin{cases} -8x - 3y = -5x \\ -6x + y = -5y \end{cases} \Rightarrow y = x \Rightarrow \mathbf{v} = (1, 1).$$

Since there are two distinct real eigenvalues, the general solution is

$$\begin{aligned} \mathbf{y} &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= C_1 e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-5t}\begin{pmatrix}1\\1\end{pmatrix}. \end{aligned}$$

Now plug in the initial condition $\mathbf{y}(0) = (-3, 2)$ to get

$$\begin{cases} -3 = C_1 + C_2 \\ 2 = 2C_1 + C_2 \end{cases} \Rightarrow C_1 = 5, C_2 = -8.$$

Thus the particular solution is

$$\mathbf{y} = 5e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} - 8e^{-5t}\begin{pmatrix}1\\1\end{pmatrix} = \boxed{\begin{pmatrix} 5e^{-2t} - 8e^{-5t} \\ 10e^{-2t} - 8e^{-5t} \end{pmatrix}}.$$

4. Let $A = \begin{pmatrix} -3 & -13 \\ 1 & 3 \end{pmatrix}$. To solve $\mathbf{y}' = A\mathbf{y}$, start with eigenvalues of A:

$$\det(A - \lambda I) = (-3-\lambda)(3-\lambda) + 13 = \lambda^2 + 4 = 0$$

has solution $\lambda = \frac{0 \pm \sqrt{0-4(4)}}{2} = \frac{\pm\sqrt{-4}}{2} = \pm 2i$. Therefore $\alpha = 0$ and $\beta = 2$. Next, we find the eigenvector $\mathbf{v} = (x, y)$ corresponding to $\lambda = 2i$:

$$A\mathbf{v} = \mathbf{v} \Rightarrow \begin{cases} -3x + 13y = 2ix \\ x + 3y = 2iy \end{cases} \Rightarrow x = (-3+2i)y \Rightarrow \mathbf{v} = \begin{pmatrix} -3+2i \\ 1 \end{pmatrix}.$$

Therefore $\mathbf{a} = (-3, 1)$ and $\mathbf{b} = (2, 0)$. By the theorem from the lecture notes, the general solution is therefore

$$\begin{aligned} \mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{0t}\cos 2t\begin{pmatrix}-3\\1\end{pmatrix} - e^{0t}\sin 2t\begin{pmatrix}2\\0\end{pmatrix}\right] + C_2\left[e^{0t}\cos 2t\begin{pmatrix}2\\0\end{pmatrix} + e^{0t}\sin 2t\begin{pmatrix}-3\\1\end{pmatrix}\right] \\ &= \boxed{\begin{pmatrix} (-3C_1 + 2C_2)\cos 2t + (-2C_1 - 3C_2)\cos 2t \\ C_1\cos 2t + C_2\sin 2t \end{pmatrix}}. \end{aligned}$$

5. Start with the eigenvalues of $A = \begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}$:

$$\det(A - \lambda I) = (5 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Therefore $\lambda = 4$ is a repeated eigenvalue. Next, find an eigenvector $\mathbf{v} = (x, y)$ by solving $A\mathbf{v} = \lambda\mathbf{v}$ to obtain

$$\begin{cases} 5x + y = 4x \\ -x + 3y = 4y \end{cases} \Rightarrow y = -x \Rightarrow \mathbf{v} = (1, -1).$$

Next, find an generalized eigenvector $\mathbf{w} = (x, y)$ by solving $(A - 4I)\mathbf{w} = \mathbf{v}$ to get

$$\begin{cases} x + y = 1 \\ -x - y = -1 \end{cases} \Rightarrow x = 1 - y \Rightarrow \mathbf{w} = (1, 0).$$

