Old MATH 416 Exams

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Last updated to include Exams from 2016

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Chapter 1

General information about these exams

These are the exams I have given between 2007 and 2016 in stochastic processes courses. Each exam is given here, followed by what I believe are the solutions (there may be a small number of computational errors or typos in these answers).

I have edited these exams to remove questions that do not match the current syllabus of MATH 416; that's why some of them contain only 1 or 2 questions.

Note that I have revised my probability course several times over the years, and what was on "Exam 1" in past years may not match what is on "Exam 1" now. To help give you some guidance on what questions are appropriate, each question on each exam is followed by a section number in parenthesis (like "(3.2)"). That means that question can be solved using material from that section (or from earlier sections) in the 2021 version of my MATH 416 lecture notes, *Markov Chains and Martingales*.

Chapter 2

Exams from 2008 to 2010

2.1 Winter 2008 Exam 2

1. Fischer and Spassky are playing for the World Chess Championship. They play until one player has 6 more wins than the other. In each game there is a 40% chance that Fischer wins, a 10% chance that Spassky wins, and a 50% chance that they draw. (For simplicity we ignore the advantage of playing white. The actual match in 1972 was best of 24 games. Fischer won 12.5–8.5.)

   a) (2.5) What is the expected number of games in the match?

   b) (2.5) Suppose at some point Fischer is leading 4 wins to 2. What is the chance that Fischer wins the match?

   c) (2.5) With Fischer leading by 4 wins to 2, how many more games are expected in the match?

2. A plant has $n + 1$ flowers, of which n are arranged in a circle (label these $1, 2, ..., n$ clockwise). The last flower is in the center of the circle (label this flower 0). A bee jumps from flower to flower; its position $\{X_n\}$ is a Markov chain with the following properties:

   - If the bee is on the center flower, then on the next step it moves to any of the n outer flowers with probability $1/n$.
   - If the bee is on any of the outer flowers, then on the next step it moves to the center flower with probability $1/2$ and moves clockwise to the next flower on the circle with probability $1/2$.

   a) (1.3) Find the probability that the bee is on the center flower after 5 steps, given that it is on the center flower after 2 steps.

   b) (1.6) Find $P_0(T_1 = 3)$.

c) (1.7) Is 0 (the center flower) a recurrent or transient state? Justify your answer.

Solutions

1. a) Let $X(t)$ be the number of games Fischer is ahead or behind; this is then an escape problem for a simple random walk with $p = .4$, $q = .1$, $r = .5$, $x = 0$, $a = -6$ and $b = 6$. So

$$\begin{aligned} ET = \left(\frac{b-a}{p-q}\right)\frac{(q/p)^{x-a}-1}{(q/p)^{b-a}-1} - \frac{x-a}{p-q} &= \left(\frac{12}{.3}\right)\frac{(1/4)^6-1}{(1/4)^{12}-1} - \frac{6}{.3} \\ &= 40\left(\frac{(1/4)^6-1}{(1/4)^{12}-1}\right) - 20 \\ &= 40\left(\frac{4^6-4^{12}}{1-4^{12}}\right) - 20. \end{aligned}$$

b) The setup is as in part (a) except that $x = 4 - 2 = 2$. We have

$$P(X(T) = 6) = \frac{(q/p)^{x-a}-1}{(q/p)^{b-a}-1} = \frac{(1/4)^8-1}{(1/4)^4-1} = \frac{1-4^8}{4^4-4^8}.$$

c) The setup is the same as part (b):

$$\begin{aligned} ET = \left(\frac{b-a}{p-q}\right)\frac{(q/p)^{x-a}-1}{(q/p)^{b-a}-1} - \frac{x-a}{p-q} &= \left(\frac{12}{.3}\right)\frac{(1/4)^8-1}{(1/4)^{12}-1} - \frac{8}{.3} \\ &= 40\left(\frac{(1/4)^8-1}{(1/4)^{12}-1}\right) - \frac{80}{3} \\ &= 40\left(\frac{4^4-4^{12}}{1-4^{12}}\right) - \frac{80}{3}. \end{aligned}$$

2. a) By stationarity, $P(X_5 = 0|X_2 = 0) = P(X_3 = 0|X_0 = 0)$. If you are the bee and you start at 0, then if you are on 0 after the third step you must have:

i. stepped to some outer flower on the first step (this happens with probability 1),
ii. then *not* stepped back to the center flower on the second step (this happens with probability $\frac{1}{2}$ no matter what outer flower you are at),
iii. and then stepped back to the center flower on the third step (this happens with probability $\frac{1}{2}$ no matter what outer flower you are at).

So $P(X_3 = 0|X_0 = 0) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

You could have done this problem by writing the transition matrix P and calculating $(P^3)_{0,0}$. This is not a very good method of reasoning, however.

b) First, by symmetry it doesn't matter which flower you call 1 as long as it is an outer one. If you start at 0, and the first time you hit flower 1 is on the third step, then one of two things must have happened:

- *Case 1:* Your first step is to an outer flower other than 1 (this has probability $\frac{n-1}{n}$), then your second step is back to the center (this has probability $\frac{1}{2}$), and your third step is to 1 (this has probability $\frac{1}{n}$).
- *Case 2:* Your first step is to flower $n-1$ (this happens with probability $\frac{1}{n}$), your second step is to flower n (this has probability $\frac{1}{2}$), and your third step is to flower 1 (this has probability $\frac{1}{2}$). Notice that flower 1 follows flower n clockwise around the circle.

So $P_0(T_1 = 3) = P(Case\ 1) + P(Case\ 2) = \frac{n-1}{2n^2} + \frac{1}{4n} = \frac{3n-2}{4n^2}$.

c) Whether or not 0 is recurrent depends on the value of $f_0 = f_{0,0}$. We calculate this directly:

$$f_0 = P_0(T_0 < \infty) = 1 - P_0(T_0 = \infty).$$

Now if you start at 0 and $T_0 = \infty$, then you never return to 0. This happens only if every step after the first step is around the circle clockwise (and not back to the center). Each of these steps occurs with probability $\frac{1}{2}$, so the probability of never returning to 0 is "$\left(\frac{1}{2}\right)^\infty$", in other words $P_0(T_0 = \infty) = 0$. Thus $f_0 = 1 - 0 = 1$ and therefore 0 is a recurrent state.

2.2 Winter 2008 Final

1. You buy some stock in a company that is currently worth $\$50$ per share. Each day, the stock price (per share) goes up by $\$1$ with probability $1/4$, goes down by $\$1$ with probability $1/4$, and stays the same with probability $1/2$.

   a) (2.5) How much should you expect your stock to be worth (per share) in 30 days?

   b) (2.5) If you hold on to your stock indefinitely, what is the probability that it is eventually worthless?

   c) (2.5) Suppose you sell your stock when the price either hits $\$20$ or $\$70$ per share. Under these conditions, what is the probability you will be able to sell your stock at $\$70$ per share?

   d) (2.5) Suppose you decide to sell your stock when the price hits $\$K$ per share. What should you choose K to be if you want to be exactly 80% sure that your stock doesn't become worthless before you sell it?

2. Here is a greatly simplified version of the game "Snakes and Ladders" (also called "Chutes and Ladders"). There are six squares in a column numbered from 1 at the bottom through 6 at the top. There is a ladder from square 2 to square 4 and a snake from square 5 to square 3. At each turn you toss a fair coin; one side of the coin is labelled 1 and the other is labelled 2. At each turn you toss the coin, then move up the column by the number that you roll, and finally climb the ladder or go down the snake if applicable.

   In other words, if you are on square x and roll a, you work out what $x + a$ is. If $x + a = 1, 3, 4$ or 6 you end up at square $x + a$ after your turn. If $x + a = 2$ you end up at square 4, if $x + a = 5$, you end up at square 3, and if $x + a > 6$, you end up at square 6.

   a) (1.3) Write down the transition matrix for this process (please arrange it so that state j corresponds to the j^{th} column and j^{th} row of the matrix).

   b) (1.7) Which states are transient? recurrent? absorbing?

   c) (1.3) Find the second and fourth powers of the transition matrix.

   d) (1.3) What is the probability of getting to square 6 after 4 turns if you start at square 1?

   e) (1.7) What is the probability of reaching state 4 (at some time) if you start at state 1?

Solutions

1. Let X_n represent the price of the stock (per share) after the n^{th} day; this is a simple random walk with $X_0 = x = 50$, $p = q = 1/4$, and $r = 1/2$. Therefore $\mu = p - q = 0$.

   a) $EX_n = x + \mu n = 50 + 0 = 50$.

   b) This refers to a gambler's ruin with $p = q$; this probability is 1.

   c) This is an escape with $a = 20$, $b = 70$. The question asks for $P(X(T) = b)$ which is (since $p = q$)
   $$\frac{x-a}{b-a} = \frac{30}{50} = \frac{3}{5}.$$

   d) This is an escape with $a = 0$, $b = K$. We want $P(X(T) = b) = .80$ so
   $$\frac{x-a}{b-a} = .80 \Rightarrow \frac{50}{K} = \frac{4}{5} \Rightarrow 4K = 250 \Rightarrow K = 62.5.$$

2. a) If you are on square 1, then if you roll a 1 you end up at square 4 (because you follow the ladder) and if you roll a 2 you end up at square 3. If you are on square 2, then if you roll a 1 you end up at square 3 and if you roll a 2 you end up at square 4. If you are on square 3, then if you roll a 1 you end up at square 4 and if you roll a 2 you end up at square 3 (because you follow the snake). If you are on square 4, then if you roll a 1 you end up at square 3 (because you follow the snake) and if you roll a 2 you end up at square 6. If you are on square 5 or square 6, then no matter what you roll you end up on square 6. So the matrix is
   $$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   b) Since $P_{6,6} = 1$, 6 is absorbing, hence also a recurrent state. States 1 and 2 are clearly transient because there is no way you can move to either of these states. State 5 is transient because if you start there, you then move to state 6 and cannot return to state 5. If you start at state 4 then there is a probability of $1/2$ that you move to state 6 (and in this case you would never return to state 4). So $f_{4,4} \le 1/2$ and 4 is therefore transient. If you start at state 3 then there is a probability of $1/4$ that you move to state 4 then to state 6 on the first two steps, in which case you would never return to state 3. So $f_{3,3} \le 3/4$ and 3 is therefore a transient state. To summarize, 6 is absorbing and recurrent; all other states are transient.

c) Notice that since the first three rows of P are the same, the first three rows of P^n are the same for every n. Also since the first, second and fifth columns of P are all zeros, those columns will also be all zeros for all powers of P.

$$P^2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; P^4 = (P^2)^2 = \begin{pmatrix} 0 & 0 & \frac{5}{16} & \frac{3}{16} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{5}{16} & \frac{3}{16} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{5}{16} & \frac{3}{16} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{16} & \frac{1}{8} & 0 & \frac{11}{16} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

d) This is $(P^4)_{1,6} = \frac{1}{2}$.

e) The question is asking for $f_{1,4} = P_1(T_4 < \infty)$. Suppose that you started at state 1 and did *not* reach state 4 in any amount of time. Then you must have moved to state 3 and stayed there forever. The probability of this happening is "$(1/2)^\infty$", i.e. is zero. So $f_{1,4} = 1 - 0 = 1$.

2.3 Spring 2008 Exam 1

1. Let X_n be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, ...\}$ and transition function defined by

$$P(x, y) = \begin{cases} \frac{1}{3} & \text{if } y = 0, y = x + 1 \text{ or } y = x + 2 \\ 0 & \text{otherwise} \end{cases}.$$

   a) (1.10) Show this chain is positive recurrent.

   b) (1.10) Let π be the stationary distribution of X_n. Find $\pi(0)$ and $\pi(2)$.

   c) (1.10) Is π steady-state? Why or why not?

2. Let X_n be a Markov chain with state space $\mathcal{S} = \{..., -2, -1, 0, 1, 2, ...\}$ and transition function

$$P(x, y) = \begin{cases} \frac{1}{2} & \text{if } y = x + 1 \\ \frac{1}{2} & \text{if } y = x - 1 \\ 0 & \text{otherwise} \end{cases}.$$

   (1.8) This irreducible Markov chain is called an *unbiased simple random walk on the integers*; in class we showed that this chain is recurrent. Is this Markov chain positive recurrent or null recurrent? Explain your answer.

3. Let $\{X_n\}$ be a Markov chain with state space $\{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

   a) (1.5) Find all stationary distributions.

   b) (1.10) Suppose $\pi_0 = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$. Estimate the distribution π_n for large n. *Hint:* This answer may involve separate cases.

4. Suppose $\{X_t\}$ is a continuous-time Markov process with state space $\mathcal{S} = \{0, 1, 2\}$ and time t transition function

$$P(t) = \frac{1}{9} \begin{pmatrix} 1 + 6te^{-3t} + 8e^{-3t} & 2 - 6te^{-3t} - 2e^{-3t} & 6 - 6e^{-3t} \\ 1 + 6te^{-3t} - e^{-3t} & 2 - 6te^{-3t} + 7e^{-3t} & 6 - 6e^{-3t} \\ 1 - 3te^{-3t} - e^{-3t} & 2 + 3te^{-3t} - 2e^{-3t} & 6 + 3e^{-3t} \end{pmatrix}.$$

   Assume that $Q_{xx} = 0$ for all x. *Note:* Do not forget the $1/9$ that is in front of this matrix.

a) (3.2) Find $P(X_6 = 1 | X_3 = 2)$.

b) (3.2) Find q_{01}.

c) (3.4) Find the steady-state distribution π.

d) (3.4) Find m_1.

Solutions

1. a) Construct a factor of the Markov chain: let $Y_n = 0$ if $X_n = 0$ and $Y_n = 1$ otherwise. Then $\{Y_n\}$ is a two-state Markov chain with transition matrix

$$P_Y = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

$\{Y_n\}$ is obviously irreducible and therefore positive recurrent; it has stationary distribution $\pi_Y = (1/3, 2/3)$. In particular 0 has mean return time $(1/3)^{-1} = 3$ for the Markov chain $\{Y_n\}$. But 0 returns in $\{Y_n\}$ if and only if it returns in $\{X_n\}$ so $m_0 = 3$ for the Markov chain $\{X_n\}$ as well. Hence 0 is positive recurrent. But 0 leads to every other state so the entire chain $\{X_n\}$ is positive recurrent.

b) Since $m_0 = 3$, $\pi(0) = m_0^{-1} = 1/3$. Now since $P(x, 1) = 0$ unless $x = 1$, we have

$$\pi(1) = \sum_{x \in \mathcal{S}} \pi(x)P(x,1) = \pi(0)P(0,1) = (1/3)(1/3) = 1/9.$$

Similarly, since $P(x, 2) = 0$ unless $x = 0$ or 1, we have

$$\begin{aligned} \pi(2) = \sum_{x \in \mathcal{S}} \pi(x)P(x,2) &= \pi(0)P(0,2) + \pi(1)P(1,2) \\ &= (1/3)(1/3) + (1/9)(1/3) = 10/27. \end{aligned}$$

c) This Markov chain is irreducible and aperiodic since one can get from 0 to itself in two steps ($0 \to 1 \to 0$) or in three steps ($0 \to 1 \to 2 \to 0$). Hence π is steady-state.

2. If $\{X_n\}$ is positive recurrent, then since it is irreducible there is a unique stationary distribution π. By symmetry, $\pi(x) = \pi(y)$ for all $x, y \in \mathcal{S}$ (since all points look the same in the Markov chain). In other words, π must be uniform. But there is no uniform distribution on an infinite countable set, so there is no stationary distribution. Therefore $\{X_n\}$ is null recurrent.

(Why is there no uniform distribution on an infinite countable set? If π assigns 0 probability to each state $x \in \mathcal{S}$, then $\sum_{x \in \mathcal{S}} \pi(x) = \sum_x 0 = 0$. But if π assigns probability $\alpha > 0$ to each state $x \in \mathcal{S}$, then $\sum_{x \in \mathcal{S}} \pi(x) = \sum_x \alpha = \infty$. Either way this contradicts the fact that the sum of the probabilities of the states must be 1.)

3. a) Notice that $0 \to 1 \to 2 \to 3 \to 0$ so the chain is irreducible. Hence there is one stationary distribution which we write as $\pi = (a, b, c, d)$. Setting $\pi P = \pi$ we obtain the equations

$$\frac{1}{2}b + \frac{3}{4}d = a, \quad a = b, \quad \frac{1}{2}b + \frac{1}{4}d = c, \quad c = d$$

which together with $a+b+c+d=1$ can be solved to give $a=.3, b=.3, c=.2, d=.2$. So $\pi=(.3,.3,.2,.2)$.

b) Notice this chain has period 2 because one always alternates between the two sets $\{0,2\}$ and $\{1,3\}$. Therefore if n is odd we have

$$\begin{aligned} P^n(0,0)=P^n(0,2)=P^n(2,0) &= P^n(2,2)= \\ =P^n(1,1) &= P^n(1,3)=P^n(3,1)=P^n(3,3)=0 \end{aligned}$$

and if n is even we have

$$\begin{aligned} P^n(0,1)=P^n(0,3)=P^n(2,1) &= P^n(2,3)= \\ =P^n(1,0) &= P^n(1,2)=P^n(3,0)=P^n(3,2)=0. \end{aligned}$$

The nonzero entries of $P^n(x,y)$ must approach $d\cdot\pi(y)$ as n is large; since $d=2$ and π is known from part (a), we see that

$$P^n \approx \begin{cases} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \end{pmatrix} & \text{if } n \text{ even} \\ \\ \begin{pmatrix} 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{3}{5} & 0 & \frac{2}{5} & 0 \end{pmatrix} & \text{if } n \text{ odd} \end{cases}$$

and finally, since $\pi_n=\pi_0 P^n$, by multiplying matrices we see that

$$\pi_n \approx \begin{cases} \left(\frac{2}{5},\frac{1}{5},\frac{4}{15},\frac{2}{15}\right) & \text{if } n \text{ even} \\ \\ \left(\frac{1}{5},\frac{2}{5},\frac{2}{15},\frac{4}{15}\right) & \text{if } n \text{ odd} \end{cases}.$$

4. a) By definition, $P(X_6=1|X_3=2)=P_{21}(3)$. Then

$$P_{21}(3)=\frac{1}{9}[2-6te^{-3t}-2e^{-3t}]_{t=3}=\frac{2}{9}-\frac{20}{9}e^{-9}.$$

b) Notice $q_{01}=P'_{01}(0)=\left[\frac{1}{9}(2-6te^{-3t}-2e^{-3t})\right]'(0)$. Evaluate this to get

$$P'_{01}(0)=\left[\frac{1}{9}(-6(e^{-3t}-3te^{-3t})+6e^{-3t})\right]_{t=0}=\frac{1}{9}(-6+6)=0.$$

c) Since π is steady-state, $\pi(y)=\lim_{t\to\infty}P_{xy}(t)$ so by taking limits on any row of P as $t\to\infty$ we see $\pi=\left(\frac{1}{9},\frac{2}{9},\frac{2}{3}\right)$.

d) We know $2/9 = \pi(1) = [q_1 m_1]^{-1}$ so $m_1 = 9/(2q_1)$. It remains to find q_1. Now $q_{11} = P'_{11}(0) = (1/9)[-9te^{-3t} + 3e^{-3t} - 21e^{-3t}]_{t=0} = -2$ and we also know $q_{11} = -q_1(1 - Q_{11})$. By assumption, $Q_{11} = 0$ so $q_1 = -q_{11} = 2$. Finally we see that $m_1 = 9/(2q_1) = 9/4$.

2.4 Spring 2008 Final

1. Consider the stationary Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, 2\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

   a) (1.5) Find the stationary distribution π of $\{X_n\}$.
   b) (1.10) Approximate $P^n(0,1)$ for large n.
   c) (1.10) Find the mean return time to state 2.
   d) (1.8) For how many times n in the time interval $[1, 7000000]$ would you expect X_n to be equal to 1?