Now by the theorem from the lecture notes, the general solution of $\mathbf{y}' = A\mathbf{y}$ is

$$\begin{aligned} \mathbf{y}_h &= C_1 e^{\lambda t}\mathbf{v} + C_2\left[e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{v}\right] \\ &= C_1 e^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\left[e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \end{aligned}$$

which when written coordinate-wise, is

$$\begin{cases} x_h(t) & = (C_1 + C_2)e^{4t} + C_2 te^{4t} \\ y_h(t) & = -C_1 e^{4t} - C_2 te^{4t} \end{cases}$$

Now for the particular solution. Since $\mathbf{q} = (-13e^{-2t}, 7e^{-2t})$, we guess $\mathbf{y}_p = (Ae^{-2t}, Be^{-2t})$. Plugging in the original equation, we get

$$\begin{aligned} \begin{cases} -2Ae^{-2t} & = 5Ae^{-2t} + Be^{-2t} - 13e^{-2t} \\ -2Be^{-2t} & = -Ae^{-2t} + 3Be^{-2t} + 7e^{-2t} \end{cases} \quad & \Rightarrow \begin{cases} -2A & = 5A + B - 13 \\ -2B & = -A + 3B + 7 \end{cases} \\ & \Rightarrow \begin{cases} 13 & = 7A + B \\ -7 & = -A + 5B \end{cases} \\ & \Rightarrow A = 2, B = -1. \end{aligned}$$

Therefore $\mathbf{y}_p = (2Ae^{-2t}, -e^{-2t})$, so the general solution of the system is

$$\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h = \boxed{\begin{pmatrix} 2e^{-2t} + (C_1 + C_2)e^{4t} + C_2 te^{4t} \\ -e^{-2t} - C_1 e^{4t} - C_2 te^{4t} \end{pmatrix}}.$$

5.3 Fall 2023 Final Exam

1. (1.6, 3.3, 5.1) Explain what is meant by "existence/uniqueness" in the context of differential equations.

2. (5.1) Perform the "reduction of order" trick on the system

$$\begin{cases} x'' \quad\quad +3x' \quad\quad\quad -4x \quad +2y \quad = e^{4t} \\ \quad\quad y'' \quad -x' \quad +2y' \quad +7x \quad -5y \quad = 0 \end{cases}$$

converting this second-order system to a first-order system of the form $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$. In particular, what is $\mathbf{y}$, what is A and what is $\mathbf{q}$?

3. (3.2) Use Euler's method with two steps to estimate $\mathbf{y}(7)$, where $\mathbf{y} = \mathbf{y}(t)$ is the solution of the initial value problem

$$\begin{cases} \mathbf{y}' = (tx + 2y, -ty) \\ \mathbf{y}(1) = (2, -1) \end{cases}$$

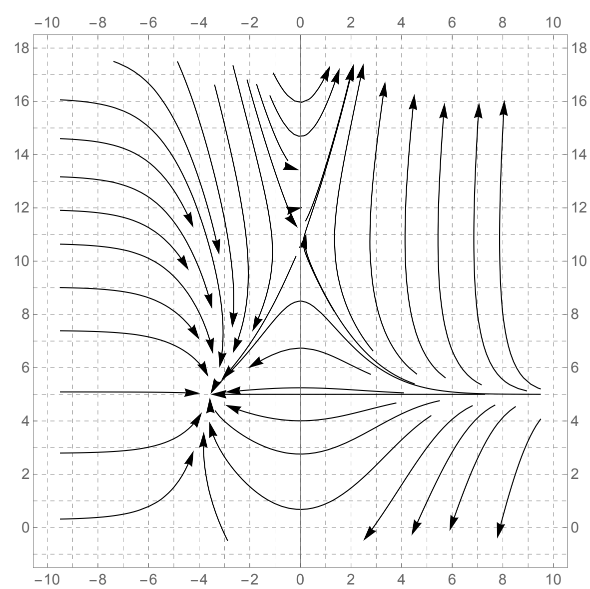
4. (1.7) Sketch the phase line associated to this autonomous differential equation:

$$\frac{dy}{dt} = y^4 - 25y^2$$

5. (4.7) Find and classify the equilibria of the system

$$\begin{cases} x' = (x-2)(y+3) \\ y' = (x+1)(y-5) \end{cases}$$

6. The phase plane of a first-order, autonomous 2×2 system of ODEs $\mathbf{y}' = \Phi(\mathbf{y})$ is given below (where $\mathbf{y} = (x, y)$). Use the picture to answer the questions below.



a) (4.7) This system has an equilibrium at $(0, 11)$. Which of these best describes $D\Phi(0, 11)$?