2. Classify each of the following statements as true or false.

   a) (1.5) Any steady-state distribution of a Markov chain must also be stationary.
   b) (1.6) If a state y of a Markov chain is absorbing, and some state $x \neq y$ leads to y, then x is a transient state.
   c) (3.3) If q_{xy} is an infinitesimal parameter of a continuous-time Markov chain (and $x \neq y$), then $q_{xy} \geq 0$.
   d) (1.8) If a bounded sequence of integers has Cesáro limit as $n \to \infty$, then that sequence converges as $n \to \infty$.
   e) (1.6) Every aperiodic, positive recurrent Markov chain has a steady-state distribution.
   f) (1.6) If a Markov chain has two (and only two) closed and irreducible positive recurrent sets A and B, then for any state x, $f_{x,A} + f_{x,B} = 1$.

   Justify any one of your answers above, either by proving the statement (if true) or by providing a counterexample (if false).

3. The parts of this question are unrelated.

   a) (2.6) Let $\{X_n\}$ be a birth and death chain with state space $\mathcal{S} = \{0, 1, 2, ...\}$ with birth rate $p_x = 1/2$ for all x and death rates $q_x = 1/(2x)$ for each $x \geq 1$. Is this chain positive recurrent, null recurrent or transient?
   b) (3.3) Suppose $\{X_t\}$ is a continuous-time Markov process with $P_{00}(t) = \frac{2}{3} + Ke^{-7t}$ and $P_{01}(t) = L - e^{-7t}$ where K and L are constants. Find K and L, and calculate $P(X_5 = 1 \text{ and } X_3 = 0 \,|\, X_2 = 0)$.

4. For each part of this question, describe a Markov chain $\{X_n\}$ with the given properties (give a separate chain for each part). You may describe each Markov chain by giving a transition function/matrix, or by drawing a graph and labelling arrows with the appropriate probabilities.

   a) (1.6) The state space $\mathcal{S}$ has seven elements; all states have period 4.

   b) (1.8) X_n is irreducible and positive recurrent; $\mathcal{S} = \{0, 1, 2, ...\}$; the mean return time to state zero is 4.

Solutions

1. a) Set $\pi = (a, b, c)$ and solve $\pi P = \pi$ (together with $a + b + c = 1$ to obtain the system of equations

$$\begin{cases} a/3 + b/2 + c/2 = a \\ a/3 + b/2 = b \\ a/3 + c/2 = c \\ a + b + c = 1 \end{cases}$$

which has solution $a = 3/7, b = 2/7, c = 2/7$. So $\pi = (\frac{3}{7}, \frac{2}{7}, \frac{2}{7})$.

b) $\{X_n\}$ is irreducible and aperiodic since $P(0, 0) > 0$, hence for large n, $P^n(0, 1) \approx \pi(1) = \frac{2}{7}$.

c) $m_2 = [\pi(2)]^{-1} = \frac{7}{2}$.

d) Let $N = 7000000$. By the ergodic theorem, the expected number of times in $[1, N]$ one would expect $X_n = 1$ is $N \cdot \pi(1) = 7000000 \cdot \frac{2}{7} = 2000000$.

2. a) TRUE (Proof: done in class; uses definitions and the Bounded Convergence Theorem.)

b) TRUE (Proof: if x is recurrent and $x \to y$, then by a theorem from class $y \to x$. But y does not lead to x since y is absorbing (y leads only to itself). Hence x is transient.)

c) TRUE (Proof: $q_{xy} = P'_{xy}(0)$. If $q_{xy} < 0$, then P_{xy} has negative slope at zero. But since $P_{xy}(0) = 0$, this means that for some $t > 0$, $P_{xy}(t) < 0$. But this is impossible since $P_{xy}(t)$ is by definition a probability.)

d) FALSE (Counterexample: $a_n = (-1)^n$ is bounded; has Cesàro limit 0 but has no limit as $n \to \infty$.)

e) FALSE (Counterexample: any aperiodic, positive recurrent, reducible Markov chain (for example, the two-state Markov chain with $P(0, 0) = P(1, 1) = 1$).)

f) FALSE (Counterexample: let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, ...\}$ and let 0 and 1 be absorbing (these will be the two closed and irreducible, positive recurrent sets A and B). For $x > 1$, define $P(x, x+1)$ to be 1. Therefore, every point other than 0 and 1 is transient, but for such a point x, $f_{x,A} + f_{x,B} = 0 + 0 = 0$.

3. a) Since $\{X_n\}$ is a birth and death chain, $\{X_n\}$ is recurrent if and only if

$$\sum_{x=1}^{\infty} \frac{q_1 \cdots q_x}{p_1 \cdots p_x} = \infty.$$

So we evaluate this expression:

$$\begin{aligned}
\sum_{x=1}^{\infty} \frac{q_1 \cdots q_x}{p_1 \cdots p_x} &= \sum_{x=1}^{\infty} \frac{\frac{1}{2}\frac{1}{4}\frac{1}{6} \cdots \frac{1}{(2x)}}{\frac{1}{2}\frac{1}{2} \cdots \frac{1}{2}} \\
&= \sum_{x=1}^{\infty} 1 \frac{1}{2}\frac{1}{3} \cdots \frac{1}{x} \\
&= \sum_{x=1}^{\infty} \frac{1}{x!} \\
&= \sum_{x=1}^{\infty} \frac{1^x}{x!} = e^1 = e < \infty.
\end{aligned}$$

Therefore $\{X_n\}$ is transient.

b) Since $P_{00}(0) = \delta_{00} = 1$, we have $\frac{2}{3} + Ke^{-7(0)} = 1$, i.e. $K = \frac{1}{3}$ and since $P_{01}(0) = \delta_{01} = 0$ we have $L - e^{-7(0)} = 0$, i.e. $L = 1$. Then

$$\begin{aligned}
P(X_5 = 1, X_3 = 0 \,|\, X_2 = 0) &= P(X_5 = 1 \,|\, X_3 = 0) \cdot P(X_3 = 0 \,|\, X_2 = 0) \\
&= P_{01}(2) P_{00}(1) \\
&= (1 - e^{-14}) \left(\frac{2}{3} + \frac{1}{3} e^{-7} \right).
\end{aligned}$$

4. a) Draw two squares which intersect in one point (which is a vertex of both squares). Set up the Markov chain so that you move around both squares in periodic cycles.

   b) There are many solutions; one is to define $P(x, 0) = 1/4$ for all x and $P(x, x+1) = 3/4$ for all x.

2.5 Spring 2009 Exam 1

1. Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and transition function P defined by

$$P(x,y) = \begin{cases} 2 \cdot (3^{-y}) & \text{if } x = 0, y > 0 \\ 1 & \text{if } x > 0, y = -x \\ 1 & \text{if } x < 0, y = 0 \\ 0 & \text{otherwise} \end{cases}$$

   a) (1.10) Show this Markov chain is positive recurrent and find the stationary distribution π.

   b) (1.10) Approximate the values of $P^n(0, y)$ for all $y \in \mathcal{S}$ where n is large (there will be different cases here).

2. Suppose $\{X_t\}$ is a continuous-time Markov process with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and time t transition function

$$P(t) = \frac{1}{12} \begin{pmatrix} 3 + 8e^{-t} + e^{-4t} & 3 - 3e^{-4t} & 3 - 4e^{-t} + e^{-4t} & 3 - 4e^{-t} + e^{-4t} \\ 3 - 3e^{-4t} & 3 + 9e^{-4t} & 3 - 3e^{-4t} & 3 - 3e^{-4t} \\ 3 - 4e^{-t} + e^{-4t} & 3 - 3e^{-4t} & 3 + 2e^{-t} + 6e^{-3t} + e^{-4t} & 3 + 2e^{-t} - 6e^{-3t} + e^{-4t} \\ 3 - 4e^{-t} + e^{-4t} & 3 - 3e^{-4t} & 3 + 2e^{-t} - 6e^{-3t} + e^{-4t} & 3 + 2e^{-t} + 6e^{-3t} + e^{-4t} \end{pmatrix}.$$

   Note: Do not forget the $1/12$ that is in front of this matrix.

   a) (3.4) Find the stationary distribution π.

   b) (3.4) Find the mean return time to state 3.

   c) (3.2) Suppose $X_0 = 1$. Given this, what is the probability that $X_t \neq 1$ for some $t \leq 3$?

   d) (3.2) Suppose $X_0 = 2$. What is the probability that the first state X_t visits other than 2 is state 1?

Solutions

1. a) Consider state 0; this state must return to itself after three steps and cannot return to itself before the third step; therefore $m_0 = 3 < \infty$ and 0 is therefore positive recurrent (furthermore, we know $\pi(0) = \frac{1}{m_0} = \frac{1}{3}$). Since 0 leads to every other state, every state is positive recurrent and hence X_n is positive recurrent.

   Alternate method of showing $\{X_n\}$ is positive recurrent: Define a factor $\{Z_n\}$ of this Markov chain as follows:

$$Z_n = \begin{cases} 0 & \text{if } X_n = 0 \\ 1 & \text{if } X_n > 0 \\ 2 & \text{if } X_n < 0 \end{cases};$$

   $\{Z_n\}$ has transition matrix

$$P^Z = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};$$

   clearly $\{Z_n\}$ is irreducible (since $0 \to 1 \to 2 \to 0$ in $\{Z_n\}$) and therefore positive recurrent since it has finite state space. Therefore 0 has finite mean return time m_0 in $\{Z_n\}$, and since 0 hasn't been grouped with any other states to form $\{X_n\}$, 0 must have the same finite mean return time in $\{X_n\}$, so 0 is positive recurrent in $\{X_n\}$. Since 0 leads to all the other states of $\{X_n\}$, all states are positive recurrent and therefore $\{X_n\}$ is positive recurrent.

   Let π be the stationary distribution of $\{X_n\}$. The stationary distribution of $\{Z_n\}$, denoted π^Z, can be found by setting $\pi^Z P^Z = \pi^Z$; this gives $\pi^Z = (1/3, 1/3, 1/3)$. So reasoning as before, since 0 wasn't grouped with any other state, $\pi(0) = \pi^Z(0) = 1/3$ (from the first paragraph of this solution, you can get this far without defining a factor). Now by the stationarity equation, if $y > 0$,

$$\pi(y) = \sum_{x \in \mathcal{S}} \pi(x)P(x,y) = \pi(0)P(0,y) = \frac{1}{3}2(3^{-y}) = \frac{2}{3^{y+1}}.$$

   Finally, if $y < 0$, the stationarity equation gives

$$\pi(y) = \sum_{x \in \mathcal{S}} \pi(x)P(x,y) = \pi(-y)P(-y,y) = \pi(-y) \cdot 1 = \frac{2}{3^{-y+1}}.$$

   So to summarize, the stationary distribution is

$$\pi(y) = \begin{cases} \frac{1}{3} & \text{if } y = 0 \\ \frac{2}{3^{|y|+1}} & \text{else} \end{cases}.$$

b) This chain has period $d = 3$; it rotates between states in the following fashion:

$$0 \to \{\text{positive numbers}\} \to \{\text{negative numbers}\} \to 0 \to \cdots$$

Notice that 0 always returns to itself every third step, and never returns on any number of steps which is not a multiple of three. So:

$$P^n(0,0) = \begin{cases} 1 & \text{if } n = 3m \\ 0 & \text{else} \end{cases}.$$

Next, 0 can only hit a positive number y on the n^{th} step if n has remainder 1 when divided by 3. For large n, $P^n(0,y)$ is either 0 or approximately $d \cdot \pi(y) = 3\pi(y)$ where π is the stationary distribution. So

$$y > 0 \Rightarrow P^n(0,y) \approx \begin{cases} 3\pi(y) = \frac{2}{3^y} & \text{if } n = 3m+1 \\ 0 & \text{else} \end{cases}.$$

Finally, 0 can only hit a negative number y on the n^{th} step if n has remainder 2 when divided by 3. For large n, $P^n(0,y)$ is either 0 or approximately $d \cdot \pi(y) = 3\pi(y)$ where π is the stationary distribution. So

$$y < 0 \Rightarrow P^n(0,y) \approx \begin{cases} 3\pi(y) = \frac{2}{3^{|y|}} = \frac{2}{3^{-y}} & \text{if } n = 3m+2 \\ 0 & \text{else} \end{cases}.$$

2. a) The stationary distribution π is steady-state, so it satisfies $\pi(y) = \lim_{t\to\infty} P_{xy}(t)$ for all $x, y \in \mathcal{S}$. As $t \to \infty$, $e^{-kt} \to 0$ for any $k > 0$, so

$$\lim_{t\to\infty} P(t) = \frac{1}{12}\begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}.$$

So $\pi = (1/4, 1/4, 1/4, 1/4)$.

b) First, find the infinitesimal matrix $Q = P'(0)$ (we don't actually need all the entries of Q):

$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}.$$

So $q_3 = -q_{33} = 2$. Finally, $\pi(3) = \frac{1}{m_3 q_3}$ so substituting in the known values $\pi(3) = 1/4$ and $q_3 = 2$, we get $m_3 = \frac{1}{q_3\pi(3)} = \frac{1}{2(1/4)} = 2$.

c) $P(X_t \neq 1 \text{ for some } t \leq 3) = P(W_1 \leq 3)$ where W_1 is the waiting time at state 1. Since W_1 is exponential with parameter $q_1 = -q_{11} = 3$, this is $F_{W_1}(3) = 1 - e^{-3\cdot 3} = 1 - e^{-9}$.

d) This is $\pi_{21} = \frac{q_{21}}{q_2} = \frac{1}{2}$.

2.6 Winter 2009 Exam 1

1. (2.5) Suppose $\{X_n\}$ is a random walk with $X_0 = x$. For some $a \leq x \leq b$, let $T = T_{a,b}$, the hitting time to set $\{a, b\}$. State the first and second Wald Identities for this random walk.

2. A gambler plays a series of games of chance in which he is equally likely to win or lose each game (there are no ties; assume the gambler bets $\$1$ per game). Suppose that he starts with a capital of \$20. He decides to quit the game if he goes broke or wins \$30.

   a) (2.5) Find the probability that when he quits he will have won \$30.

   b) (2.5) Find the expected number of games he will play before quitting.

Solutions

1. Wald's First Identity is $EX_T = x + ET$; Wald's Second Identity is Var $X_T = \sigma^2 ET$.

2. Let X_t represent the gambler's holdings after the t^{th} game. X_t is then a simple random walk with $X_0 = x = 20$ and $p = q = 1/2$.

   a) This is a stopping time/escape problem with $a = 0$ and $b = 20+30 = 50$. We have $P(X_T = b) = \frac{x-a}{b-a} = \frac{2}{5}$.

   b) From Wald's Second Identity, we can derive $ET = \frac{(b-x)(x-a)}{p+q} = \frac{(30)(20)}{1} = 600$.

2.7 Winter 2009 Final

1. Suppose the current value of a rare stamp is $\$500$. Each day, the value of the stamp either stays the same (with probability 95%), increases by $\$5$ (with probability 3%), or decreases by $\$5$ (with probability 2%).

   a) (2.5) How much should one expect the stamp to be worth 20 days from now?

   b) (2.5) What is the probability that the stamp is eventually worthless?

   c) (2.5) Suppose the stamp's owner decides to sell the stamp when its value reaches either $\$600$ or $\$300$. What is the probability that when the stamp is sold, it is worth $\$600$? (You only need to give an expression that gives the answer.)

2. Consider a Markov chain $\{X_n\}$ with state space $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \frac{3}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & \frac{5}{7} & \frac{2}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{6}{7} & \frac{1}{7} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

   a) (1.5) For each closed and irreducible class, find the stationary distribution supported on that class.

   b) (1.6) For every possible pair (x, C) of a transient state x and an irreducible closed set C, compute the absorption probability $f_{x,C}$.

   c) (1.10) Find the mean return time to state 1.

3. Let $P(x, y)$ be the transition function of a Markov chain $\{X_n\}$. A probability distribution η on the state space $\mathcal{S}$ is called **symmetric** (or said to be **in detailed balance**) if for every $x, y \in \mathcal{S}$,

$$\eta(x)P(x, y) = \eta(y)P(y, x).$$

   a) (1.5) Verify that every symmetric distribution is also stationary.

   b) (1.5) Suppose that the initial distribution π_0 of a Markov chain $\{X_n\}$ is symmetric. Show that

$$P(X_n = y \mid X_{n+1} = x) = P(x, y)$$

   for all $n \geq 0$. (This means the time-reversed chain is also a Markov chain with the same transition function as the original chain.)

4. Consider a Markov chain on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with transition function P defined by

$$P(0,0) = 0, \quad P(0,x) = \frac{C}{x(x+1)^2} \quad \text{and} \quad P(x, x-1) = 1$$

for all $x \geq 1$, where $C = \left[\sum_{x=1}^{\infty} \frac{1}{x(x+1)^2}\right]^{-1}$.

a) (1.10) Show this Markov chain is positive recurrent.

b) (1.10) Let $\pi = (\pi(0), \pi(1), \pi(2), \pi(3), \cdots)$ be the unique stationary distribution of the Markov chain. Calculate the values of $\pi(0), \pi(1)$, and $\pi(2)$ in terms of C.

Hint: The following identity may be useful:

$$\frac{1}{(x-1)x} = \frac{1}{x-1} - \frac{1}{x}.$$

Solutions

1. We model this situation by a random walk. To make the random walk simple, we re-scale so that the step size is $1 rather than $5 as follows: let $X_0 = 100$ and let $\{S_t\}$ be i.i.d. random variables taking the value 1 with probability $p = .03$, the value -1 with probability $q = .02$, and the value 0 with probability $r = .95$. Then if we let
$$X_n = X_0 + \sum_{t=1}^{n} S_t$$
be the simple random walk defined thusly, $5X_n$ gives the value of the stamp at time n.

   a) First, by direct calculation,
   $$E[S_t] = (1)(.03) + (-1)(.02) = .01$$
   for every t. Therefore
   $$E[X_{20}] = EX_0 + \sum_{t=1}^{20} E[S_t] = 100 + 20(.01) = 100.2$$
   and the stamp should be worth $5(100.2) = \$501$ twenty days from now.

   b) Since $p > q$, this probability (i.e., the gambler's ruin probability) is
   $$P_{100}(T_0 < \infty) = (q/p)^{100} = (2/3)^{100}.$$

   c) If the stamp is worth $600 at time n, then $X_n = 600/5 = 120$. Similarly, if the stamp is worth $300 at time n, then $X_n = 300/5 = 60$. So (using the language of Markov chains) the problem asks is for the probability $P_{100}(T_{120} < T_{60})$. We let $x = 100, a = 60, b = 120$ in the formula for the simple random walk escape probabilities and obtain
   $$P_{100}(T_{120} < T_{60}) = \frac{(q/p)^{x-a} - 1}{(q/p)^{b-a} - 1} = \frac{(2/3)^{40} - 1}{(2/3)^{60} - 1}.$$

2. a) First of all, observe $0 \leftrightarrow 1$ but neither 0 nor 1 lead anywhere else, so $C_1 = \{0, 1\}$ is a communicating class. Next, observe $3, 4, 5$ all lead to one another but none of these three points lead anywhere else, so $C_2 = \{3, 4, 5\}$ is a second communicating class. Finally, notice $2 \to 1$ but 1 does not lead back to 2, so 2 is transient; notice also that $6 \to 0$ but 0 does not lead back to 6 so 6 is transient.