   A. $D\Phi(0, 11)$ has two real, positive eigenvalues.
   B. $D\Phi(0, 11)$ has two real eigenvalues, one of which is positive and one of which is negative.
   C. $D\Phi(0, 11)$ has two real, negative eigenvalues.
   D. $D\Phi(0, 11)$ has two non-real eigenvalues.

b) (4.8) This system has an equilibrium at $(-3.5, 5)$. If $t = tr D\Phi(-3.5, 5)$ and $d = \det D\Phi(-3.5, 5)$, which of these inequalities best describe t and d?

   A. $0 < t^2/4 < d$.
   B. $0 < d < t^2/4$.
   C. $t < 0$ and $t^2/4 < d$.
   D. $t < 0$ and $d < t^2/4$.

c) (3.7) Suppose you know $x(0) = 5$ and $y(0)$ is between 2 and 4. Is this sufficient to determine $\lim_{t\to\infty} x(t)$? If so, what is $\lim_{t\to\infty} x(t)$?

d) (3.7) Suppose you know $x(0) = 5$ and $y(0)$ is between 4 and 6. Is this sufficient to determine $\lim_{t\to\infty} x(t)$? If so, what is $\lim_{t\to\infty} x(t)$?

e) (3.7) Let $(x(t), y(t))$ be the solution to this system satisfying $(x(0), y(0)) = (5, 4)$. Estimate $y(t)$ at the value of t where $x(t) = 0$.

7. (2.2) Find the general solution of this differential equation:

$$ty'' + 5y' = 14t^2$$

8. (2.4) Find the particular solution of this initial value problem:

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{t^2 \cos y} \\ y(1) = 0 \end{cases}$$

9. (5.2) Find the particular solution of this initial value problem:

$$\begin{cases} y'' - 7y' - 18y = 0 \\ y(0) = 5 \\ y'(0) = 12 \end{cases}$$

10. (5.2) Find the general solution of this ODE:

$$y''(t) - 5y'(t) - 24y(t) = 33e^{8t}$$

11. (4.5) Find the general solution of this system:

$$\begin{cases} x' = -5x - 6y \\ y' = 3x + y \end{cases}$$

12. (4.6) Find the general solution of this system:

$$\mathbf{y}' = (-9x + 3y + e^{-t}, -7x + y - e^{-t})$$

13. (5.3) A 3 kg mass is attached to a spring with spring constant 9 kg/sec^2. The system is stretched to a length of $\frac{1}{2}$ m beyond its initial position and then released (with zero initial velocity). The mass experiences friction as it moves, with damping coefficient 12 kg·m/sec^2. If the system is subject to an external force of $15\cos 3t$ Newtons, what is the position of the mass at time t?

14. (2.5) At time $t = 0$, the temperature inside a room is 40°F. The room is heated by a furnace that generates heat $U(t) = 20 - 6e^{-2t}$ °F/hr. If the outside temperature is a constant $M = 30$°F and the constant from Newton's Law of Heating and Cooling is $K = \frac{1}{2}$, find the temperature inside the room at time t. (Assume that there are no other heat sources other than those described in this problem.)

Solutions

1. "Existence/uniqueness" means that under reasonable assumptions (about the smoothness of the function ϕ), the initial value problem $y' = \phi(t, y)$; $y(t_0) = y_0$ has one and only one solution of the form $y = f(t)$. Analogous results hold for systems and higher-order equations.