   There is one stationary distribution π_1 supported on the closed, irreducible class $C_1 = \{0, 1\}$; write the restriction of P to C_1 as
   $$P|_{C_1} = \begin{pmatrix} 1/4 & 3/4 \\ 1/2 & 1/2 \end{pmatrix}.$$

The stationary distribution here is

$$\pi_1 = \left(\frac{1/2}{1/2+3/4}, \frac{3/4}{1/2+3/4}\right) = \left(\frac{2}{5}, \frac{3}{5}\right)$$

(using the formula for nontrivial two-state Markov chains). Thinking of this as a distribution on the entire state space, we have

$$\pi_1 = \left(\frac{2}{5}, \frac{3}{5}, 0, 0, 0, 0, 0\right).$$

Now, consider the other closed, irreducible class C_2 and find the stationary distribution π_2 supported on this class by considering the matrix $P|_{C_2}$, writing $\pi_2 = (a, b, c)$ and solving $\pi_2 \, P|_{C_2} = \pi_2$ (remembering that $a + b + c = 1$):

$$\begin{aligned}
\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{5}{7} & \frac{2}{7} & 0 \\ 0 & \frac{6}{7} & \frac{1}{7} \end{pmatrix} &= \begin{pmatrix} a & b & c \end{pmatrix} \\
&\Rightarrow \begin{cases} (2/7)a + (5/7)b = a \\ (3/7)a + (2/7)b + (6/7)c = b \\ (2/7)a + (1/7)c = c \end{cases} \\
&\Rightarrow \begin{cases} a = b \\ 3a + 6c = 5b \\ a = 3c \end{cases}
\end{aligned}$$

Remembering that $a + b + c = 1$, writing this in terms of c we have $3c + 3c + c = 1$ so $c = \frac{1}{7}$, and consequently $\pi_2 = (\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$. Thinking of this as a distribution on the whole state space, we have

$$\pi_2 = \left(0, 0, 0, \frac{3}{7}, \frac{3}{7}, \frac{1}{7}, 0\right).$$

b) First, observe $\mathcal{S}_T = \{2, 6\}$ since 2 and 6 lead to 0 but 0 does not lead back to 2 or 6. Consider $C = C_1$, the closed, irreducible, positive recurrent set $\{0, 1\}$. Let $a = f_{2,C}$ and $b = f_{6,C}$. We have the system of equations

$$\begin{cases} a = \frac{1}{5}(1) + \frac{1}{5}a \\ b = \frac{1}{4}(1) + \frac{1}{4}a + \frac{1}{4}b \end{cases}$$

Solving these equations for a and b yield $a = f_{2,C_1} = 1/4$ and $b = f_{6,C_1} = 5/12$. Since C_1 and C_2 are the only closed irreducible classes, we have $f_{2,C_2} = 1 - f_{2,C_1}$ and $f_{6,C_2} = 1 - f_{6,C_1}$. Therefore we obtain

$$f_{2,C_1} = \frac{1}{4}, \quad f_{2,C_2} = \frac{3}{4}, \quad f_{6,C_1} = \frac{5}{12}, \quad f_{6,C_2} = \frac{7}{12}.$$

c) The state 1 is positive recurrent since it belongs to a closed, irreducible, finite class; the stationary distribution supported on this class is π_1. So

$$m_1 = \frac{1}{\pi_1(1)} = \frac{1}{(2/5)} = \frac{5}{2}.$$

3. a) We need to show that $\eta P = \eta$, i.e.,

$$\sum_{x \in \mathbb{S}} \eta(x) P(x,y) = \eta(y).$$

Using the detailed balance equations we have

$$\begin{aligned}
\sum_x \eta(x) P(x,y) &= \sum_x \eta(y) P(y,x) \\
&= \eta(y) \sum_x P(y,x) \\
&= \eta(y) \cdot 1 \\
&= \eta(y)
\end{aligned}$$

So η is stationary by definition.

b) Since the initial distribution is symmetric, the distribution is stationary and thus $P(X_n = x) = \eta(x)$ for any $n \geq 0$ and any $x \in \mathcal{S}$.

$$\begin{aligned}
P(X_n = y \,|\, X_{n+1} = x) &= \frac{P(X_{n+1} = x, X_n = y)}{P(X_{n+1} = x)} \\
&= \frac{\eta(y) P(y,x)}{\eta(x)} \\
&= \frac{\eta(x) P(x,y)}{\eta(x)} \quad \text{(since } \eta \text{ is symmetric)} \\
&= P(x,y).
\end{aligned}$$

4. a) We calculate $m_0 = E_0 T_0$, the mean return time to state 0. Notice that if you start in state 0 and return for the first time on the n^{th} step, then your first step must be to state $n-1$, i.e. $P_0(T_0 = n) = P(0, n-1)$. Therefore we have $P_0(T_0 = 1) = P(0,0) = 0$ and $P_0(T_0 = n) = \frac{C}{(n-1)n^2}$ if $n \geq 2$. It

follows that

$$\begin{aligned} m_0 &= \sum_{n=1}^{\infty} n P_0(T_0 = n) \\ &= \sum_{n=2}^{\infty} n \cdot \frac{C}{(n-1)n^2} \\ &= C \sum_{n=2}^{\infty} \frac{1}{(n-1)n} \\ &= C \sum_{n=2}^{\infty} \left[\frac{1}{n-1} - \frac{1}{n} \right] \\ &= C < \infty \end{aligned}$$

so 0 is positive recurrent (to evaluate the series, write it out and observe how the terms cancel). Since the chain is obviously irreducible, the chain is positive recurrent.

b) There is one stationary distribution π since the chain is irreducible and positive recurrent. We have $\pi(0) = \frac{1}{m_0} = \frac{1}{C}$. Now by the definition of stationary distribution,

$$\sum_{x \in \mathcal{S}} \pi(x) P(x, y) = \pi(y), \qquad \forall y \in \mathcal{S}.$$

Plugging in $y = 0$ here, we have

$$\pi(0)P(0,0) + \pi(1)P(1,0) = \pi(0).$$

Since $P(0,0) = 0$ and $P(1,0) = 1$, we have $\pi(1) == \pi(0) = \frac{1}{C}$. Plugging $y = 1$ into the stationarity equation, we get

$$\begin{aligned} \pi(0)P(0,1) + \pi(2)P(2,1) &= \pi(1) \\ \frac{1}{C} \cdot \frac{C}{1 \cdot (1+1)^2} + \pi(2) \cdot 1 &= \frac{1}{C} \\ \frac{1}{4} + \pi(2) &= \frac{1}{C}. \end{aligned}$$

Hence $\pi(2) = \frac{1}{C} - \frac{1}{4}$. So the stationary distribution is

$$\pi = \left(\frac{1}{C}, \frac{1}{C}, \frac{1}{C} - \frac{1}{4}, \quad \ldots \right).$$

2.8 Winter 2010 Final

1. Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3\}$ and transition matrix
$$P = \begin{pmatrix} 0 & 1/5 & 0 & 4/5 \\ 2/5 & 0 & 3/5 & 0 \\ 0 & 4/5 & 0 & 1/5 \\ 1/5 & 0 & 4/5 & 0 \end{pmatrix}$$

   a) (1.5) Find the stationary distribution π of $\{X_n\}$.

   b) (1.10) Is the stationary distribution steady-state? Why or why not?

   c) (1.8) Find the mean return time to state 0.

   d) (1.10) For each state y, estimate $P^n(0, y)$ for n large (there may be different answers depending on the values of n and y).

2. Let $\{X_n\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, \dots\}$ and transition probabilities $P(k, 1) = \frac{1}{k+1}$ and $P(k, k+1) = \frac{k}{k+1}$. All other transition probabilities are 0.

   a) (1.3) Show that $P^{n-1}(1, n) = \dfrac{1}{n}$ for $n \geq 1$.

   b) (1.6) Find $P_1(T_1 \geq n)$ for $n \geq 1$.

   c) (1.7) Show that the process is recurrent.

   d) (1.8) Is the process null recurrent or positively recurrent? Justify your answer.

3. You buy a stock at \$100. Each day there is a 25% chance that the value of the stock goes up by \$1, a 25% chance that it goes down by \$1 and a 50% chance that its value stays the same (unless the price drops to \$0, in which case there is a 100% that the price remains \$0). You plan to hold the stock until it reaches \$300 (when you will sell it).

   a) (2.5) What is the probability that the story ends happily with you selling at \$300?

   b) (2.5) What is the expected time until you know the fate of your investment (i.e. until the price reaches \$0 or \$300)?

   c) (2.5) Suppose it turns out that a year later your stock is still worth \$100, but there is now a 20% chance that if goes up by \$1, a 30% chance that it goes down by \$1 and a 50% chance that its value stays the same. What is the chance of a happy ending (at \$300) now?

Solutions

1. a) Set $\pi = (\pi(0), \pi(1), \pi(2), \pi(3)) = (a, b, c, d)$. Notice that this Markov chain is periodic with period 2 and has periodic structure $\{0, 2\} \leftrightarrow \{1, 3\}$. This means that every other state is either a 0 or a 2, so by the ergodic theorem

$$\pi(0) + \pi(2) = \lim_{n\to\infty} \frac{N_n(0)}{n} + \lim_{n\to\infty} \frac{N_n(2)}{n} \lim_{n\to\infty} \frac{1}{n}(N_n(0) + N_n(2)) = \frac{1}{2}.$$

Therefore $a + c = \frac{1}{2}$, and also $b + d = \frac{1}{2}$.

Now if π is stationary, then we use the matrix equation $\pi P = \pi$ together with $a + b + c + d = 1$ to obtain the system

$$\left\{\begin{array}{rcl} a &=& (2/5)b + (1/5)d \\ b &=& (1/5)a + (4/5)c \\ c &=& (3/5)b + (4/5)d \\ d &=& (4/5)a + (1/5)c \\ 1 &=& a + b + c + d \end{array}\right. \Rightarrow \left\{\begin{array}{rcl} 5a &=& 2b + d \\ 5b &=& a + 4c \\ 5c &=& 3b + 4d \\ 5d &=& 4a + c \\ 1 &=& a + b + c + d \end{array}\right. .$$

Substitute $d = \frac{1}{2} - b$ and $c = \frac{1}{2} - a$ in the first two equations to obtain

$$\left\{\begin{array}{rcl} 5a &=& b + 1/2 \\ 5b &=& (-1/2)a + 2 \end{array}\right. ;$$

these can be solved to obtain $a = 9/56, b = 17/56$. Finally, $c = 1/2 - a = 19/56$ and $d = 1/2 - b = 11/56$. So $\pi = \left(\frac{9}{56}, \frac{17}{56}, \frac{19}{56}, \frac{11}{56}\right)$.

b) This Markov chain has period 2, so the stationary distribution is not steady state.

c) Let π be the stationary distribution; then $m_0 = \frac{1}{\pi(0)} = \frac{56}{9}$.

d) As stated above, X_t is periodic with period $d = 2$ and has periodic structure $\{0, 2\} \leftrightarrow \{1, 3\}$. So by the theory developed in class, for large n,

$$P^n(0,0) \approx \left\{\begin{array}{cl} d \cdot \pi(0) = \frac{9}{28} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array}\right. ;$$

$$P^n(0,1) \approx \left\{\begin{array}{cl} d \cdot \pi(1) = \frac{17}{28} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{array}\right. ;$$

$$P^n(0,2) \approx \left\{\begin{array}{cl} d \cdot \pi(2) = \frac{19}{28} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array}\right. ;$$

$$P^n(0,3) \approx \left\{\begin{array}{cl} d \cdot \pi(3) = \frac{11}{28} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{array}\right. .$$

2. a) There is one and only one way to get from 1 to n in $n-1$ steps: $1 \to 2 \to 3 \to \cdots \to n$. The probability of this sequence of steps is
$$P(1,2)P(2,3)\cdot\dots\cdot P(n-1,n) = \frac{1}{2}\cdot\frac{2}{3}\cdot\dots\cdot\frac{n-1}{n} = \frac{1}{n}.$$
b) Observe that if $X_0 = 1$, $T_1 \geq n$ if and only if your first $n-1$ steps are $1 \to 2 \to 3 \to \cdots \to n$. The probability of this is $1/n$ by (a).

c) By (b), $P_1(T_1 = \infty) = \lim_{n\to\infty} P_1(T_1 \geq n) = \lim_{n\to\infty} 1/n = 0$. Therefore $f_{1,1} = P_1(T_1 < \infty) = 1 - 0 = 1$, so the state 1 is recurrent. It is obvious that the process is irreducible, so all states are recurrent.

d) Calculate the mean return time to state 1:
$$\begin{aligned} m_1 = E_1(T_1) &= \sum_{n=1}^{\infty} n \cdot P_1(T_1 = n) \\ &= P_1(T_1 = 1) + 2\cdot P_1(T_1 = 2) + 3\cdot P_1(T_1 = 3) + ... \\ &= P_1(T_1 = 1) + P_1(T_1 = 2) + P_1(T_1 = 3) + ... \\ &\qquad\qquad +P_1(T_1 = 2) + P_1(T_1 = 3) + ... \\ &\qquad\qquad\qquad\qquad +P_1(T_1 = 3) + ... \\ &\qquad\qquad\qquad\qquad\qquad\qquad +... \\ &= P_1(T_1 \geq 1) + P_1(T_1 \geq 2) + P_1(T_1 \geq 3) + ... \\ &= \sum_{n=1}^{\infty} P_1(T_1 \geq n) \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad \text{(by part (a))} \\ &= \infty. \end{aligned}$$
Therefore 1 is null recurrent and since X_n is irreducible, the entire chain is null recurrent.

3. a) This is modeled by a random walk with $X_0 = x = 100$; $p = q = .25$. Let $T = \min(T_0, T_{300})$. Then
$$P(X_T = 300) = \frac{x-a}{b-a} = \frac{100-0}{300-0} = \frac{1}{3}.$$
b) Repeating the notation as in part (a),
$$ET = \frac{(x-a)(b-x)}{p+q} = \frac{(100-0)(300-100)}{1-1/2} = 2\cdot 100\cdot 200 = 40000.$$
c) This time, we model the situation by a random walk with $X_0 = x = 100$, $p = .3$, $q = .2$. Now if $T = \min(T_0, T_{300})$, then
$$P(X_T = 300) = \frac{(q/p)^{x-a} - 1}{(q/p)^{b-a} - 1} = \frac{(3/2)^{100} - 1}{(3/2)^{300} - 1}.$$

2.9 Spring 2009 Exam 2

1. Let $\{W_t\}$ be a one-dimensional Brownian motion with parameter $\sigma^2 = 3$.

   a) Describe the random variable W_4, either by giving its density function, or by describing the distribution in words or symbols (giving parameters if necessary).

   b) (4.1, 4.4) Find $Cov(W_6, W_5)$.

   c) (4.1, 4.4) Find $Cov(W_9 - W_6, W_5 - W_2)$.

   d) (4.1) Find $P(W_0 < W_1 < W_2 < W_3)$.

   e) (4.2) Let $M = \max\{W_s : 0 \leq s \leq t\}$. Find the density of M (you may assume that M is a continuous random variable).

2. Let $\{W_t\}$ be standard one-dimensional Brownian motion. Define, for $t \geq 0$, a process $\{Y_t\}$ by $Y_t = e^{-t}W_{e^{2t}}$. ($\{Y_t\}$ is called an **Ornstein-Uhlenbeck process**.)

   a) (4.4) Find the mean function of $Y(t)$.

   b) (4.4) Find the covariance function of $Y(t)$.

   c) (4.4) Is $Y(t)$ a Gaussian process? Why or why not?

Solutions

1. a) W_4 is normal $n(0, 12)$ (the mean of W_t is always zero, and the variance of W_t is $\sigma^2 t = 3 \cdot 4 = 12$.

   b) $Cov(W_s, W_t)) = r_W(s, t) = \sigma^2 \min(s, t)$; in this case we have $\sigma^2 = 3$, $s = 6$ and $t = 5$ so the covariance is $3 \cdot 5 = 15$.

   c) By the independent increment property, these two random variables are independent. Therefore $Cov(W_9 - W_6, W_5 - W_2) = 0$.

   d) First, observe that for $j = 0, 1, 2$ we see that $W_{j+1} - W_j$ is normal with mean 0 and variance 3, so

$$P(W_{j+1} > W_j) = P(W_{j+1} - W_j > 0) = P(n(0, 3) > 0) = 1/2$$

   since a normal random variable is symmetric about its mean. Now we have

$$\begin{aligned} P(0 < W_1 < W_2 < W_3) &= P(0 < W_1) \cdot P(W_1 < W_2) \cdot P(W_2 < W_3) \\ &\qquad \text{(by the } \perp \text{ increment property)} \\ &= (1/2)(1/2)(1/2) \\ &= 1/8. \end{aligned}$$

   e) Start by finding the cumulative distribution function of M.

$$\begin{aligned} F_M(m) &= P(M \leq m) \\ &= P(M < m) \\ &\qquad \text{(by the assumption that } M \text{ is continuous)} \\ &= P(T_m > t) \\ &\qquad \text{(where } T_m = \min\{t > 0 : W_t = m\}) \\ &= 1 - P(T_m \leq t) \\ &= 1 - \left[2 - 2\Phi\left(\frac{m}{\sqrt{3t}}\right)\right] \\ &\qquad \text{(by the reflection principle)} \\ &= 2\Phi\left(\frac{m}{\sqrt{3t}}\right) - 1. \end{aligned}$$

   This holds for $m \geq 0$; if $m < 0$ it is clear that $P(M \leq m) = 0$ since $W_0 = 0$. Next, find the density function of M (recall that $\Phi'(x) = \phi(x) =$

$\frac{1}{\sqrt{2\pi}}e^{(-x^2/2)}$, the density of the standard normal). For $m \geq 0$, we have:

$$\begin{aligned} f_M(m) &= \frac{d}{dm}\left[2\Phi\left(\frac{m}{\sqrt{3t}}\right) - 1\right] \\ &= 2\left(\frac{1}{\sqrt{2\pi}}\right)\exp\left(\frac{-m^2}{2\cdot 3t}\right)\left(\frac{1}{\sqrt{3t}}\right) \\ &= \sqrt{\frac{2}{3\pi t}}\exp\left(\frac{-m^2}{6t}\right) \end{aligned}$$

This holds for $m \geq 0$. If $m < 0$, it is clear that $f_M(m) = 0$.

2. a) This is a direct calculation using linearity of expectation:

$$\mu_Y(t) = E[Y_t] = E[e^{-t}W_{e^{2t}}] = e^{-t}E[W_{e^{2t}}] = e^{-t}\cdot 0 = 0.$$

b) The covariance function is a direct calculation:

$$\begin{aligned} r_Y(s,t) &= Cov(Y_s, Y_t) \\ &= Cov(e^{-s}W_{e^{2s}}, e^{-t}W_{e^{2t}}) \\ &= e^{-s}e^{-t}Cov(W_{e^{2s}}, W_{e^{2t}}) \\ &= e^{-s-t}r_W(e^{2s}, e^{2t}) \\ &= e^{-s-t}\min(e^{2s}, e^{2t}) \\ &= \begin{cases} e^{-s-t+2s} & \text{if } s \leq t \\ e^{-s-t+2t} & \text{if } t < s \end{cases} \\ &= \begin{cases} e^{s-t} & \text{if } s \leq t \\ e^{t-s} & \text{if } t < s \end{cases} \\ &= e^{-|t-s|}. \end{aligned}$$

c) We know by definition that $\{W_t\}$ is Gaussian. By a theorem from Chapter 6, if $\{W_t\}$ is a Gaussian process, then for any real-valued functions f and g, $Y_t = f(t)W_{g(t)}$ is also Gaussian. So $\{Y_t\}$ is indeed Gaussian.

2.10 Spring 2009 Final

1. Classify each of the following statements as true or false:

   a) (4.7) Standard two-dimensional Brownian motion is point recurrent.

   b) (1.10) If a factor of a Markov chain is positive recurrent, then the original chain is also positive recurrent.

   c) (4.4) A Gaussian process is determined completely by its mean and covariance functions.

2. Consider a Markov chain $\{X_n\}$ where the state space consists of the eight vertices of a cube. The transition probabilities are given as follows: $P(x,y) = 1/3$ if there is an edge of the cube running from x to y, and $P(x,y) = 0$ otherwise. (To avoid confusion, note that this implies that $P(x,x) = 0$ for all vertices x.)

   a) (1.10) Find the stationary distribution π of this Markov chain.

   b) (1.10) Are there any steady-state distributions of this Markov chain? Why or why not?

   c) (1.10) Let x be any vertex of the cube. Approximate $P^n(x,x)$ for large n.