2. Let $\boxed{\mathbf{y} = (x, y, x', y')}$ and solve each equation for the left-most term. This gives
$$\begin{cases} x'' &= 4x - 2y - 3x' + e^{4t} \\ y'' &= -7x + 5y + x' - 2y' \end{cases}$$
so
$$\mathbf{y}' = \begin{pmatrix} x' \\ y' \\ x'' \\ y'' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ -3x' + 4x - 2y \\ x' - 2y - 7x + 5y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e^{4t} \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & -3 & 0 \\ -7 & 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ x'' \\ y'' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e^{4t} \\ 0 \end{pmatrix}$$
$$= A\mathbf{y} + \mathbf{q}$$
where $\boxed{A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & -2 & -3 & 0 \\ -7 & 5 & 1 & -2 \end{pmatrix}}$ and $\boxed{\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ e^{4t} \\ 0 \end{pmatrix}}$.

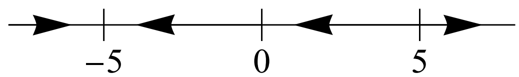
3. Let $\Phi(t, \mathbf{y}) = (tx+2y, -ty)$ so that the ODE is $\mathbf{y}' = \Phi(t, \mathbf{y})$. We are given $n = 3$, which means $\Delta t = \dfrac{t_n - t_0}{n} = \dfrac{7-1}{2} = 3$. We are also given the initial value $(t_0, \mathbf{y}_0) = (1, (2, -1))$, so $\Phi(t_0, \mathbf{y}_0) = (1(2) + 2(-1), -1(-1)) = (0, 1)$. Therefore
$$\begin{cases} t_1 &= t_0 + \Delta t = 1 + 3 = 4 \\ \mathbf{y}_1 &= \mathbf{y}_0 + \Phi(t_0, \mathbf{y}_0)\Delta t = (2, -1) + (0, 1)3 = (2, -1) + (0, 3) = (2, 2) \end{cases}$$
so $(t_1, \mathbf{y}_1) = (4, (2, 2))$. Next, $\Phi(t_1, \mathbf{y}_1) = (4(2) + 2(2), -4(2)) = (12, -8)$ so
$$\begin{cases} t_2 &= t_1 + \Delta t = 4 + 3 = 7 \\ \mathbf{y}_2 &= \mathbf{y}_1 + \Phi(t_1, \mathbf{y}_1)\Delta t = (2, 1) + (12, -8)3 = (2, 2) + (36, -24) = \boxed{(38, -22)}. \end{cases}$$

4. Let $\phi(y) = y^4 - 25y^2$ so that the ODE is $y' = \phi(y)$. To find equilibria, set $\phi(y) = 0$ and solve for y:
$$0 = \phi(y) = y^4 - 25y^2 = y^2(y^2 - 25) = y^2(y-5)(y+5)$$

so the equilibria are $y = 0$, $y = 5$ and $y = -5$. To classify these, use the derivative $\phi'(y) = 4y^3 - 50y$:

$$\begin{array}{ll} \phi'(0) = 0 \text{ but } \phi''(0) = 12(0)^2 - 50 \neq 0 & \Rightarrow 0 \text{ is semistable;} \\ \phi'(5) = 4(125) - 50(5) = 250 > 0 & \Rightarrow y = 5 \text{ is unstable;} \\ \phi'(-5) = 4(-125) - 50(-5) = -250 < 0 & \Rightarrow y = -5 \text{ is stable.} \end{array}$$

Therefore the phase line looks like this:



5. To find the equilibria, set x' and y' equal to 0 and solve for x and y:

$$\begin{cases} 0 &= (x-2)(y+3) \Rightarrow x = 2 \text{ or } y = -3 \\ 0 &= (x+1)(y-5) \end{cases}$$

If $x = 2$, then from the second equation $y = 5$, giving the equilibrium $(2,5)$. If $y = -3$, then from the second equation $x = -1$, giving the equilibrium $(-1,-3)$. To classify these, use the total derivative $D\Phi = \begin{pmatrix} y+3 & x-2 \\ y-5 & x-1 \end{pmatrix}$:

$D\Phi(2,5) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix has eigenvalues $\lambda = 8$ and $\lambda = 1$ which are both positive and real, so $\boxed{(2,5) \text{ is an unstable node}}$.