3. Let $\{X_t\}$ be a CTMC with $\mathcal{S} = \{0,1,2\}$ such that $\pi_{01} = \pi_{12} = \pi_{20} = 1$ and $q_0 = 2$, $q_1 = 3$, $q_2 = 10$.

   a) (3.3) Find the infinitesimal matrix Q of this CTMC.

   b) (3.2) Suppose $X_0 = 1$. What is the probability that $X_t = 1$ for all $t < 4$?

   c) (3.2) Assume that -7 is one eigenvalue of Q. What are the other eigenvalues of Q? (*Hint:* You may not need to solve $\det(Q - \lambda I) = 0$ here.)

   d) (3.4) Find the stationary distribution of this Markov process.

   e) (3.4) Find the mean return time to state 2.

   f) (3.4) For what total length of time in the first 56 million units of time would you expect $X_t = 1$?

4. Let $\{W_t\}$ and $\{V_t\}$ be independent, standard one-dimensional Brownian motions.

   a) (4.4) Let $Z_t = W_t - V_t$. Show that $\{Z_t\}$ is a Brownian motion; what is its parameter?

   b) (4.4) Use part (a) to show that

   $$P(W_t = V_t \text{ for infinitely many } t) = 1.$$

5. The parts of this question are unrelated.

a) (3.3) Consider a pure birth CTMC on $\mathcal{S} = \{0, 1, 2, ...\}$ with birth rates $\lambda_x = \lambda x$ for all $x \in \mathcal{S}$. Write the forward equation for this process.

b) (3.3) Let $\{X_t\}$ be a continuous-time Markov chain with finite state space such that $q_{00} = -1$, $q_{01} = 1$ and $q_{11} = -1$. Suppose $P_{01}(t) = A + Be^{-6t} + Ce^{-4t}$. Find A, B and C.

Solutions

1. a) FALSE; Brownian motion in two dimensions is neighborhood recurrent but not point recurrent.

   b) FALSE; a Markov chain with state space $\mathcal{S} = \{0\}$ and transition function $P(0,0) = 1$ is obviously positive recurrent and is a factor of every Markov chain (by grouping all the states together), no matter whether the original chain was positive recurrent or not.

   c) TRUE; this is a central fact about Gaussian processes.

2. a) By symmetry, π must assign the same probability to all the vertices, so π is uniform on the eight vertices: $\pi = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$.

   b) For any vertex x, $P^n(x,x) > 0$ implies that n is a multiple of 2, so this Markov chain is not aperiodic; hence there are no steady-state distributions.

   c) The period of this Markov chain is 2, since $P^2(x,x) > 0$ (to see this, think about starting at x, travelling along an edge, then going back to x along the same edge on the second step). Since $P^2(x,x) > 0$ and $d = 2$, we have
$$P^n(x,x) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ d \cdot \pi(x) = 2 \cdot \frac{1}{8} = \frac{1}{4} & \text{if } n \text{ is even} \end{cases}.$$

3. a) From the formulas $q_{xx} = -q_x$ and $q_{xy} = q_x \pi_{xy}$ for $x \neq y$, we have
$$Q = (q_{xy}) = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -3 & 3 \\ 10 & 0 & -10 \end{pmatrix}.$$

   b) Let W_1 be the waiting time for state 1; W_1 is exponential with parameter $q_1 = 3$. So:
$$\begin{aligned} P(X_t = 1 \,\forall t < 4 \,|\, X_0 = 1) &= P(W_1 \geq 4) \\ &= 1 - \left[1 - e^{-3 \cdot 4}\right] \\ &= e^{-12}. \end{aligned}$$

   c) The trace of Q is -15 (sum the diagonal entries to get this); this must also be the sum of the eigenvalues. Now 0 is an eigenvalue of any $Q-$matrix, and we are given -7 is an eigenvalue, so the third eigenvalue must be $-15 - (0 + (-7)) = -8$. To summarize, Q has eigenvalues $0, -7$ and -8.

   d) Let π be the stationary distribution of the process; write $\pi = (a,b,c)$ and set $\pi Q = \overrightarrow{0}$ to obtain
$$\begin{cases} -2a + 10c = 0 \\ 2a - 3b = 0 \\ 3b - 10c = 0 \end{cases}$$

Solving this together with $a+b+c=1$, we obtain $\pi=(a,b,c)=\left(\frac{15}{28},\frac{5}{14},\frac{3}{28}\right)$.

e) If m_2 is the mean return time to state 2, then for the stationary distribution π we have $\pi(2)=(m_2q_2)^{-1}$. We know $\pi(2)=3/28$ from (d) and $q_2=10$ is given; by substituting this in we obtain $\frac{3}{28}=\frac{1}{10m_2}$. Solving this, we get $m_2=\frac{14}{15}$.

f) By the ergodic theorem, this should be roughly $\pi(1)\cdot(56\,000\,000)=30\,000\,000$ units of time.

4. a) First, $\{Z_t\}$ is Gaussian since it is the difference of the Gaussian processes $\{W_t\}$ and $\{V_t\}$. We find the mean and covariance functions of $\{Z_t\}$:

$$\mu_Z(t)=E[W_t]-E[V_t]=0-0=0.$$

$$\begin{aligned}
r_Z(s,t) &= Cov[Z_s,Z_t]\\
&= Cov[W_s-V_s,W_t-V_t]\\
&= r_W(s,t)-Cov(W(s),V(t))-Cov(W(t),V(s))+r_V(s,t)\\
&= \min(s,t)-0-0+\min(s,t)\\
&\quad \text{(since } \{W_t\}\perp\{V_t\})\\
&= 2\min(s,t).
\end{aligned}$$

These are the same mean and covariance functions as those of a Brownian motion with parameter $\sigma^2=2$, so $\{Z_t\}$ is in fact Brownian motion with parameter $\sigma^2=2$ since the mean and covariance functions completely determine a Gaussian process.

b) Since $\{Z_t\}$ is a one-dimensional Brownian motion, it is recurrent. This means that the probability that $Z_t=0$ for infinitely many t is 1. But

$$W_t=V_t \Leftrightarrow W_t-V_t=0 \Leftrightarrow Z_t=0$$

so therefore $P(W_t=V_t$ for infinitely many $t)$ is 1.

5. a) The forward equation for a general CTMC is $P'_{xy}(t)=\sum_{z\in\mathcal{S}}P_{xz}(t)q_{zy}$. For the process in this problem, the values q_{zy} are zero unless $z=y-1$, in which case $q_{zy}=\lambda_z=\lambda z=\lambda(y-1)$, or $z=y$, in which case $q_{zy}=q_{yy}=-\lambda y$. So the forward equation simplifies to

$$P'_{xy}(t)=\lambda(y-1)P_{x,y-1}(t)-\lambda yP_{xy}(t).$$

(Of course, this simplifies if $y<x$ to obtain $P_{xy}(t)=0$.)

b) Observe that $P_{01}(0) = \delta_{01} = 0$ since $0 \neq 1$ and $P'_{01}(0) = q_{01} = 1$. Finally, $P'_{01}(0)$ must equal the $0, 1-$entry of the matrix Q^2 since $P^{(n)}(0) = Q^n$ for all positive integers n. We can find the $0, 1-$entry of Q^2 from the given information: since $q_{00} + q_{01} = 0$, we know that $q_{0y} = 0$ for all $y > 1$. Now, letting v be the $0, 1-$entry of Q^2, by matrix multiplication we have

$$v = \sum_y q_{0y}q_{y1} = (-1)(1) + (1)(-1) + 0 + 0 + ... + 0 = -2.$$

Putting this together, we have the system of equations

$$\begin{cases} P_{01}(0) = 0 \\ P'_{01}(0) = q_{01} = 1 \\ P''_{01}(0) = v = -2 \end{cases} \Rightarrow \begin{cases} A + B + C = 0 \\ -6B - 4C = 1 \\ 36B + 16C = -2 \end{cases}$$

The equations on the right were found by differentiating the given expression $P_{01}(t) = A + Be^{-4t} + Ce^{-6t}$ with respect to t (differentiating once for the second equation and twice for the last equation), then plugging in $t = 0$ to obtain expressions in terms of A, B and C. Finally, this linear system of equations can be solved for the unknowns to obtain $A = 1/3$, $B = 1/6$ and $C = -1/2$.

2.11 Spring 2010 Exam 1

1. Let $\{X_t\}$ be a continuous-time Markov chain with state space $\mathcal{S} = \{0, 1, 2\}$ and transition function

$$P(t) = \frac{1}{18} \begin{pmatrix} 5 + 5e^{-6t} + 8e^{-3t} & 2 - 10e^{-6t} + 8e^{-3t} & 11 + 5e^{-6t} - 16e^{-3t} \\ 5 - 7e^{-6t} + 2e^{-3t} & 2 + 14e^{-6t} + 2e^{-3t} & 11 - 7e^{-6t} - 4e^{-3t} \\ 5 - e^{-6t} - 4e^{-3t} & 2 + 2e^{-6t} - 4e^{-3t} & 11 - e^{-6t} + 8e^{-3t} \end{pmatrix}.$$

   a) (3.2) Suppose $X_0 = X_1 = X_2 = 1$. Given this, find the probability that $X_4 = 2$.
   b) (3.2) Suppose $X_0 = X_3 = 1$. Given this, find the probability that $X_2 = 1$.
   c) (3.3) Suppose $X_0 = 1$; find the probability that when the process jumps for the first time, it jumps from state 1 to state 2.
   d) (3.2) Suppose $X_0 = 1$; find the probability that $X_t \neq 1$ for some $t \leq 3$.
   e) (3.4) Find all stationary distributions of X_t.

2. (3.7) Find the mean return time to each state in the infinite server queue.

Solutions

1. a) By the Markov property, this probability is

$$P(X_4 = 2 \,|\, X_2 = 1) = P_{12}(2) = \frac{1}{18}\left(11 - 7e^{-12} - 4e^{-6}\right).$$

b) By the definition of conditional probability:

$$\begin{aligned}
& P(X_2 = 1 \,|\, X_0 = X_3 = 1) \\
= \ & \frac{P(X_0 = X_2 = X_3 = 1)}{P(X_0 = X_3 = 1)} \\
= \ & \frac{P(X_0 = 1)\cdot P_{11}(2)\cdot P_{11}(1)}{P(X_0 = 1)\cdot P_{11}(3)} \\
= \ & \frac{P_{11}(2)\cdot P_{11}(1)}{P_{11}(3)} \\
= \ & \frac{\frac{1}{18}\left(2 + 14e^{-12} + 2e^{-6}\right)\cdot \frac{1}{18}\left(2 + 14e^{-6} + 2e^{-3}\right)}{\frac{1}{18}\left(2 + 14e^{-18} + 2e^{-9}\right)} \\
= \ & \frac{\left(2 + 14e^{-12} + 2e^{-6}\right)\left(2 + 14e^{-6} + 2e^{-3}\right)}{18\left(2 + 14e^{-18} + 2e^{-9}\right)}.
\end{aligned}$$

c) First, find the infinitesimal parameter q_{12}:

$$\begin{aligned}
q_{12} = P'_{12}(0) &= \frac{d}{dt}\left[\frac{1}{18}\left(11 - 7e^{-6t} - 4e^{-3t}\right)\right]_{t=0} \\
&= \frac{1}{18}\left(42e^{-6(0)} + 12e^{-6(0)}\right) = \frac{54}{18} = 3.
\end{aligned}$$

Next, find the holding rate q_1:

$$\begin{aligned}
q_1 = -q_{11} = -P'_{11}(0) &= -\frac{d}{dt}\left[\frac{1}{18}\left(2 + 14e^{-6t} + 2e^{-3t}\right)\right]_{t=0} \\
&= \frac{-1}{18}\left(-84e^{-6(0)} - 6e^{-6(0)}\right) = \frac{90}{18} = 5.
\end{aligned}$$

Finally, the jump probability is $\pi_{12} = q_{12}/q_1 = 3/5$.

d) From part (c), we see that W_1 is exponential with parameter $q_1 = 5$. So the probability that the process jumps before time 3 is $P(W_1 \leq 5) = 1 - e^{-5(3)} = 1 - e^{-15}$.

e) This process is clearly irreducible and since the state space is finite, it is positive recurrent. Therefore it has one stationary distribution which is steady state, i.e. the stationary distribution satisfies $\pi(y) = \lim_{t\to\infty} P_{xy}(t)$ for all $x, y \in \mathcal{S}$. By evaluating this limit, we see that $\pi = \left(\frac{5}{18}, \frac{1}{9}, \frac{11}{18}\right)$.

2. We know from class that the stationary distribution of the infinite server queue is Poisson with parameter λ/μ, i.e.

$$\pi(x) = e^{-\lambda/\mu}\frac{(\lambda/\mu)^x}{x!} = \frac{\lambda^x}{e^{\lambda/\mu}\mu^x x!}.$$

Now the holding rate q_x at state x must satisfy $q_x = -q_{xx} = \sum_{y\neq x} q_{xy}$; since the infinite server queue is a birth/death process, the only nonzero q_{xy} for $x \neq y$ are $q_{x,x+1} = \lambda_x = \lambda$ and $q_{x,x-1} = \mu_x = \mu x$. Therefore

$$q_x = \lambda + \mu x;$$

finally since $\pi(x) = 1/(m_x\, q_x)$ we see that

$$\frac{\lambda^x}{e^{\lambda/\mu}\mu^x x!} = \frac{1}{m_x(\lambda+\mu x)} \quad\Rightarrow\quad m_x = \frac{e^{\lambda/\mu}\mu^x x!}{\lambda^x(\lambda+\mu x)}.$$

2.12 Spring 2010 Exam 2

1. Let $\{W_t\}$ be standard, one-dimensional Brownian motion.
   a) (4.1, 4.4) Find $Cov[W_3, W_5 - 2W_2]$.
   b) (4.5) What is the probability that $W_t < 0$ for all $t \in (0, 1)$?
   c) (4.2) What is the probability that $W_t > -1$ for all $t \in (0, 1)$?
   d) (4.4) Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that $X_t = f(t)W_{e^t}$ is a Brownian motion.

2. Let $\{Y_t : t \geq 0\}$ be a Gaussian process with $\mu_Y(t) = 0$ and $r_Y(s, t) = e^{-|t-s|}$.
   a) (4.4) For a fixed t, describe the random variable Y_t, either by giving its density function or by describing it in words or symbols (giving parameters if necessary).
   b) (4.4) Let $Z_t = \sqrt{t}\, Y(\frac{1}{2} \ln t)$. Show that Z_t is standard Brownian motion.

Solutions

1. a) We calculate the covariance using linearity and the fact that

$$Cov(W_s, W_t) = r_W(s,t) = \min(s,t) :$$

$$\begin{aligned} Cov(W_3, W_5 - 2W_2) &= Cov(W_3, W_5) - 2Cov(W_3, W_2) \\ &= 3 - 2 \cdot 2 = -1. \end{aligned}$$

b) Let $Z = \{t : W_t = 0\}$; it was proven in Section 6.6 that with probability one, $Z \cap (0, \epsilon)$ is infinite for every $\epsilon > 0$. Hence

$$P(W_t < 0 \,\forall t \in (0,1)) = 0.$$

c) This is an application of the reflection principle:

$$\begin{aligned} P(W_t > -1 \;\forall t \in (0,1)) &= P(W_t < 1 \;\forall t < 1) \\ &\qquad \text{(by symmetry of } W_t\text{)} \\ &= P(T_1 \geq 1) \\ &\qquad \text{(where } T_1 = \min\{t > 0 : W(t) = 1\}\text{)} \\ &= 1 - P(T_1 \leq 1) \\ &= 1 - \left[2 - 2\Phi\left(\frac{1}{\sqrt{1}}\right)\right] \\ &= 2\Phi(1) - 1. \end{aligned}$$

d) Suppose such a function exists. Then $\{X_t\}$ must have the same covariance function as a Brownian motion, i.e. $r_X(s,t) = \sigma^2 \min(s,t)$ for some constant σ^2. But by direct calculation,

$$\begin{aligned} r_X(s,t) &= Cov(X_s, X_t) \\ &= Cov(f(s)W_{e^s}, f(t)W_{e^t}) \\ &= f(s)f(t)r_W(e^s, e^t) \\ &= f(s)f(t)e^{\min(s,t)}. \end{aligned}$$

We conclude $f(s)f(t)e^{\min(s,t)} = \sigma^2 \min(s,t)$ for all s and t. In particular, this equation must hold for $s = t$, i.e. $[f(s)]^2 e^s = \sigma^2 s$ and therefore

$$f(s) = \sqrt{\frac{\sigma^2 s}{e^s}} = \frac{\sigma\sqrt{s}}{e^{s/2}}.$$

But this choice of f does not work. For example,

$$r_X(1,9) = f(1)f(9)e^{\min(1,9)} = \left(\frac{\sigma\sqrt{1}}{e^{1/2}}\right) \cdot \left(\frac{\sigma\sqrt{9}}{e^{9/2}}\right) \cdot e^1 = 3e^{-4}\sigma^2 \neq \sigma^2 \min(1,9).$$

Therefore there is no way for $\{X_t\}$ to have the same covariance function as a BM, so $\{X_t\}$ cannot be a BM.

2. a) Since $\{Y_t\}$ is a Gaussian process, Y_t is normal for each t. Y_t has mean $\mu_Y(t) = 0$ and variance $r_Y(t,t) = e^{-|t-t|} = e^0 = 1$ so in fact Y_t is standard normal for every t.

b) By a theorem from Chapter 6, since $\{Y_t\}$ is Gaussian, so is $\{Z_t\}$ since Z_t is of the form $f(t)Y_{g(t)}$ for functions f and g. Now

$$\mu_Z(t) = E[Z_t] = E\left[\sqrt{t}\,Y\left(\frac{1}{2}\ln t\right)\right] = \sqrt{t}\mu_Y\left(\frac{1}{2}\ln t\right) = \sqrt{t}\cdot 0 = 0$$

and

$$\begin{aligned}
r_Z(s,t) &= Cov(Z_s, Z_t) \\
&= Cov\left(\sqrt{s}\,Y_{\left(\frac{1}{2}\ln s\right)}, \sqrt{t}\,Y_{\left(\frac{1}{2}\ln t\right)}\right) \\
&= \sqrt{st} r_Y\left(\frac{1}{2}\ln s, \frac{1}{2}\ln t\right) \\
&= \sqrt{st}\cdot e^{-|\frac{1}{2}(\ln t - \ln s)|} \\
&= \begin{cases} \sqrt{st}\cdot e^{-\frac{1}{2}(\ln t - \ln s)} & \text{if } s \le t \\ \sqrt{st}\cdot e^{-\frac{1}{2}(\ln s - \ln t)} & \text{if } t < s \end{cases} \\
&= \begin{cases} \sqrt{st}\cdot e^{\ln\sqrt{s/t}} & \text{if } s \le t \\ \sqrt{st}\cdot e^{\ln\sqrt{t/s}} & \text{if } t < s \end{cases} \\
&= \begin{cases} \sqrt{st}\cdot\sqrt{\frac{s}{t}} & \text{if } s \le t \\ \sqrt{st}\cdot\sqrt{\frac{t}{s}} & \text{if } t < s \end{cases} \\
&= \begin{cases} s & \text{if } s \le t \\ t & \text{if } t < s \end{cases} \\
&= \min(s,t).
\end{aligned}$$

Since $\{Z_t\}$ is a Gaussian process with the same mean and covariance functions as standard Brownian motion, $\{Z_t\}$ is standard Brownian motion, since mean and covariance functions determine a Gaussian process..

2.13 Spring 2010 Final

1. Classify each of the following statements as true or false:

   a) (Group Presentations) A (discrete-time) branching chain is recurrent if and only if the expected number of offspring each organism has is at least 1.

   b) (3.5) A birth-death CTMC on $\mathcal{S} = \{0, 1, 2, ...\}$ is determined completely by its birth and death rates.

   c) (3.4) A state in a continuous-time Markov chain is transient if and only if it is transient in the jump chain of the CTMC.

   d) (3.4) Any stationary distribution of an irreducible, continuous-time Markov chain is steady-state.