$D\Phi(-1,-3) = \begin{pmatrix} 0 & -3 \\ -8 & 0 \end{pmatrix}$. This matrix has determinant $-24 < 0$, which implies that $\boxed{(-1,-3) \text{ is an (unstable) saddle}}$.

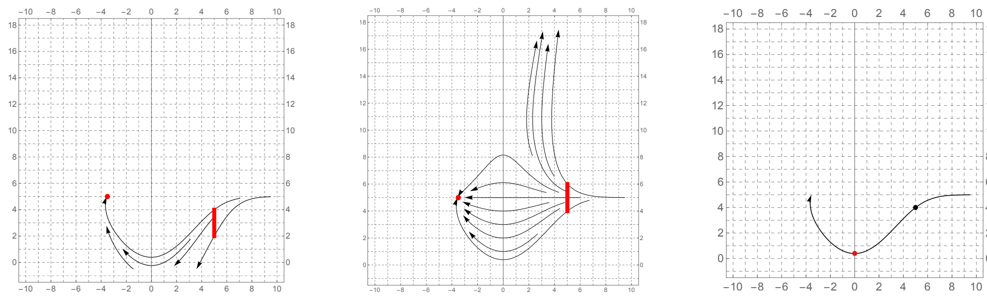
6. a) $(0,11)$ is a saddle, so it must have two real eigenvalues of opposite sign. This is choice $\boxed{\textbf{B}}$.

   b) $(-3.5,5)$ is a stable node, so $0 < d < t^2/4$. This is choice $\boxed{\textbf{B}}$.

   c) $\lim\limits_{t\to\infty} x(t)$ must be $\boxed{-3.5}$ since all solution curves with initial value $(5,y)$ where $2 \leq y \leq 4$ (indicated by the red line segment in the picture below at left) go to the stable node as $t \to \infty$.

   d) Here, you $\boxed{\text{cannot determine}}$ $\lim\limits_{t\to\infty} x(t)$ because some solution curves with initial value $(5,y)$ where $4 \leq y \leq 6$ (indicated by the red line segment in the middle picture below) go to the stable node as $t \to \infty$, but others tend to ∞.

e) Using the picture below at right, we see that when $x = 0$ for the solution curve passing through $(5,4)$, $y \approx \boxed{\frac{1}{3}}$.



7. This is a second-order equation with no y in it, so we let $v = y' = \frac{dy}{dt}$. Thus the equation becomes
$$tv' + 5v = 14t^2.$$
We solve this with integrating factors. First, divide through by t to get
$$v' + \frac{5}{t}v = 14t$$
and the integrating factor is therefore $\mu(t) = \exp\left(\int \frac{5}{t}dt\right) = \exp(5 \ln t) = t^5$. After multiplying through by t^5, the equation becomes
$$\begin{aligned} t^5v' + 5t^4v &= 14t^6 \\ \frac{d}{dt}\left(t^5 v\right) &= 14t^6 \\ t^5 v &= \int 14t^6\, dt \\ t^5 v &= 2t^7 + C \\ v &= 2t^2 + Ct^{-5}. \end{aligned}$$
Finally, since $v = y'$, $y = \int v\, dt = \int \left(2t^2 + Ct^{-5}\right)\, dt = \frac{2}{3}t^3 + \frac{C}{-4}t^{-4} + D$, which after renaming constants can be written as $\boxed{y = \frac{2}{3}t^3 + Ct^{-4} + D}$.