2. Let $\{X_t\}$ be a continuous-time Markov chain with state space $\mathcal{S} = \{0, 1, 2\}$ and infinitesimal matrix

$$Q = \begin{pmatrix} -7 & 3 & 4 \\ 4 & -6 & 2 \\ 4 & 0 & -4 \end{pmatrix}$$

   a) (3.4) Find the stationary distribution of $\{X_t\}$.

   b) (3.4) Find the mean return time to state 2.

   c) (3.4) Find the holding rate of state 1.

   d) (3.3) The forward equation associated to this CTMC is a system of how many differential equations?

   e) (3.3) Write down any one of the differential equations which comprise the forward equation, plugging in any numbers which are entries of Q.

3. Let $\{W_t\}$ be a standard, two-dimensional Brownian motion; for each t set $W_t = (X_t, Y_t)$.

   a) (4.7) What is $P(W_t = (0, 0)$ for some $t > 0)$?

   b) (4.1, 4.7) Find $P(W_5$ lies in Quadrant IV$)$.

   c) (4.2) Let $\gamma = P(|X_2| > |X_1|)$. Is γ greater than $1/2$, less than $1/2$, or equal to $1/2$ (one of these three statements is true with probability one)? Briefly explain your answer.

   d) (4.4) Show that $3X_t - 2Y_t$ is a Brownian motion. What is its parameter?

4. Let $\{W_t\}$ be a standard one-dimensional Brownian motion and let $\mu \in \mathbb{R}$ be a constant. Define $X_t = W_t + \mu t$; $\{X_t\}$ is called **Brownian motion with drift**; it is the continuous analogue of biased random walk.

   a) (4.4) Describe the random variable X_r, either by giving its density function explicitly or by describing it in words or symbols (giving parameters if necessary).

   b) (4.4) Find the mean and covariance functions of X_t.

Solutions

1. a) FALSE; for example, if every organism has exactly two offspring, then the number of organisms in the n^{th} generation increases without bound.

   b) TRUE; any CTMC is determined (via the forward or backward equation) by its infinitesimal parameters, and the birth and death rates of a birth-death process determine its infinitesimal parameters.

   c) TRUE; this is a basic property of the class structure of CTMCs.

   d) TRUE; because CTMCs have no periodicity properties, all stationary distributions of irreducible CTMCs are steady-state.

2. a) Write $\pi = (a, b, c)$; if π is stationary then $\pi Q = 0$. Writing this out we obtain the system of equations

$$\begin{cases} -7a + 4b + 4c = 0 \\ 3a - 6b = 0 \\ 4a + 2b - 4c = 0 \\ a + b + c = 1 \end{cases}$$

   From the second equation we see $a = 2b$; substituting this into the third equation we obtain $10b - 4c = 0$, i.e. $c = \frac{5}{2}b$. Plugging all this into the last equation we see $2b + b + \frac{5}{2}b = 1$, i.e. $b = \frac{2}{11}$. Finally the stationary distribution is

$$\pi = \left(\frac{4}{11}, \frac{2}{11}, \frac{5}{11}\right).$$

   b) The mean return time m_2 satisfies $\pi(2) = \frac{1}{m_2 q_2}$; from (a) we see $\pi(2) = \frac{5}{11}$ and $q_2 = -q_{22} = 4$. So $\frac{5}{11} = \frac{1}{4m_2}$, i.e. $m_2 = \frac{11}{20}$.

   c) The holding rate is $q_1 = -q_{11} = 6$.

   d) For every $x, y \in \mathcal{S}$ there is a differential equation in the forward equation giving an expression for $P'_{xy}(t)$. Since there are 3 states, there are $3 \cdot 3 = 9$ equations in the forward equation.

   e) The forward equation says $P'_{xy}(t) = \sum_{z \in \mathcal{S}} P_{xz}(t) q_{zy}$. Plugging in $x = y = 0$, for example, we obtain the equation

$$\begin{aligned} P'_{00}(t) &= P_{00}(t)q_{00} + P_{01}(t)q_{10} + P_{02}(t)q_{20} \\ P'_{00}(t) &= -7P_{00}(t) + 4P_{01}(t) + 4P_{02}(t). \end{aligned}$$

   A different choice of x and/or y would give a different (but similar looking) equation.

3. a) Since two-dimensional Brownian motion is point transient, this probability is zero.

b) W_5 lies in Quadrant IV if and only if $X_5 > 0$ and $Y_5 < 0$; each of these events has probability $1/2$ since X_5 and Y_5 are normal with mean zero; since the events $X_5 > 0$ and $Y_5 < 0$ are independent, $P(W_5$ lies in Quadrant IV$) = (1/2)(1/2) = 1/4$.

c) Suppose $X_1 > 0$. Now

$$\begin{aligned}
P(|X_2| > |X_1|) &= P(X_2 > X_1) + P(X_2 < -X_1) \\
&= P(X_2 - X_1 > 0) + P(X_2 - X_1 < -2X_1) \\
&\geq 1/2 + 0 \\
&\qquad \text{since } X_2 - X_1 \text{ is normal with mean zero}
\end{aligned}$$

Therefore with probability one, $\gamma > 1/2$.

d) Let $Z_t = 3X_t - 2Y_t$. First, we show $\{Z_t\}$ is Gaussian. Let $t_1, ..., t_n$ be arbitrary nonnegative real numbers and let $b_1, ..., b_n$ be arbitrary real numbers. Now

$$\sum_{j=1}^{n} b_j Z_{t_j} = \sum_{j=1}^{n} 3b_j X_{t_j} - \sum_{j=1}^{n} 2b_j Y_{t_j};$$

since $\{X_t\}$ and $\{Y_t\}$ are Gaussian processes, each of the individual sums are normal and since $X_t \perp Y_t$, the entire sum is normal as it is the sum of two independent normal r.v.s. Hence $\{Z_t\}$ is Gaussian by definition. Next, the mean function of $\{Z_t\}$ is $\mu_Z(t) = 3\mu_X(t) - 2\mu_Y(t) = 3 \cdot 0 - 2 \cdot 0 = 0$. The covariance function is

$$\begin{aligned}
r_Z(s,t) &= Cov(3X_s - 2Y_s, 3X_t - 2Y_t) \\
&= 9r_X(s,t) - 6Cov(X_s, Y_t) - 6Cov(X_t, Y_s) + 4r_{YY}(s,t) \\
&= 9\min(s,t) + 0 + 0 + 4\min(s,t) \\
&= 13\min(s,t).
\end{aligned}$$

Therefore $\{Z_t\}$ is a BM with parameter $\sigma^2 = 13$, since mean and covariance functions uniquely determine a Gaussian process.

4. a) W_4 is $n(0,4)$ by elementary property of Brownian motion. Therefore $X_4 = W_4 + 4\mu = n(0,4) + 4\mu = n(4\mu, 4)$.

b) First, the mean function:

$$\mu_X(t) = EX_t = E[W_t + \mu t] = 0 + \mu t = \mu t.$$

Next, the covariance function (note that μt, being a constant for each t, is independent of any other random variable):

$$\begin{aligned}
r_X(s,t) &= Cov(W_s + \mu s, W_t + \mu t) \\
&= r_W(s,t) + Cov(W_s, \mu t) + Cov(\mu s, W_t) + Cov(\mu s, \mu t) \\
&= \min(s,t) + 0 + 0 + 0 = \min(s,t).
\end{aligned}$$

Chapter 3

Exams from 2013 to 2016

3.1 Spring 2013 Final

1. Two basketball teams play a series where the first team to have won three more games than its opponent will be declared the "champion". Suppose that Team A is 60% likely to win any game the teams play, that there are no ties, and that the results of the individual games are independent.

   a) (2.5) What is the probability that Team A ends up being the champion? (Your answer need not be simplified.)

   b) (2.5) What is the expected number of games that will be played? (Your answer need not be simplified.)

   c) (2.5) Would Team A's probability of becoming the champion increase or decrease if the two teams played until one team has won four more games than its opponent (as opposed to three)?

2. Suppose $\{X_t\}$ is a Markov chain with state space $\{0, 1, 2\}$ whose transition matrix is

$$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

   a) (1.3) Find $P(X_3 = 1 \,|\, X_2 = 0)$.

   b) (1.3) Find $P(X_7 = 2 \,|\, X_5 = 0)$.

   c) (1.3) Suppose the initial distibution is $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Find $P(X_2 = 1)$.

3. Suppose $\{X_t\}$ is a Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ whose transition matrix is

$$\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{4} & \frac{5}{12} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

a) (1.7) Classify all the states as recurrent or transient.

b) (1.7) List all the absorbing states of this Markov chain.

c) (1.7) Find $f_{1,2}$, f_2, $f_{1,4}$ and $f_{5,2}$.

d) (1.6) Find $f_{4,3}$. *Hint:* write this probability as a geometric series.

e) (1.8) Find $E_4(V_4)$. *Hint:* Identify V_4 as a common random variable.

4. Consider a simple, unbiased random walk $\{X_t\}$ on $\mathbb{Z}^2$ starting at the origin.

   a) (2.7) Find $P(X_{10} = (1, 0))$.

   b) (2.7) Find $P(X_4 = (1, -1))$.

   c) (2.7) Find $P(X_t = (0, 0)$ for some $t > 0)$.

   d) (2.5) Let $T = \min\{t > 0 :$ the $x-$coordinate of X_t is $\pm 10\}$. Find ET.

   e) **Bonus** (2.7) Let $T_v = \min\{t > 0 : X_t$ lies on the line $x = 2\}$ and let $T_h = \min\{t > 0 : X_t$ lies on the line $y = 2\}$. Find $P(T_v < T_h)$.

Solutions

1. To model this, let $\{X_t\}$ be a simple random walk starting at 0 with $p = .6$ and $q = .4$. Let $T = \min\{t > 0 : X_t \in \{-3, 3\}$ be the escape time corresponding to $a = -3$ and $b = 3$. $X(T) = 3$ corresponds to Team A becoming champion, and $X(T) = -3$ corresponds to Team A not becoming champion.

    a) By formula developed in class for biased simple random walk,

    $$P(X(T) = 3) = \frac{(q/p)^{x-a} - 1}{(q/p)^{b-a} - 1} = \frac{(2/3)^3 - 1}{(2/3)^6 - 1}.$$

    b) By formula developed in class,

    $$ET = \left(\frac{b-a}{p-q}\right)\frac{(q/p)^{x-a} - 1}{(q/p)^{b-a} - 1} - \frac{x-a}{p-q} = \left(\frac{6}{.2}\right)\frac{(2/3)^3 - 1}{(2/3)^6 - 1} - \frac{3}{.2}$$

    c) It would increase; the more games that are played, the more likely the 'better' team (Team A) is to become the champion.

2. a) $P(X_3 = 1 \mid X_2 = 0) = P_{0,1} = \frac{2}{5}$.

    b) First, we find the time 2 transition matrix P^2 by usual matrix multiplication:

    $$P^2 = P\,P = \begin{pmatrix} \frac{9}{25} & \frac{8}{25} & \frac{8}{25} \\ \frac{1}{5} & \frac{12}{25} & \frac{8}{25} \\ \frac{11}{25} & \frac{1}{5} & \frac{9}{25} \end{pmatrix}.$$

    Now $P(X_7 = 2 \mid X_5 = 0) = P^2_{0,2} = \frac{8}{25}$.

    c) We have $\pi_0 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$ and $\pi_2 = \pi_0 P^2 = \left(\frac{23}{75}, \frac{28}{75}, \frac{24}{75}\right)$. Thus $P(X_2 = 1) = \pi_2(1) = \frac{28}{75}$.

3. a) (4 pts) $\{0, 1, 2\}$ is a finite irreducible set, so states $0, 1$ and 2 are recurrent. State 3 is absorbing, so it is recurrent. States 4 and 5 lead to 3 but 3 is absorbing, so states 4 and 5 are transient.

    b) State 3 is the only absorbing state.

    c) $f_{1,2} = 1$ since 1 and 2 belong to the same irreducible set. $f_2 = f_{2,2} = 1$ since 2 is recurrent. $f_{1,4} = 0$ since 1 does not lead to 4. $f_{5,2} = 1$ since the only way to start at 5 and never get to 2 is to continually follow the arrow going from 5 to itself. This situation has probability $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \ldots = 0$.

    d) By definition,

    $$f_{4,3} = P_4(T_3 < \infty) = \sum_{t=1}^{\infty} P_4(T_3 = t) = \sum_{t=1}^{\infty} \frac{1}{3}\left(\frac{1}{4}\right)^{t-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{4}{9}.$$

e) Notice that if you start in state 4, once you leave you never come back. Therefore $P_4(V_4 = n) = P_4(X_1 = ... = X_n = 4, X_{n+1} \neq 4) = \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)$. This is the density function of a geometric random variable with parameter $\frac{3}{4}$, so its expected value is $E_4(V_4) = \frac{1-3/4}{3/4} = \frac{1}{3}$.

4. a) After an even number of steps, the sum of the $x-$ coordinate and $y-$coordinate of the point you are at must be even. Therefore this probability is 0.

b) In order for X_4 to be $(1, -1)$, your first four steps must be (in some order):

- two to the right, one down, and one to the left; or
- two down, one up and one to the right.

Each of these probabilities can be found using the multinomial formula. They are both $\frac{4!}{2!1!1!}\left(\frac{1}{4}\right)^2\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{12}{4^4} = \frac{3}{4^3}$. Since there are two cases, the total probability is $P(X_4 = (1, -1)) = 2 \cdot \frac{3}{4^3} = \frac{3}{32}$.

c) We showed in class that two-dimensional simple unbiased random walk is recurrent, so this probability is 1.

d) Keeping track of only the $x-$coordinate of the random walk, we get a one-dimensional simple random walk with $p = q = \frac{1}{4}$ and $r = \frac{1}{2}$. This walk is unbiased, so by the escape time formulas derived in class we have

$$ET = \frac{(x-a)(b-x)}{p+q} = \frac{(0 - -10)(10-0)}{\frac{1}{4}+\frac{1}{4}} = 200.$$

e) We have

$$1 = P(T_v < T_h) + P(T_v = T_h) + P(T_v > T_h)$$

and the middle probability above is zero, since the random walk never moves diagonally in one step and therefore cannot hit both lines at the same time. By symmetry, the left- and right- hand probabilities above must be equal, so they must both equal $\frac{1}{2}$.

3.2 Spring 2014 Exam 1

1. Abe has three shirts that he wears to work: a blue shirt, a white shirt, and a yellow shirt. Assume that:

   - Abe never wears the blue or yellow shirt two days in a row;
   - If Abe wears the white shirt, then the chances he wears the white shirt the next day is $1/4$;
   - If Abe wears the white shirt, then the chances he wears the blue shirt the next day is $1/2$;
   - If Abe wears the blue shirt, then he is equally likely to wear the white or yellow shirt on the following day;
   - If Abe wears the yellow shirt, then the chances he wears the white shirt on the next day is $3/4$.

   a) (1.3) Suppose that today, Abe is equally likely to wear any of his three shirts. What is the probability that he wears the white shirt tomorrow?

   b) (1.3) If Abe is wearing the yellow shirt today, what is the probability that Abe is wearing the white shirt two days from now?

   c) (1.3) Find the probability that Abe is wearing the white shirt today, given that he wore the white shirt yesterday and that he will wear the white shirt tomorrow.

2. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{5}{8} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   a) (1.6) List all the absorbing states of this Markov chain.

   b) (1.7) Determine which states of this Markov chain are recurrent, and which states are transient.

   c) (1.7) Compute $f_{1,3}$, $f_{3,5}$ and $f_{4,1}$.

   d) (1.6) Compute $P_1(T_3 = 4)$.

   e) (1.8) Compute $E_5(V_4)$.

3. (1.7) Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, ...\}$ and transition function defined by

$$P(x,y) = \begin{cases} 1 & \text{if } x = 0, y = 1 \\ 0 & \text{else} \\ \frac{1}{x+1} & \text{if } x \neq 0, y = 0 \\ \frac{x}{x+1} & \text{if } x \neq 0, y = x+1 \\ 0 & \text{otherwise} \end{cases}.$$

Determine, with justification, whether or not this chain is recurrent or transient. (You may assume without proof that the chain is irreducible.)

Solutions

1. a) Let X be the length of the movie. We have $\mu = EX = 110$ and $\sigma^2 = Var(X) = 36$.

$$P(X \geq 200) = P(X - 110 \geq 90) \leq P(|X - 110| \geq 90) \leq \frac{36}{90^2} = \frac{36}{8100} = \frac{1}{225}.$$

   b) Let A_9 be the average length of the movies watched. By the CLT, $A_9 \approx n(110, \frac{36}{9}) = n(110, 4) = 110 + 2Z$ where Z is standard normal. Thus

$$\begin{aligned} P(109 \leq A_9 \leq 113) &\approx P(109 \leq 110 + 2Z \leq 113) \\ &= P\left(\frac{-1}{2} \leq Z \leq \frac{3}{2}\right) = \Phi\left(\frac{3}{2}\right) - \Phi\left(\frac{-1}{2}\right). \end{aligned}$$

2. Based on the given information, the transition matrix for this Markov chain is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

where the rows/columns correspond to blue, white and yellow respectively.

   a) We are given that $\pi_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Thus the distribution tomorrow is $\pi_1 = \pi_0 P = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. Therefore the probability he wears the white shirt tomorrow is the second entry of this vector, namely $\frac{1}{2}$.

   b) First, we compute P^2:

$$P^2 = \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{3}{16} & \frac{1}{2} & \frac{5}{16} \\ \frac{3}{8} & \frac{5}{16} & \frac{5}{16} \end{pmatrix}$$

   The question is asking for $(P^2)_{3,2} = \frac{5}{16}$.

   c) Let time 0 correspond to yesterday. This question is therefore asking for

$$\begin{aligned} P(X_1 = W \mid X_0 = X_2 = W) &= \frac{P(X_0 = X_1 = X_2 = W)}{P(X_0 = X_2 = W)} \\ &= \frac{\pi_0(W)P(W,W)P(W,W)}{\pi_0(W)P^2(W,W)} \\ &= \frac{(P_{2,2})^2}{(P^2)_{2,2}} = \frac{(1/4)^2}{1/2} = \frac{1}{8}. \end{aligned}$$

3. a) 6 is the only absorbing state.

b) The communicating classes are $\{1, 2, 3\}$ and $\{6\}$, so those states are recurrent (as they belong to finite communicating classes). States 4 and 5 do not belong to a communicating class, so they are transient.

c) $f_{1,3} = 1$ because 1 and 3 belong to the same communicating class of recurrent states.

$f_{3,5} = 0$ since 3 is recurrent but 5 does not belong to the same communicating class as 3.

To compute $f_{4,1}$, use a system of equations involving $f_{4,1}$ and $f_{5,1}$. Notice that since 1 and 3 lie in the same communicating class, we know $f_{4,1} = f_{4,3}$ and $f_{5,1} = f_{5,3}$. We therefore have

$$\begin{cases} f_{4,1} = \frac{1}{8} \cdot 1 + \frac{5}{8} f_{4,1} + \frac{1}{4} f_{5,1} \\ f_{5,1} = \frac{1}{4} \cdot 1 + \frac{1}{4} f_{4,1} + \frac{1}{2} \cdot 0 \end{cases}$$

Multiply through by 8 in the top equation and 4 in the bottom equation to get

$$\begin{cases} 3f_{4,1} = 1 + 2f_{5,1} \\ 4f_{5,1} = 1 + f_{4,1} \end{cases}$$

Solving for $f_{4,1}$ in the bottom equation, we get $f_{4,1} = 4f_{5,1} - 1$. Substituting this into the top equation, we get

$$3(4f_{5,1} - 1) = 1 + 2f_{5,1}$$

which reduces to $12f_{5,1} - 3 = 1 + 2f_{5,1}$, i.e. $10f_{5,1} = 4$ i.e. $f_{5,1} = \frac{2}{5}$. Substituting this into one of the original equations to solve for $f_{4,1}$, we get $f_{4,1} = \frac{3}{5}$.

d) From looking at the directed graph of the Markov chain, the only way this condition is achieved if the first four states are $1 \to 2 \to 1 \to 2 \to 3$. This probability is $P(1, 2)P(2, 1)P(1, 2)P(2, 3) = 1 \cdot \frac{2}{3} \cdot 1 \cdot \frac{1}{3} = \frac{2}{9}$.

e) First, we note that if you start in state 5, the only way you can hit state 4 in any amount of finite time is if your first step is to state 4. Therefore $f_{5,4} = P_5(T_4 < \infty) = \frac{1}{4}$.