8. This equation is separable; separate variables to get $\cos y\, dy = \frac{1}{t^2}\, dt$. Integrate both sides to get the general solution $\sin y = -\frac{1}{t} + C$. To solve for C, plug in the initial condition $(1,0)$ to get $\sin 0 = -1 + C$; this leads to $C = 1$ so the particular solution is $\boxed{\sin y = -\frac{1}{t} + 1}$.

9. This is second-order and constant-coefficient, so we solve the characteristic equation:

$$0 = \lambda^2 - 7\lambda - 18 = (\lambda - 9)(\lambda + 2) \Rightarrow \lambda = 9, \lambda = -2.$$

Therefore the general solution is $y = C_1 e^{9t} + C_2 e^{-2t}$, which means $y' = 9C_1 e^{9t} - 2C_2 e^{-2t}$. Plug in the initial value to solve for C_1 and C_2:

$$\begin{cases} y(0) = 5 \\ y'(0) = 12 \end{cases} \Rightarrow \begin{cases} 5 = C_1 + C_2 \\ 12 = 9C_1 - 2C_2 \end{cases} \Rightarrow C_1 = 2, C_2 = 3.$$

Thus the particular solution is $\boxed{y = 2e^{9t} + 3e^{-2t}}$.

10. First, solve the corresponding homogeneous equation by factoring the characteristic equation:

$$0 = \lambda^2 - 5\lambda - 24 = (\lambda - 8)(\lambda + 3) \Rightarrow \lambda = 8, \lambda = -3.$$

Thus $y_h = C_1 e^{8t} + C_2 e^{-3t}$.

Next, find y_p using undetermined coefficients. Since e^{8t} is already part of y_h, we need to guess $y_p = Ate^{8t}$; that means $y_p' = Ae^{8t} + 8Ate^{8t}$ and $y_p'' = 8Ae^{8t} + 8Ae^{8t} + 64Ate^{8t} = 16Ae^{8t} + 64Ate^{8t}$. Now, plug all this in the original equation to get

$$\begin{aligned} y_p''(t) - 5y_p'(t) - 24y_p(t) &= 33e^{8t} \\ (16Ae^{8t} + 64Ate^{8t}) - 5(Ae^{8t} + 8Ate^{8t}) - 24Ate^{8t} &= 33e^{8t} \\ 11Ae^{8t} &= 33e^{8t} \\ A &= 3 \end{aligned}$$

Therefore $y_p = 3te^{8t}$ so the general solution is

$$y = y_p + y_h, \text{ i.e. } \boxed{y = 3te^{8t} + C_1 e^{8t} + C_2 e^{-3t}}.$$

11. Think of this system as $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} -5 & -6 \\ 3 & 1 \end{pmatrix}$. To get started, find eigenvalues of A:

$$0 = \det(A - \lambda I) = (-5-\lambda)(1-\lambda) + 18 = \lambda^2 + 4\lambda + 13 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i.$$

Therefore $\alpha = -2$ and $\beta = 3$. We need an eigenvector corresponding to $\lambda = -2 + 3i$. Writing $\mathbf{v} = (x, y)$, we have

$$\begin{aligned} A\mathbf{v} = (-2+3i)\mathbf{v} \Rightarrow \begin{cases} -5x - 6y &= (-2+3i)x \\ 3x + y &= (-2+3i)y \end{cases} &\Rightarrow 3x = (-3+3i)y \\ &\Rightarrow x = (-1+i)y \\ &\Rightarrow \mathbf{v} = (-1+i, 1) \end{aligned}$$

so $\mathbf{a} = (-1, 1)$ and $\mathbf{b} = (1, 0)$. By the standard formula that gives the solution for complex eigenvalues, we have

$$\begin{aligned}\mathbf{y} &= C_1\left[e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right] + C_2\left[e^{\alpha t}\cos(\beta t)\mathbf{b} + e^{\alpha t}\sin(\beta t)\mathbf{a}\right] \\ &= C_1\left[e^{-2t}\cos 3t\begin{pmatrix}-1\\1\end{pmatrix} - e^{-2t}\sin 3t\begin{pmatrix}1\\0\end{pmatrix}\right] + C_2\left[e^{-2t}\cos 3t\begin{pmatrix}1\\0\end{pmatrix} + e^{-2t}\sin 3t\begin{pmatrix}-1\\1\end{pmatrix}\right] \\ &= \boxed{\begin{pmatrix}(-C_1 + C_2)e^{-2t}\cos 3t + (-C_1 - C_2)e^{-2t}\sin 3t \\ C_1e^{-2t}\cos 3t + C_2e^{-2t}\sin 3t\end{pmatrix}}.\end{aligned}$$

12. We think of this system as $\mathbf{y}' = A\mathbf{y} + \mathbf{q}$ where $A = \begin{pmatrix}-9 & 3\\-7 & 1\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}e^{-t}\\-e^{-t}\end{pmatrix}$. Start with eigenvalues of A:

$$0 = \det(A-\lambda I) = (-9-\lambda)(1-\lambda)+21 = \lambda^2+8\lambda+12 = (\lambda+2)(\lambda+6) \Rightarrow \lambda = -2, \lambda = -6.$$

Next, eigenvectors. Write $\mathbf{v} = (x, y)$:

$$\lambda = -2 : A\mathbf{v} = -2\mathbf{v} \Rightarrow \begin{cases}-9x + 3y &= -2x\\ -7x + y &= -2y\end{cases} \Rightarrow 3y = 7x \Rightarrow (3, 7)$$

$$\lambda = -6 : A\mathbf{v} = -6\mathbf{v} \Rightarrow \begin{cases}-9x + 3y &= -6x\\ -7x + y &= -6y\end{cases} \Rightarrow 3y = 3x \Rightarrow y = x \Rightarrow (1, 1)$$

Therefore the solution of the homogeneous is

$$\mathbf{y}_h = C_1e^{-2t}\begin{pmatrix}3\\7\end{pmatrix} + C_2e^{-6t}\begin{pmatrix}1\\1\end{pmatrix}.$$

Now we find the particular solution using undetermined coefficients: guess $\mathbf{y}_p = \begin{pmatrix}Ae^{-t}\\Be^{-t}\end{pmatrix}$ so that $\mathbf{y}_p' = \begin{pmatrix}-Ae^{-t}\\-Be^{-t}\end{pmatrix}$ and plug this into the original system to get

$$\begin{cases}-Ae^{-t} = -9Ae^{-t} + 3Be^{-t} + e^{-t}\\ -Be^{-t} = -7Ae^{-t} + Be^{-t} - e^{-t}\end{cases} \Rightarrow \begin{cases}-1 &= -8A + 3B\\ 1 &= -7A + 2B\end{cases} \Rightarrow A = -1, B = -3.$$

Therefore $\mathbf{y}_p = \begin{pmatrix}-e^{-t}\\-3e^{-t}\end{pmatrix}$ so the solution of the system is $\mathbf{y} = \mathbf{y}_p + \mathbf{y}_h$, i.e.

$$\boxed{\mathbf{y} = \begin{pmatrix}-e^{-t}\\-3e^{-t}\end{pmatrix} + C_1e^{-2t}\begin{pmatrix}3\\7\end{pmatrix} + C_2e^{-6t}\begin{pmatrix}1\\1\end{pmatrix}}.$$

Coordinate-wise, this is

$$\boxed{\begin{cases}x(t) &= -e^{-t} + 3C_1e^{-2t} + C_2e^{-6t}\\ y(t) &= -3e^{-t} + 7C_1e^{-2t} + C_2e^{-6t}\end{cases}}.$$

13. Let $x(t)$ be the position of the mass at time t. We are given $m = 3$, $b = 12$, $k = 9$ and $F_{ext}(t) = 15\cos 3t$, so the oscillator equation is

$$mx'' + bx' + kx = F_{ext}(t) \quad \Rightarrow \quad 3x'' + 12x' + 9x = 15\cos 3t.$$

Divide through the entire equation by 3 to get $x'' + 4x' + 3x = 5\cos 3t$. To solve this, first solve the corresponding homogeneous equation by factoring the characteristic equation:

$$0 = \lambda^2 + 4\lambda + 3 = (\lambda+3)(\lambda+1) \Rightarrow \lambda = -3, \lambda = -1$$

so $x_h = C_1e^{-3t} + C_2e^{-t}$.