Next, we compute $f_{4,4}$. If you start in state 4, there are two ways to hit state 4 in some amount of finite time. You can either return immediately to state 4 (this has probability $\frac{5}{8}$), or you can go to state 5 and then immediately come back to state 4 (this has probability $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$. Therefore $f_{4,4} = \frac{5}{8} + \frac{1}{16} = \frac{11}{16}$.

Now, by a theorem in class (properties of recurrent and transient states),

$$E_5(V_4) = \frac{f_{5,4}}{1 - f_{4,4}} = \frac{1/4}{1 - 11/16} = \frac{1}{4} \cdot \frac{16}{5} = \frac{4}{5}.$$

4. First, notice that if you start in state zero, to say that $T_0 = \infty$ (which means you never return to zero) means that your steps must be $0 \to 1 \to 2 \to 3 \to 4 \to$ .... Therefore

$$
\begin{aligned}
f_{0,0} &= P_0(T_0 < \infty) \\
&= 1 - P_0(T_0 = \infty) \\
&= 1 - \lim_{n\to\infty} P_0(T_0 > n) \\
&= 1 - \lim_{n\to\infty} P(0,1)P(1,2)P(2,3)\cdots P(n-1,n) \\
&= 1 - \lim_{n\to\infty} 1 \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} \\
&= 1 - \lim_{n\to\infty} \frac{2}{n} \\
&= 1.
\end{aligned}
$$

Since $f_{0,0} = 1$, 0 is recurrent, and since the chain is irreducible, $\{X_t\}$ is recurrent.

3.3 Spring 2014 Exam 2

1. Suppose Mexico and Portugal are playing a series of soccer matches. The results of each match are independent of one another and in each match, both teams have a 40% probability of winning. The series of matches will end when one team has won seven more matches than the other; that team will have "won" the series.

   a) (2.5) What is the probability that Mexico wins the series?

   b) (2.5) How many matches should the teams expect to play?

   c) (2.5) Suppose Portugal wins the first three matches. What is the probability that Mexico subsequently wins the series?

2. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/4 & 0 & 3/4 & 0 \\ 3/7 & 0 & 4/7 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   a) (1.5) Find all stationary distributions (if any) of this Markov chain.

   b) (1.10) Find all steady-state distributions (if any) of this Markov chain.

   c) (1.8) Find the mean return times to states 2, 3 and 5.

   d) (1.8) Suppose you start in state 4. How many times in the first 4000 steps would you expect to visit state 2?

   e) (1.10) Let n be large. Estimate $P^n(1, 3)$ (there may be cases depending on the value of n).

3. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ and transition function P defined by setting

$$P(x, y) = \begin{cases} \frac{1}{3^y} & \text{if } x = 0, y > 0 \\ 3^y & \text{if } x = 0, y < 0 \\ \frac{1}{2} & \text{if } x \neq 0, y = -x \\ \frac{1}{2} & \text{if } x \neq 0, y = 0 \\ 0 & \text{else} \end{cases}$$

   a) (1.6) Find the period of this Markov chain.

   b) (1.10) Find the mean return time to state 0.

c) (1.10) Does $\lim_{n\to\infty} P^n(4,0)$ exist? If so, what is it? If not, why does this limit not exist?

d) (1.10) Find the mean return time to state 22.

4. (1.10) Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1,2,3,...\}$ and transition function P defined by setting

$$P(x,y) = \begin{cases} \frac{C}{y^2} & \text{if } x = 1 \\ 1 & \text{if } x > 1, y = x-1 \\ 0 & \text{else} \end{cases}$$

where C is some constant that makes the values of the transition function add up to 1.

Determine, with justification, whether this Markov chain is positive recurrent, null recurrent, or transient.

Solutions

1. Let $\{X_t\}$ be the number of games Mexico is ahead after t matches. $\{X_t\}$ is therefore a a simple random walk with $p = .4$ and $q = .4$; let T be the number of matches played, we have $T = T_{\{7,-7\}}$.

   a) Mexico wins if $T_7 < T_{-7}$. So by the escape time probabilities for unbiased random walk, $P_0(T_7 < T_{-7}) = \frac{0-(-7)}{7-(-7)} = \frac{1}{2}$.

   b) Since $p = q$, the random walk is unbiased. Furthermore, if we set ξ_t to be 1 if Mexico wins and -1 if Portugal wins, we have $Var(\xi_j) = p + q = \frac{4}{5}$. Next, by Wald's First Identity, $E[X(T)] = x + (p - q)ET = 0 + (.4 - .4)ET = 0$. Last, by Wald's Second Identity, $Var[X(T)] = Var(\xi_j)ET$. Therefore since $Var[X(T)] = E[X(T)^2] - E[X(T)]^2 = 49 - 0^2 = 49$ and $Var(\xi_j) = \frac{4}{5}$, we have $49 = \frac{4}{5}ET$ so $ET = \frac{245}{4} =$ **61.25 matches**.

   c) Since Portugal wins the first three matches, we should assume $X_0 = x = -3$. Therefore, by the escape time probabilities for unbiased random walk, the answer is $P_{-3}(T_7 < T_{-7}) = \frac{-3-(-7)}{7-(-7)} = \frac{4}{14} = \frac{2}{7}$.

2. a) This chain is irreducible since $1 \to 2 \to 3 \to 5 \to 1$ and $1 \to 4 \to 1$. Since the state space is finite, the chain is positive recurrent so there is one stationary distribution. Let $\pi = (a, b, c, d, e)$ be the stationary distribution. By setting $\pi P = \pi$, we get

$$\begin{cases} \frac{3}{7}b + d = a \\ \frac{1}{4}a + \frac{1}{4}c = b \\ \frac{4}{7}b + e = c \\ \frac{3}{4}a + \frac{1}{4}c = d \\ \frac{1}{2}c = e \end{cases}$$

   Substitute the fifth equation into the third to get $\frac{4}{7}b + \frac{1}{2}c = c$, so $c = \frac{8}{7}b$. Substituting this into the second equation, we get $\frac{1}{4}a + \frac{2}{7}b = b$, so $a = \frac{20}{7}b$. Substituting this into the first equation, we get $\frac{3}{7}b + d = \frac{20}{7}b$, so $d = \frac{17}{7}b$. We have

$$a = \frac{20}{7}b \quad b = b = \frac{7}{7}b \quad c = \frac{8}{7}b \quad d = \frac{17}{7}b \quad e = \frac{1}{2}c = \frac{4}{7}b$$

   so since $a+b+c+d+e = 1$, we have $\frac{20+7+8+17+4}{7}b = 1$ so $b = \frac{1}{8}$. We have

$$a = \frac{20}{56} = \frac{5}{14} \quad b = \frac{1}{8} \quad c = \frac{1}{7} \quad d = \frac{17}{56} \quad e = \frac{1}{14}$$

   so $\pi = \left(\frac{5}{14}, \frac{1}{8}, \frac{1}{7}, \frac{17}{56}, \frac{1}{14}\right)$.

   b) The period of this Markov chain is 2, so this Markov chain has **no steady-state distribution**.

c) $m_2 = \frac{1}{\pi(2)}$ =**8**; $m_3 = \frac{1}{\pi(3)}$ =**7**; $m_5 = \frac{1}{\pi(5)} =$ **14**.

d) By the ergodic theorem, this is $4000\,\pi(2) = (4000)\,\frac{1}{8} =$ **500 times**.

e) The chain alternates between the sets $\{1,3\}$ and $\{2,4,5\}$ (this is why the period is 2). Therefore

$$P^n(1,3) = \begin{cases} d\,\pi(3) = (2)\frac{1}{7} = \frac{2}{7} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

3. a) Notice that we can go $0 \to 1 \to 0$ and $0 \to 1 \to -1 \to 0$ so $d_0|2$ and $d_0|3$ so $d_0 = 1$. The chain is irreducible, since zero leads to and from any other number, so the Markov chain has **period 1**.

b) Define a factor $\{Y_t\}$ of $\{X_t\}$ as follows: set $Y_t = 0$ if $X_t = 0$; set $Y_t = 1$ if $X_t > 0$; set $Y_t = 2$ if $X_t < 0$. The transition matrix of $\{Y_t\}$ is

$$\begin{pmatrix} 0 & a & b \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

where $a = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ... = \frac{1}{3}\left(\frac{1}{1-1/3}\right) = \frac{1}{2}$ and $b = 1 - a = \frac{1}{2}$. Since $a = b = \frac{1}{2}$, the transition matrix of $\{Y_t\}$ is doubly stochastic so the stationary distribution of $\{Y_t\}$ is uniform $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Thus the mean return time of 0 in $\{Y_t\}$, hence the mean return time of 0 in $\{X_t\}$, is **3**.

c) From part (a), 0 is positive recurrent. Since $\{X_t\}$ is irreducible, $\{X_t\}$ is positive recurrent and since the chain is aperiodic, $\lim_{n\to\infty} P^n(4,0) = \pi(0) = \frac{1}{m_0} = \frac{1}{3}$.

d) Notice first that by symmetry, $\pi(22) = \pi(-22)$ since the Markov chain acts the same on positive numbers as it does on negative numbers. By the stationarity equation,

$$\pi(22) = \sum_{x\in\mathcal{S}} \pi(x)P(x,y) = \pi(-22)\frac{1}{2} + \pi(0)\frac{1}{3^{22}}$$

Since $\pi(22) = \pi(-22)$, we have $\pi(22) = \frac{1}{2}\pi(22) + \frac{1}{3}\,\frac{1}{3^{22}}$ so solving for $\pi(22)$ we get $\pi(22) = \frac{2}{3^{23}}$. Therefore the mean return time to state 22 is $m_{22} = \frac{1}{\pi(22)} = \frac{3^{23}}{2}$.

(In general, for any $y \neq 0$, $\pi(y) = \frac{2}{3^{|y|+1}}$ and $m_y = \frac{3^{|y|+1}}{2}$ by the same reasoning.)

4. Suppose you start in state 1. Then T_1, the time of your first return to 1, is equal to n if and only if your first step is to state n (because you will then

step one unit to the left $n-1$ times and return to 1 at that point). So $P_1(T_1 = n) = P(1,n) = \frac{C}{n^2}$.

Now, $f_1 = P_1(T_1 < \infty) = \sum_{n=1}^{\infty} P_1(T_1 = n) = \sum_{n=1}^{\infty} \frac{C}{n^2} < \infty$ so 1 is recurrent. Since 1 leads to every other state, the Markov chain is recurrent.

Now let's compute the mean return time of state 1.

$$m_1 = E_1(T_1) = \sum_{n=1} n \cdot P_1(T_1 = n) = \sum_{n=1}^{\infty} n \frac{C}{n^2} = \sum_{n=1}^{\infty} \frac{C}{n} = \infty.$$

Therefore 1 is null recurrent; since 1 leads everywhere else, the Markov chain is **null recurrent**.

3.4 Spring 2014 Final

1. Suppose $\{X_t\}$ is a Markov chain with state space $\{1,2,3,4,5\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & \frac{5}{8} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{8} & \frac{3}{8} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   a) (1.3) Suppose the initial distribution is $(\frac{1}{2},0,0,\frac{1}{2},0)$. Find the distribution at time 2.
   b) (1.3) Find $P(X_5 = 2 \,|\, X_3 = 2, X_2 = 4)$.
   c) (1.3) Suppose the initial distribution is $(\frac{1}{2},0,0,\frac{1}{2},0)$. Find $P(X_1 = 2 \,|\, X_2 = 2)$.
   d) (1.7) Find $f_{4,3}$, f_4, $f_{5,2}$, and $f_{1,3}$.
   e) (1.10) Find all stationary distributions of this Markov chain.

2. Suppose $\{X_t\}$ is a Markov chain with state space $\mathcal{S} = \{0,1,2,3,...\}$ and transition function

$$P(x,y) = \begin{cases} 12 \cdot 4^{-y} & \text{if } x = 0, y > 1 \\ 2 \cdot 2^{-y} & \text{if } x = 1, y > 1 \\ \frac{3}{4} & \text{if } x > 1, y = 0 \\ \frac{1}{4} & \text{if } x > 1, y = 1 \end{cases}$$

   a) (1.6) Find the period of this Markov chain.
   b) (1.3) Find $P^2(3,5)$.
   c) (1.10) Find the mean return time to state 10.

3. Suppose that from one week to the next, the value of an investment increases by $\$10$ with probability 40%, stays the same with probability 20%, and decreases by $\$10$ with probability 40%.

   a) (2.5) If the initial investment is $\$100$, what is the probability that the value of the investment reaches $\$150$ before it falls to $\$80$?
   b) (2.5) Suppose that after an initial investment of $\$200$, the investor will withdraw the money the first time the investment reaches either $\$150$ or $\$300$. How many weeks should the investor expect to her money in the account?

c) (2.5) If the investor leaves their initial investment of $\$1000$ in this scheme forever, what is the probability that their investment will eventually be worth $\$0$?

d) (2.5) Suppose that an investor puts $\$100$ in the scheme and after 10 weeks, the investment is worth $\$120$. Suppose that at this point, due to market changes, the probability of a weekly $\$10$ increase becomes 30% and the probability of a weekly $\$10$ decrease becomes 60%. What is the probability that the value of the investment subsequently reaches $\$150$ before it falls to $\$80$?

4. Let $\{X_t\}$ be a CTMC with state space $\{1,2,3\}$ and time t transition matrix

$$P(t) = \frac{1}{30}\begin{pmatrix} 18+12e^{-5t} & 7+5e^{-6t}-12e^{-5t} & f(t) \\ A+Be^{-5t} & g(t) & C+De^{-6t} \\ 18-18e^{-5t} & 7-25e^{-6t}+18e^{-5t} & 5+25e^{-6t} \end{pmatrix}$$

a) (3.2) Find the function $f(t)$.

b) (3.2) Suppose you start in state 1. Find the probability that after 4 units of time, you are in state 1.

c) (3.2) Suppose you start in state 1. Find the probability that you remain in state 1 for at least 4 units of time before jumping.

d) (3.2) Suppose you start in state 3. Find the probability that when you jump, your first jump is to state 2.

e) (3.3) Find the stationary distribution π.

f) (3.3) Find the mean return time to state 3.

g) (3.3) Find the values of the constants A, B, C and D.

h) (3.3) Find the function $g(t)$.

5. Let $\{W_t\}$, $\{\widehat{W}_t\}$ and $\{\widetilde{W}_t\}$ be independent, standard Brownian motions.

a) (4.2) Compute $P(W_8 > 4 \,|\, W_3 = 4)$.

b) (4.2) Compute $P(W_t = 3 \text{ for some } t \leq 2)$.

c) (4.4) Let $X_t = (1-t)W_{t/(1-t)}$. Compute the mean and covariance functions of X_t.

d) (4.7) Determine, with justification, whether the following statement is true or false: $P\left(W_t + \widehat{W}_t = \widetilde{W}_t \text{ for an unbounded set of } t\right) = 1$.

6. Classify the following statements as true or false:

a) (1.7) If x and y are recurrent states in a Markov chain with finite state space, then $f_{x,y} = 1$.

b) (1.7) If x is a transient state in a Markov chain $\{X_t\}$, then no matter the value of X_0, the probability you visit x infinitely many times is zero.

c) (1.8) If x is a null recurrent state in a Markov chain and $x \to y$, then y must be null recurrent.

d) (2.7) Two-dimensional unbiased random walk is recurrent.

e) (3.4) A stationary distribution of an irreducible CTMC must also be steady-state.

f) (4.4) If a stationary process has mean function $\mu_X(t) = 0$ and covariance function $r_X(s,t) = \sigma^2 \min(s,t)$, then the process is a Brownian motion.

g) (4.7) Two-dimensional Brownian motion is neighborhood recurrent.

h) (3.7) The infinite server queue has a stationary distribution.

i) (Group Presentations) In a branching chain where each organism has either 0 or 3 offspring (uniformly), the extinction probability is 1.

Solutions

1. a) Let $\pi_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$. Then

$$\pi_2 = \pi_0 P^2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & \frac{1}{2} & \frac{3}{8} & 0 & 0 \\ \frac{1}{8} & 0 & \frac{7}{16} & 0 & 0 \\ \frac{3}{8} & \frac{3}{16} & \frac{7}{16} & 0 & 0 \\ \frac{1}{8} & \frac{3}{16} & \frac{13}{64} & \frac{9}{64} & \frac{11}{32} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{8} & \frac{11}{32} & \frac{37}{128} & \frac{9}{128} & \frac{11}{64} \end{pmatrix}.$$

b) By the Markov property, $P(X_5 = 2 \,|\, X_3 = 2, X_2 = 4) = P^2(2,2) = \frac{7}{16}$.

c) $P(X_1 = 2 \,|\, X_2 = 2) = \frac{P(X_1=2, X_2=2)}{P(X_2=2)} = \frac{\pi_1(2)P(2,2)}{\pi_2(2)} = \frac{(3/16)(1/4)}{11/48} = \frac{9}{44}$.

d) Since $\{1,2,3\}$ is a communicating class of recurrent states, $f_{43} = f_{4,\{1,2,3\}} = \sum_{n=1}^{\infty} P_4(T_{\{1,2,3\}} = n) = \sum_{n=1}^{\infty} \left(\frac{3}{8}\right)^n = \frac{3}{8} \cdot \left(\frac{1}{1-3/8}\right) = \frac{3}{5}$.

$f_{44} = 3/8$ since the only way for you to return from state 4 to state 4 is to return there on the first step.

$f_{52} = 0$ since 5 and 2 belong to disjoint communicating clases.

$f_{13} = 1$ since 1 and 3 belong to the same communicating class of recurrent states.

e) There are two communicating classes of recurrent states: $C_1 = \{1,2,3\}$ and $C_2 = \{5\}$. The stationary distribution supported on C_2 is $\pi_2 = (0,0,0,0,1)$ and the stationary distribution π_1 supported on C_1 is solved for by setting $\pi_1 = (a,b,c,0,0)$ and setting $\pi_1 P|_{\{1,2,3\}} = \pi_1$ to obtain

$$\begin{cases} \frac{1}{4}a + \frac{1}{2}b = a \\ \frac{1}{8}a + \frac{3}{4}c = b \\ \frac{5}{8}a + \frac{1}{2}b + \frac{1}{4}c = c \\ a + b + c = 1 \end{cases} \Rightarrow \pi_1 = \left(\frac{3}{13}, \frac{9}{26}, \frac{11}{26}, 0, 0\right).$$

Thus all stationary distributions are convex combinations of π_1 and π_2, i.e. are of the form

$$\left(\frac{3}{13}\alpha, \frac{9}{26}\alpha, \frac{11}{26}\alpha, 0, 1-\alpha\right)$$

where $\alpha \in [0,1]$.

2. a) The period is 2 since the chain has the structure $\{0,1\} \leftrightarrow \{2,3,4,...\}$

b) $P^2(3,5) = \sum_{x \in \S} P(3,x)P(x,5) = P(3,0)P(0,5) + P(3,1)P(1,5) = \frac{3}{4} \cdot \frac{12}{4^5} + \frac{1}{4} \cdot \frac{2}{2^5} = \frac{9}{4^5} + \frac{1}{64} = \frac{25}{4^5}$.

c) We have $m_{10} = \frac{1}{\pi(10)}$ where π is the stationary distribution. Now, first define a factor $\{Z_t\}$ of $\{X_t\}$ by setting $Z_t = 0$ if $X_t = 0$; $Z_t = 1$ if $X_t = 1$; $Z_t = 2$ if $X_t > 1$. $\{Z_t\}$ is a finite state-space Markov chain with transition matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

The stationary distribution of $\{Z_t\}$ is $(\frac{3}{8}, \frac{1}{8}, \frac{1}{2})$. So $\pi(0) = \frac{3}{8}$ and $\pi(1) = \frac{1}{8}$ (and since these states weren't grouped to create the factor $\{Z_t\}$, these values of the stationary distribution hold for $\{X_t\}$ as well). So by the stationarity equation,

$$\pi(10) = \pi(0)P(0,10) + \pi(1)P(1,10) = \frac{3}{8} \cdot \frac{12}{4^{10}} + \frac{1}{8} \cdot \frac{2}{2^{10}} = \frac{9}{2^{21}} + \frac{1}{2^{12}}$$

so $m_{10} = \left(\frac{9}{2^{21}} + \frac{1}{2^{12}}\right)^{-1} = \frac{2^{21}}{2^9+9}$.