Next, find x_p using undetermined coefficients. Guess $x_p = A\cos 3t + B\sin 3t$ so that $x_p' = -3A\sin 3t + 3B\cos 3t$ and $x_p'' = -9A\cos 3t - 9B\sin 3t$. Plug all this in the original equation to get

$$\begin{aligned} x_p'' + 4x_p' + 3x_p &= 5\cos 3t \\ (-9A\cos 3t - 9B\sin 3t) + 4(-3A\sin 3t + 3B\cos 3t) + 3(A\cos 3t + B\sin 3t) &= 5\cos 3t \\ (-6A + 12B)\cos 3t + (-6B - 12A)\sin 3t &= 5\cos 3t \end{aligned}$$

Therefore

$$\begin{cases} -6A + 12B &= 5 \qquad\qquad\quad \Rightarrow -6A + 12(-2A) = 5 \Rightarrow -30A = 5 \\ -6B - 12A &= 0 \Rightarrow B = -2A \end{cases}$$

Therefore $A = -\frac{1}{6}$ and $B = \frac{1}{3}$, so the general solution of the ODE is

$$x = x_p + x_h = -\frac{1}{6}\cos 3t + \frac{1}{3}\sin 3t + C_1e^{-3t} + C_2e^{-t}.$$

To find the particular solution, we need $x' = \frac{1}{2}\sin 3t + \cos 3t - 3C_1e^{-3t} - C_2e^{-t}$. Plug in the initial conditions $x(0) = \frac{1}{2}$ and $x'(0) = 0$ to get

$$\begin{cases} x(0) = \frac{1}{2} \\ x'(0) = 0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2} &= -\frac{1}{6} + C_1 + C_2 \\ 0 &= 1 - 3C_1 - C_2. \end{cases} \Rightarrow \begin{cases} \frac{2}{3} &= C_1 + C_2 \\ -1 &= -3C_1 - C_2. \end{cases}$$

Add the equations to get $-\frac{1}{3} = -2C_1$, so $C_1 = \frac{1}{6}$ and it follows that $C_2 = \frac{1}{2}$. All together, the particular solution which gives the position of the mass is

$$\boxed{x(t) = -\frac{1}{6}\cos 3t + \frac{1}{3}\sin 3t + \frac{1}{6}e^{-3t} + \frac{1}{2}e^{-t}}.$$

14. From our work in Section 2.5 of the lecture notes, we know the solution to the heating and cooling equation coming from Newton's Law is

$$\begin{aligned} T(t) &= e^{-Kt}\left(\int e^{Kt}\left[KM(t) + H(t) + U(t)\right]\,dt\right) \\ &= e^{-t/2}\left(\int e^{t/2}\left[\frac{1}{2}(30) + 0 + 20 - 6e^{-2t}\right]\,dt\right) \\ &= e^{-t/2}\left(\int \left[35e^{t/2} - 6e^{-3t/2}\right]\,dt\right) \\ &= e^{-t/2}\left(70e^{t/2} + 4e^{-3t/2} + C\right) \\ &= 70 + 4e^{-2t} + Ce^{-t/2}. \end{aligned}$$

To find the particular solution, plug in the initial condition $T(0) = 40$ to get

$$40 = 70 + 4(1) + C(1) \Rightarrow C = -34.$$

Therefore the temperature of the room at time t is

$$\boxed{T(t) = 70 + 4e^{-2t} - 34e^{-t/2}}.$$