3. a) Since this is an unbiased random walk, $P_{10}(T_{15} < T_8) = \frac{10-8}{15-8} = \frac{2}{7}$.

b) Let $T = \min\{T_{-5}, T_{10}\}$; this question is asking for ET which by the Wald Identities for unbiased random walk is $\frac{(x-a)(b-x)}{p+q} = \frac{(0-(-5))(10-0)}{.4+.4} = \frac{50}{.8} = \frac{125}{2}$ weeks.

c) Since unbiased random walk is recurrent (gambler's ruin phenomenon), this probability is 1.

d) Now the situation is modeled by a random walk starting at \$120 with $p = .3$ and $q = .6$, so $q/p = 2$. Now $P_{12}(T_{15} < T_8) = \frac{1-2^4}{1-2^7}$.

4. a) Since the rows of $P(t)$ add to 1, we have $f(t) = 5 - 5e^{-6t}$.

b) This is $P_{11}(4) = \frac{18+12e^{-20}}{30}$.

c) The holding rate of state 1 is $q_1 = -q_{11} = -P'_{11}(0) = -\frac{-60e^{-5(0)}}{30} = 2$ so the probability you wait 4 units of time before a jump is $P(W_1 > 4) = 1 - P(W_1 \leq 4) = 1 - [1 - e^{-2\cdot 4}] = e^{-8}$.

d) This is $\pi_{32} = \frac{q_{32}}{-q_{33}} = \frac{P'_{32}(0)}{-P'_{33}(0)} = \frac{2}{5}$.

e) $\pi(y) = \lim_{t\to\infty} P(x,y)$ so $\pi = \left(\frac{18}{30}, \frac{7}{30}, \frac{5}{30}\right) = \left(\frac{3}{5}, \frac{7}{30}, \frac{1}{6}\right)$.

f) We have $\frac{1}{6} = \pi(3) = \frac{1}{m_3 q_3} = \frac{1}{5m_3}$ so $m_3 = \frac{6}{5}$.

g) Since $\lim_{t\to\infty} P_{xy}(t) = \pi(y)$, we have $\frac{1}{30}A = \lim_{t\to\infty} \frac{1}{30}(A + Be^{-5t}) = \frac{3}{5}$ so $A = 18$. Similarly, $C = 5$. Since $P_{21}(0) = \frac{1}{30}(A + B) = 0$, we have $A + B = 0$ so $B = -18$. Since $P_{23}(0)\frac{1}{30}(C + D) = 0$, we have $C + D = 0$ so $D = -5$.

h) Since the rows of $P(t)$ must add to 1, $g(t) = 7 + 5e^{-6t} + 18e^{-5t}$.

5. a) $P(W_8 > 4 \,|\, W_3 = 4) = P(W_8 - W_3 > 0) = P(n(0,5) > 0) = \frac{1}{2}$.

   b) By the reflection principle, $P(W_t = 3 \text{ for some } t \le 2) = P(T_3 \le 2) = 2 - 2\Phi\left(\frac{3}{\sqrt{2}}\right)$.

   c) $\mu_X(t) = (1-t)E_W(\frac{t}{1-t}) = (1-t)0 = 0$.

   $r_X(s,t) = (1-s)(1-t)r_W(\frac{s}{1-s}, \frac{t}{1-t}) = (1-s)(1-t)\min\left(\frac{s}{1-s}, \frac{t}{1-t}\right)$.

   d) Let $V_t = W_t + \widehat{W}_t - \widetilde{W}_t$; since W_t, $\widehat{W}_t$ and $\widetilde{W}_t$ are Gaussian, so is V_t since linear combinations of the V_t can be split into an independent sum of a linear combination of the W_t, the $\widehat{W}_t$ and the $\widetilde{W}_t$.

   Next, compute the mean and covariance functions of V_t: $\mu_V(t) = \mu_W(t) + \mu_{\widehat{W}}(t) + \mu_{\widetilde{W}(t)}(t) = 0 + 0 + 0 = 0$.

   $r_V(s,t) = Cov(V_s, V_t) = Cov(W_s + \widehat{W}_s + \widetilde{W}_s, W_t + \widehat{W}_t + \widetilde{W}_t)$; since the given processes are independent this reduces to $Cov(W_s, W_t) + Cov(\widehat{W}_s, \widehat{W}_t) + Cov(\widetilde{W}_s, \widetilde{W}_t) = \min(s,t) + \min(s,t) + \min(s,t) = 3\min(s,t)$.

   Thus since the mean and covariance functions of a Gaussian process determine the process, $\{V_t\}$ is a BM with parameter $\sigma^2 = 3$, so $\{V_t\}$ is recurrent so the probability referred to in the problem is 1, so the statement is TRUE.

6. a) FALSE (x and y may belong to disjoint communicating classes)

   b) TRUE (property of transient states from Chapter 1).

   c) TRUE (property of null recurrent states from Chapter 3).

   d) TRUE (proven in class; see Chapter 2)

   e) TRUE (CTMCs have no periodicity issues)

   f) FALSE (the process must also be Gaussian to conclude that it is a BM)

   g) TRUE (discussed in class; see Chapter 6)

   h) TRUE (the stationary distribution is $Pois(\lambda t p_t)$ - see Chapter 5)

   i) FALSE (E(offspring)$= 1.5 > 1$, so $\eta < 1$)

3.5 Spring 2015 Exam 1

1. Suppose you buy an antique for $\$75$. Each month, the value of the antique increases by $\$5$ (with probability $.5$), decreases by $\$5$ (with probability $.25$), or stays the same (with probability $.25$).

   a) (2.5) What is the probability that the antique is eventually worthless?

   b) (2.5) Suppose you decide to sell the antique when the value of the antique is either $\$60$ or $\$100$. What is the probability that when you sell the antique, you will sell it for $\$100$?

   c) (2.5) In the situation of part (b), what is the expected value of the antique at the time when you sell it?

2. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 0 & 1/8 & 1/4 & 1/8 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   a) (1.7) Classify the states of this Markov chain as recurrent or transient.

   b) (1.6) Find $P_3(T_2 = 2)$.

   c) (1.7) Find $f_{1,1}$, $f_{1,4}$ and $f_{4,1}$.

   d) (1.8) Find $E_4(V_4)$.

3. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1/7 & 3/7 & 3/7 \\ 4/7 & 0 & 3/7 \\ 2/7 & 1/7 & 4/7 \end{pmatrix}.$$

   a) (1.3) Find $P(X_5 = 2 \,|\, X_4 = 1, X_3 = 2, X_2 = 1, X_1 = 2, X_0 = 1)$.

   b) (1.3) Find $P(X_6 = 3 \,|\, X_4 = 3)$.

   c) (1.3) Suppose the initial distribution is uniform. Find the distribution at time 2.

Solutions

1. First, note that since the size of the step at each time is ± 5, we have to rescale the value of the antique by dividing by 5 so that the size of the step is ± 1. That means we should think of the initial state not as 75, but as $75/5 = 15$.

   Thus the value of the antique is modeled by a random walk with $X_0 = x = 15$, $p = .5$ and $q = .25$.

   a) From the Gambler's Ruin theorem, we have

   $$f_{15,0} = \left(\frac{.25}{.50}\right)^{15-0} = \left(\frac{1}{2}\right)^{15} = 2^{-15}.$$

   b) This is an escape probability with $a = 60/5 = 12$ and $b = 100/5 = 20$:

   $$P_{15}(T_{20} < T_{12}) = \frac{1 - \left(\frac{.25}{.50}\right)^{15-12}}{1 - \left(\frac{.25}{.50}\right)^{20-12}} = \frac{1 - \left(\frac{1}{2}\right)^{3}}{1 - \left(\frac{1}{2}\right)^{8}} = \frac{2^8 - 2^5}{2^8 - 1}.$$

   c) The value of the antique when you sell it is a random variable taking the value 100 with probability $\frac{2^8-2^5}{2^8-1}$ (from part (b)) and taking the value 60 with probability $1 - \frac{2^8-2^5}{2^8-1} = \frac{2^5-1}{2^8-1}$. Thus

   $$\begin{aligned} E[X_T] &= 100\left(\frac{2^8 - 2^5}{2^8 - 1}\right) + 60\left(\frac{2^5 - 1}{2^8 - 1}\right) \\ &= \frac{100 \cdot 2^8 - 40 \cdot 2^5 - 60}{255} \\ &= \frac{4852}{51} \\ &\approx 95.1373. \end{aligned}$$

2. a) Since the state space is finite, a state is recurrent if and only if it belongs to a communicating class. The communicating classes are $\{1, 2\}$ and $\{5\}$, so $\mathcal{S}_R = \{1, 2, 5\}$ and $\mathcal{S}_T = \{3, 4\}$.

   b) If you start in state 3 and hit state 2 for the first time on the second step, you must have gone $3 \to 1 \to 2$ or $3 \to 3 \to 2$ or $3 \to 4 \to 2$. So

   $$\begin{aligned} P_3(T_2 = 2) &= P(3,1)P(1,2) + P(3,3)P(3,2) + P(3,4)P(4,2) \\ &= \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{8} \\ &= \frac{5}{32}. \end{aligned}$$

c) $f_{1,1} = 1$ since 1 is recurrent.
$f_{1,4} = 0$ since 1 is recurrent and 4 is transient.
$f_{4,1}$ is an absorption probability. By the Law of Total Probability, we have

$$\begin{cases} f_{3,1} = \frac{1}{4}(1) + \frac{1}{4}(1) + \frac{1}{4}f_{3,1} + \frac{1}{4}f_{4,1} + 0 \\ f_{4,1} = 0 + \frac{1}{8}(1) + \frac{1}{4}f_{3,1} + \frac{1}{8}f_{4,1} + \frac{1}{2}(0) \end{cases}$$

i.e.

$$\begin{cases} 3f_{3,1} = 2 + f_{4,1} \\ 7f_{4,1} = 1 + 2f_{3,1} \end{cases}$$

Solving this system for $f_{3,1}$ and $f_{4,1}$, we obtain the solution $f_{3,1} = \frac{15}{19}$ and more importantly (given what was asked) $f_{4,1} = \frac{7}{19}$.

d) By the theorem on properties of recurrent and transient states, $E_4(V_4) = \frac{f_{4,4}}{1-f_{4,4}}$. That means we need to find $f_{4,4} = P_4(T_4 < \infty) = \sum_{n=1}^{\infty} P_4(T_4 = n)$. First, $P_4(T_4 = 1) = P(4,4) = \frac{1}{8}$. Now, if $n \geq 2$, notice that if you start in state 4 and return to 4 for the first time on the n^{th} step, then you must have moved $4 \to 3 \to 3 \to 3 \to \cdots \to 3 \to 4$. Thus for $n \geq 2$,

$$P_4(T_4 = n) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \cdots \cdot \frac{1}{4} = \left(\frac{1}{4}\right)^n.$$

This means

$$f_{4,4} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{8} + \sum_{n=2}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{8} + \left(\frac{1}{4}\right)^2 \left(\frac{1}{1-1/4}\right) = \frac{1}{8} + \frac{1}{12} = \frac{5}{24}$$

so finally $E_4(V_4) = \frac{f_{4,4}}{1-f_{4,4}} = \frac{5/24}{1-5/24} = \frac{5}{19}$.

3. a) By the Markov property,

$$\begin{aligned} P(X_5 = 2 \,|\, X_4 = 1, X_3 = 2, X_2 = 1, X_1 = 2, X_0 = 1) &= P(X_5 = 2 \,|\, X_4 = 1) \\ &= P(1,2) \\ &= \frac{3}{7}. \end{aligned}$$

b) By matrix multiplication, the time 2 transition matrix is

$$P^2 = \begin{pmatrix} 19/49 & 6/49 & 24/49 \\ 10/49 & 15/49 & 24/49 \\ 2/7 & 10/49 & 25/49 \end{pmatrix}.$$

Thus $P(X_6 = 3 \,|\, X_4 = 3) = P^2(3,3) = \frac{25}{49}$.

c) We are given $\pi_0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; thus $\pi_2 = \pi_0 P^2 = \left(\frac{43}{147}, \frac{31}{147}, \frac{73}{147}\right)$.

3.6 Spring 2015 Exam 2

1. Let $\{X_t\}$ be a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

a) (1.5) Find the stationary distribution of this Markov chain.

b) (1.10) Is the stationary distribution of this Markov chain steady-state? Why or why not?

c) (1.8) Find the mean return time to state 3.

d) (1.10) Estimate $P^n(2, 3)$ for large n.

e) (1.10) Suppose the initial distribution is $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$. Estimate the time 3000 distribution.

f) (1.8) If you start in state 2, how many times in the first 2300 steps would you expect to visit state 3?

2. (1.10) Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{3}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \\ 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For each state y, estimate $P^n(1, y)$ when n is large.

3. (2.6) Let c and d be positive constants with $c + d \leq 1$, and let $\{X_t\}$ be a birth-death chain with state space $\{0, 1, 2, 3, ...\}$ where $p_x = c^{x+1}$ for all x, and $q_x = d^x$ for all $x > 0$. Find conditions on c and d under which this Markov chain is positive recurrent.

4. (1.10) Let $\{X_t\}$ be a Markov chain with state space $\{0, 1, 2, 3, ...\}$ whose transition function is described by

$$\begin{aligned} P(0, y) &= \frac{1}{2}\left(\frac{2}{3}\right)^y \quad \text{if } y > 0 \\ P(x, 0) &= \frac{1}{4} \quad \text{if } x > 0 \\ P(x, y) &= \frac{3}{4} \quad \text{if } x > 0 \text{ and } y = x \\ P(x, y) &= 0 \quad \text{else} \end{aligned}$$

Show that this Markov chain is positive recurrent, and find the mean return time to state 2015.

Solutions

1. a) Let $\pi = (a, b, c)$. By setting $\pi P = \pi$, we obtain the system of equations

$$\begin{cases} \frac{1}{4}a + \frac{1}{4}b = a \\ \frac{1}{4}a + \frac{3}{4}c = b \\ \frac{1}{2}a + \frac{3}{4}b + \frac{1}{4}c = c \\ a + b + c = 1 \end{cases}$$

From the first equation, we see $\frac{1}{4}b = \frac{3}{4}a$ so $b = 3a$. Substituting this into the second equation, we get $\frac{1}{4}a + \frac{3}{4}c = 3a$ so $\frac{3}{4}c = \frac{11}{4}a$ so $c = \frac{11}{3}a$. Finally, $1 = a + b + c = a + 3a + \frac{11}{3}a = \frac{23}{3}a$ so $a = \frac{3}{23}$. Thus $\pi = \left(\frac{3}{23}, \frac{9}{23}, \frac{11}{23}\right)$.

b) Yes, because the Markov chain is irreducible, positive recurrent (since $\mathcal{S}$ is finite) and aperiodic.

c) $m_3 = \frac{1}{\pi(3)} = \frac{23}{11}$.

d) Since π is steady-state, $P^n(2, 3) \approx \pi(3) = \frac{11}{23}$ for large n.

e) Since π is steady-state, $\pi_{3000} \approx \pi = \left(\frac{3}{23}, \frac{9}{23}, \frac{11}{23}\right)$ no matter what the initial distribution is.

f) We are asked to find $E_2(V_{3,2300})$; we know from theory that since $\{X_t\}$ is irreducible and positive recurrent, $\frac{E_x(V_{y,n})}{n} \approx \pi(y)$ for large n. Thus by multiplying through by n, we see $E_x(V_{y,n}) \approx n \cdot \pi(y)$. In this case we have $E_2(V_{3,2300}) \approx 2300\pi(3) = 1100$.

2. First, find the stationary distribution. Let $\pi = (a, b, c, d, e)$ and set $\pi P = \pi$ to obtain the system of equations

$$\begin{cases} \frac{3}{5}b = a \\ a + \frac{2}{5}c + \frac{2}{5}d = b \\ \frac{1}{5}b + e = c \\ \frac{1}{5}b = d \\ \frac{3}{5}c + \frac{3}{5}d = e \end{cases} \Rightarrow \begin{cases} 3b = 5a \\ 5a + 2(c + d) = 5b \\ b + 5e = 5c \\ b = 5d \\ 3(c + d) = 5e \end{cases}$$

From the last equation, $c + d = \frac{5}{3}e$ and from the first equation $a = \frac{3}{5}b$; plug these two facts into the second equation to get $3b + \frac{10}{3}e = 5b$, i.e. $e = \frac{3}{5}b$ or $b = \frac{5}{3}e$ (so $a = \frac{3}{5}b = \frac{3}{5}\left(\frac{5}{3}e\right) = e$). Now, the third equation becomes $\frac{1}{5}b + \frac{3}{5}b = c$ so $c = \frac{4}{5}b = \frac{4}{3}e$. Next, the last equation can be solved for d to get $d = \frac{1}{3}e$. We have found

$$a = e \quad b = \frac{5}{3}e \quad c = \frac{4}{3}e \quad d = \frac{1}{3}e \quad e = e$$

so $a+b+c+d+e=1$ becomes $1=e(1+\frac{5}{3}+\frac{4}{3}+\frac{1}{3}+1)=\frac{16}{3}e$ so $e=\frac{3}{16}$. Thus

$$\pi=\left(\frac{3}{16},\frac{5}{16},\frac{1}{4},\frac{1}{16},\frac{3}{16}\right).$$

Next, the periodicity of this Markov chain is as follows: $d=2$ because states $1,3$ and 4 lead only to states 2 and 5 in one step, and vice versa. So if $X_0=1$, then after an even number of steps you must be in state $1,3$ or 4 and after an odd number of steps you must be in state 2 or 5. We have

$$P^n(1,y)=\begin{cases} d\pi(y)=2\pi(y) & \text{if it is possible to get from } 1 \text{ to } y \\ 0 & \text{else} \end{cases}$$

Therefore, the answer is

$$n \text{ even } \Rightarrow \begin{cases} P^n(1,1)=2\pi(1)=\frac{3}{8} \\ P^n(1,2)=0 \\ P^n(1,3)=2\pi(3)=\frac{1}{2} \\ P^n(1,4)=2\pi(4)=\frac{1}{8} \\ P^n(1,5)=0 \end{cases} \qquad n \text{ odd } \Rightarrow \begin{cases} P^n(1,1)=0 \\ P^n(1,2)=2\pi(2)=\frac{5}{8} \\ P^n(1,3)=0 \\ P^n(1,4)=0 \\ P^n(1,5)=2\pi(5)=\frac{3}{8} \end{cases}$$

3. Use the usual methods for birth-death chains: set

$$\pi_x=\frac{p_0p_1p_2\cdots p_{x-2}p_{x-1}}{q_1q_2\cdots q_{x-1}q_x}=\frac{c^1c^2c^3\cdots c^x}{d^1d^2d^3\cdots d^x}=\left(\frac{c}{d}\right)^{1+2+..+x}=\left(\frac{c}{d}\right)^{\frac{1}{2}x(x+1)}$$

where the last equality follows from the triangular series formula. Now the chain is positive recurrent if and only if

$$\sum_{x=0}^{\infty}\pi_x<\infty.$$

We have

$$\sum_{x=0}^{\infty}\pi_x=\sum_{x=0}^{\infty}\left(\frac{c}{d}\right)^{\frac{1}{2}x(x+1)}$$

which diverges if $c\geq d$ (by the n^{th} term test, because the individual terms being added do not go to zero) and converges if $c<d$ (by the Comparison Test, because in this case $\left(\frac{c}{d}\right)^{\frac{1}{2}x(x+1)}\leq\left(\frac{c}{d}\right)^x$ for large x, and $\sum\left(\frac{c}{d}\right)^x$ converges when $c<d$). Thus $\{X_t\}$ is positive recurrent if and only if $c<d$.

4. We can show the chain is positive recurrent and find $\pi(0)=\frac{1}{m_0}$ in either of two ways:

Method # 1: We compute the mean return time to state 0. Note that 0 returns to itself for the first time on the n^{th} step if and only if 0 goes anywhere (probability 1), then around a loop with probability $\frac{3}{4}$ $n-2$ times, then back to 0 (on an arrow with probability $\frac{1}{4}$). So

$$P_0(T_0 = n) = \begin{cases} 0 & \text{if } n = 1 \\ 1\left(\frac{3}{4}\right)^{n-2}\left(\frac{1}{4}\right) & \text{if } n > 1 \end{cases}$$

Therefore

$$\begin{aligned} m_0 = E_0(T_0) = \sum_{n=1}^{\infty} nP_0(T_0 = n) &= \sum_{n=2}^{\infty} n\left(\frac{3}{4}\right)^{n-2}\left(\frac{1}{4}\right) \\ &= \frac{1}{4}\left(\frac{3}{4}\right)^{-2}\sum_{n=2}^{\infty} n\left(\frac{3}{4}\right)^{n} \\ &= \frac{1}{4}\left(\frac{3}{4}\right)^{-2}\left[\frac{3/4}{(1-3/4)^2} - \frac{3}{4}\right] \\ &= \frac{1}{4}\cdot\frac{16}{9}\cdot\frac{3}{4}\cdot\frac{45}{4} \\ &= 5. \end{aligned}$$

Since $m_0 = 5 < \infty$, 0 is positive recurrent so since $\{X_t\}$ is irreducible, $\{X_t\}$ is positive recurrent and $\pi(0) = \frac{1}{m_0} = \frac{1}{5}$.

Method # 2: Define a factor $\{Z_t\}$ of this Markov chain by defining $Z_t = 0$ if $X_t = 0$ and $Z_t = 1$ otherwise. Then $\{Z_t\}$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix};$$

since $\{Z_t\}$ is irreducible and has finite state space, $\{Z_t\}$ is positive recurrent. Thus 0 has finite mean return time in $\{Z_t\}$, and since 0 wasn't grouped when defining $\{Z_t\}$, 0 has finite mean return time in $\{X_t\}$, so 0 is positive recurrent in $\{X_t\}$. By irreducibility, this means $\{X_t\}$ is positive recurrent.

Furthermore, the stationary distribution π_Z of $\{Z_t\}$ can be found by setting $\pi_Z P = \pi_Z$ and solving to obtain $\pi_Z = \left(\frac{1}{5}, \frac{4}{5}\right)$. But since 0 wasn't grouped when defining $\{Z_t\}$, $\pi_Z(0) = \pi(0) = \frac{1}{5}$ and $m_0 = \frac{1}{\pi(0)} = 5$ in $\{X_t\}$.

Finally, we find the mean return time to state 2015. By the stationarity equation,

$$\begin{aligned}
\pi(2015) &= \sum_{x=0}^{\infty} \pi(x) P(x, 2015) \\
\pi(2015) &= \pi(0) P(0, 2015) + \pi(2015) P(2015, 2015) \\
\pi(2015) &= \frac{1}{m_0} \left[\frac{1}{2} \left(\frac{2}{3} \right)^{2015} \right] + \pi(2015) \frac{3}{4} \\
\frac{1}{4} \pi(2015) &= \frac{1}{5} \cdot \frac{1}{2} \cdot \left(\frac{2}{3} \right)^{2015} \\
\pi(2015) &= 4 \cdot \frac{1}{5} \cdot \frac{1}{2} \cdot \left(\frac{2}{3} \right)^{2015} = \frac{2}{5} \left(\frac{2}{3} \right)^{2015}
\end{aligned}$$

Therefore $m_{2015} = \frac{1}{\pi(2015)} = \frac{5}{2} \left(\frac{3}{2} \right)^{2015}$.

3.7 Spring 2015 Exam 3

1. Let $\{X_t\}$ be a CTMC with state space $\{1, 2, 3, 4\}$ and infinitesimal matrix

$$Q = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 3 & -5 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -3 \end{pmatrix}.$$

   a) (3.4) Find the stationary distribution of this CTMC.
   b) (3.2) Use linearization (a.k.a. tangent line approximation) to estimate $P_{44}(.02)$.
   c) (3.4) Find the mean return time to state 1.
   d) (3.2) Suppose you start in state 2. What is the expected amount of time you would stay in state 2 before jumping?
   e) (3.2) Suppose you start in state 2. What is the probability that when you jump for the first time, your first jump is to state 1?
   f) (3.4) For what total length of time in the time interval $[0, 6800]$ would you expect to spend in state 4?

2. Let $\{X_t\}$ be a CTMC with state space $\{1, 2, 3\}$ and transition matrix

$$P(t) = \begin{pmatrix} \frac{3}{16} + \frac{3}{16}e^{-8t} + Ke^{-4t} & \frac{3}{8} - \frac{3}{8}e^{-8t} & \frac{7}{16} + \frac{3}{16}e^{-8t} - \frac{5}{8}e^{-4t} \\ \frac{3}{16} - \frac{5}{16}e^{-8t} + \frac{1}{8}e^{-4t} & h(t) & \frac{7}{16} - \frac{5}{16}e^{-8t} + Le^{-4t} \\ \frac{3}{16} + \frac{3}{16}e^{-8t} - \frac{3}{8}e^{-4t} & \frac{3}{8} - \frac{3}{8}e^{-8t} & \frac{7}{16} + \frac{3}{16}e^{-8t} + \frac{3}{8}e^{-4t} \end{pmatrix}$$

   where K and L are constants and $h(t)$ is an unknown function.

   a) (3.2) Find K, L and $h(t)$.
   b) (3.2) Find $P(X_4 = 2 \,|\, X_2 = 3)$.
   c) (3.3) Find the holding rate of state 3.
   d) (3.4) Find the stationary distribution of this CTMC.

3. Let $\{X_t\}$ be a birth-death CTMC with state space $\mathcal{S} = \{0, 1, 2, 3, ...\}$ with birth rates given by $\lambda_x = \lambda x$ for all $x \in \mathcal{S}$ and death rates given by $\mu_x = \mu$ for all $x \in \mathcal{S}$.

   a) (3.5) Write the forward equation of this process.

b) (3.5) Determine whether such a birth-death CTMC is "always transient", "always recurrent", or whether it could be "sometimes transient and sometimes recurrent" (depending on what λ and μ are). Justify your answer.

4. Let $\{W_t\}$ be standard, one-dimensional Brownian motion.

   a) (4.2) Find $P(W_5 \leq 2)$.

   b) (4.2) Find $P(W_8 = W_3)$.

   c) (4.1, 4.7) Find $Cov(W_3, W_7)$.

   d) (4.2) Find $P(W_t \geq -4$ for all $t \leq 16)$.

   e) (4.4) Let $X_t = 3W_{4t}$. Prove $\{X_t\}$ is a Brownian motion; what is its parameter?

Solutions

1. a) Let $\pi = (a, b, c, d)$ be the stationary distribution. Write $\pi Q = \mathbf{0}$ to get the system of equations

$$\begin{cases} -2a + 3b + c + d = 0 \\ a - 5b + d = 0 \\ b - c + d = 0 \\ a + b - 3d = 0 \end{cases}$$

From the third equation, we have $c = b + d$. Plugging this into the first equation, we get $-2a + 4b + 2d = 0$, i.e. $a = 2b + d$. Plugging this into the fourth equation, we get $3b - 2d = 0$, i.e. $d = \frac{3}{2}b$. Thus $a = 2b + d = 2b + \frac{3}{2}b = \frac{7}{2}b$ and $c = b + d = \frac{5}{2}b$.
Last, since $a + b + c + d = 1$, we see $\frac{7}{2}b + b + \frac{5}{2}b + \frac{3}{2}b = 1$, i.e. $\frac{17}{2}b = 1$, i.e. $b = \frac{2}{17}$. Therefore $\pi = \left(\mathbf{\frac{7}{17}}, \mathbf{\frac{2}{17}}, \mathbf{\frac{5}{17}}, \mathbf{\frac{3}{17}}\right)$.

b) $P_{44}(.02) \approx P_{44}(0) + P'_{44}(0)(.02 - 0) = 1 + q_{44}(.02) = 1 - 3(.02) = \mathbf{.94}$.

c) We have $q_1 = -q_{11} = 2$ and from part (a), $\pi(1) = \frac{7}{17} = \frac{1}{m_1 q_1} = \frac{1}{2q_1}$. Therefore $q_1 = \frac{1}{2} \cdot \frac{17}{7} = \mathbf{\frac{17}{14}}$.

d) $E(W_2) = E(Exp(q_2)) = E(Exp(-q_{22})) = E(Exp(5)) = \mathbf{\frac{1}{5}}$.

e) This is $\pi_{21} = \frac{q_{21}}{q_2} = \frac{q_{21}}{-q_{22}} = \mathbf{\frac{3}{5}}$.

f) By the ergodic theorem, this is $6800\pi(4) = 6800\frac{3}{17} = \mathbf{1200}$.

2. a) Since $P_{11}(0) = 1$, we have $1 = \frac{3}{16} + \frac{3}{16} + K$, i.e. $K = \mathbf{\frac{5}{8}}$.
Since $P_{23}(0) = 0$, we have $0 = \frac{7}{16} - \frac{5}{16} + L$, i.e. $L = \mathbf{\frac{-1}{8}}$.
Since the rows of $P(t)$ add to 1, $h(t) = 1 - P_{21}(t) - P_{23}(t) = \mathbf{\frac{3}{8} + \frac{5}{8}e^{-8t}}$.

b) $P(X_4 = 2 \,|\, X_2 = 3) = P_{32}(4 - 2) = P_{32}(2) = \mathbf{\frac{3}{8} - \frac{3}{8}e^{-16}}$.

c) $q_3 = -q_{33} = -P'_{33}(0) = -\left[\frac{-24}{16}e^{-8(0)} - \frac{12}{8}e^{-4(0)}\right] = -\left[\frac{-3}{2} - \frac{3}{2}\right] = \mathbf{3}$.

d) Using the fact that $\pi(y) = \lim\limits_{t\to\infty} P_{xy}(t)$, we see that $\pi = \left(\mathbf{\frac{3}{16}}, \mathbf{\frac{3}{8}}, \mathbf{\frac{7}{16}}\right)$.

3. a) In general, the forward equation is $P'_{xy}(t) = \sum\limits_{z\in\mathcal{S}} P_{xz}(t)q_{zy}$. In this situation, all the q_{zy} are zero except for q_{yy} which is $q_{yy} = -q_y = -(\lambda_y + \mu_y) = -(\lambda y + \mu)$; $q_{y-1,y} = \lambda_{y-1} = \lambda(y-1)$; and $q_{y+1,y} = \mu_{y+1} = \mu$. So the forward equation reduces to

$$P'_{x,y}(t) = \lambda(y-1)P_{x,y-1}(t) - (\lambda y + \mu)P_{x,y}(t) + \mu P_{x,y+1}(t).$$

b) We compute

$$\begin{aligned}\sum_{x\in\mathcal{S}} \frac{\mu_1\cdots\mu_x}{\lambda_1\cdots\lambda_x} &= \sum_{x\in\mathcal{S}} \frac{\mu\mu\cdots\mu}{\lambda\cdot 2\lambda\cdot 3\lambda\cdots x\lambda} \\ &= \sum_{x=0}^{\infty} \frac{\mu^x}{\lambda^x x!} \\ &= \sum_{x=0}^{\infty} \frac{\left(\frac{\mu}{\lambda}\right)^x}{x!} = e^{\mu/\lambda} < \infty.\end{aligned}$$

Since this series converges no matter what λ and μ are, the birth-death CTMC is **always transient** by a theorem from Section 4.5.

4. a) W_5 is $n(0,5)$ so $P(W_5 \le 2) = P(n(0,5) \le 2) = P\left(n(0,1) \le \frac{2}{\sqrt{5}}\right) = \Phi\left(\frac{2}{\sqrt{5}}\right)$.

b) $P(W_8 = W_3) = P(W_8 - W_3 = 0) = \mathbf{0}$ since $W_8 - W_3$ is the continuous r.v. $n(0,5)$.

c) $Cov(W_3, W_7) = r_W(3,7) = \sigma^2 \min(3,7) = 1\cdot 3 = \mathbf{3}$.

d) $P(W_t \ge -4 \text{ for all } t \le 16) = P(W_t \le 4 \text{ for all } t \le 16) = P(T_4 \ge 10) = 1 - P(T_4 \le 16) = 1 - \left[2 - 2\Phi\left(\frac{4}{1\sqrt{16}}\right)\right] = \mathbf{2\Phi(1) - 1}$. The second-to-last equality in the preceding argument comes from the Reflection Principle.

e) We know that $\{W_t\}$ is Gaussian, and since $\{X_t\}$ has the form $X_t = f(t)X_{g(t)}$, $\{X_t\}$ is also Gaussian by a result from class. Now we compute the mean and covariance functions of $\{X_t\}$:

$$\mu_X(t) = E[X_t] = E[3W_{4t}] = 3E[W_{4t}] = 3\mu_W(4t) = 3(0) = 0$$

$$\begin{aligned}r_X(s,t) = Cov(X_s, X_t) = Cov(3W_{4s}, 3W_{4t}) &= 9r_W(4s,4t) \\ &= 9\min(4s,4t) \\ &= 9\cdot 4\min(s,t) \\ &= 36\min(s,t)\end{aligned}$$

so by uniqueness of mean and covariance functions for Gaussian processes, $\{\mathbf{X_t}\}$ **is a Brownian motion with parameter** $\sigma^{\mathbf{2}} = \mathbf{36}$.

3.8 Spring 2016 Exam 1

1. Mike and Tom are equally talented ping-pong players (when they play against each other, they are each 50% likely to win each point they play). They decide to play until one of the players has won six more points than the other; that player will be declared the "winner" of their match.

   a) (2.5) What is the probability that Mike wins the match?

   b) (2.5) How many points should they expect to play before the match is over?

   c) (2.5) Suppose that of their first eight points, each player wins four. How many *more* points should they expect to play before the match is over?

   d) (2.5) If Mike wins two of the first three points, what is the probability that Mike goes on to win the match?

   e) (2.5) Suppose a third player, Ralph, shows up; Ralph is three times as good a player as either Mike or Tom. If he plays a match against Mike with the same rules (first to be ahead by six wins), what is the probability that Mike wins that match?

2. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/6 & 1/3 & 1/3 & 1/6 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/12 & 1/12 & 0 \\ 0 & 0 & 0 & 3/5 & 2/5 \\ 0 & 0 & 0 & 7/10 & 3/10 \end{pmatrix}.$$

   a) (1.7) Classify the states of this Markov chain as recurrent or transient.

   b) (1.6) Find $P_5(T_4 = 6)$.

   c) (1.7) Find $f_{3,4}$, $f_{3,2}$, $f_{4,3}$, $f_{3,3}$ and $f_{4,5}$.

   d) (1.3) If the distribution at time 2 is $(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0)$, find the distribution at time 3.

3. The winter weather on any given in Big Rapids is one of three states: cold, colder and coldest. Suppose that if it is "cold" on a given day, then it is 20% likely to be "cold" on the next day and 60% likely to be "colder" on the next day, and 20% likely to be "coldest" on the next day. If it is "colder" on a given day, then it is 40% likely to be "cold" on the next day and 60% likely to be "coldest" on the next day. If it is "coldest" on a given day, then it is not coldest on the next day–it is equally likely to be "cold" or "colder" on the next day.

a) (1.3) If it is "cold" for four days in a row, what is the probability that it is "colder" on the fifth day?

b) (1.3) If it is "cold" today, what is the probability that it is "colder" two days from now?

c) (1.3) If it is "colder" on days 1 and 3, what is the probability that it is "cold" on day 2?

Solutions

1. a) Let X_t be the number of games Mike is ahead after the t^{th} game; this makes $\{X_t\}$ an unbiased random walk with $p = q = \frac{1}{2}$. Then Mike wins if $T_6 < T_{-6}$, so
$$P_0(T_6 < T_{-6}) = \frac{0 - (-6)}{6 - (-6)} = \frac{1}{2}.$$

   b) Let $T = T_{6,-6}$. Then $ET = \frac{(b-x)(x-a)}{p+q} = \frac{(6-0)(0-(-6))}{\frac{1}{2}+\frac{1}{2}} = 36.$

   c) If each player wins the first four points, then $X_4 = 0$. But starting from this point is the same as starting from the beginning (by the Markov property). So the answer here is the same as (b), i.e. 36.

   d) If Mike wins two of the first three points, then at that point $X_t = 1$. So from this point, the probability Mike wins is
$$P_1(T_6 < T_{-6}) = \frac{1 - (-6)}{6 - (-6)} = \frac{7}{12}.$$

   e) Let Y_t be number of games Mike is ahead after the t^{th} game; this makes $\{X_t\}$ a biased random walk with $p\frac{1}{4}$ and $q = \frac{3}{4}$. Mike wins if $T_6 < T_{-6}$, so
$$P_0(T_6 < T_{-6}) = \frac{\left(\frac{q}{p}\right)^x - \left(\frac{q}{p}\right)^a}{\left(\frac{q}{p}\right)^b - \left(\frac{q}{p}\right)^a} = \frac{3^0 - 3^{-6}}{3^6 - 3^{-6}} = \frac{1 - 3^{-6}}{3^6 - 3^{-6}}.$$

2. a) First, sketch a directed graph; this will show you that $\{2\}$ and $\{4,5\}$ are the only communicating classes of this Markov chain. Elements of these finite communicating classes must be recurrent, and elements not in a communicating class must be transient; therefore $\mathcal{S}_R = \{2,4,5\}$ and $\mathcal{S}_T = \{1,3\}$.

   b) If you start at 5 and hit 4 for the first time on the sixth step, then your first six steps must be $5 \to 5 \to 5 \to 5 \to 5 \to 5 \to 4$. Thus
$$P_5(T_4 = 6) = P(5,5)^5 P(5,4) = \left(\frac{3}{10}\right)^5 \left(\frac{7}{10}\right).$$

c)
- $f_{4,5} = 1$ since 4 and 5 are members of the same recurrent communicating class.
- $f_{4,3} = 0$ since 4 is recurrent and 3 is transient.
- To find $f_{3,2}$, use a system of equations:

$$\begin{cases} f_{3,2} = \frac{1}{3}f_{1,2} + \frac{1}{2}f_{2,2} + \frac{1}{12}f_{3,2} + \frac{1}{12}f_{4,2} & = \frac{1}{3}f_{1,2} + \frac{1}{2} + \frac{1}{12}f_{3,2} \\ f_{1,2} = \frac{1}{6}f_{2,2} + \frac{1}{3}f_{3,2} + \frac{1}{3}f_{4,2} + \frac{1}{6}f_{5,2} & = \frac{1}{6} + \frac{1}{3}f_{3,2} \end{cases}$$

  Multiply through the top equation by 12 and the bottom equation by 6 to clear the fractions; then solve these two equations together to get $f_{1,2} = \frac{23}{58}$ and (more importantly) $f_{3,2} = \frac{20}{29}$.
- $f_{3,4} = 1 - f_{3,2} = 1 - \frac{20}{29} = \frac{9}{29}$.
- From considering the directed graph, if you start at 3, the only way to come back to 3 is to either return immediately ($3 \to 3$) or to go $3 \to 1 \to 3$. Otherwise you enter one of the recurrent communicating classes. Therefore

$$f_{3,3} = P_3(T_3 < \infty) = \sum_{n=1}^{\infty} P_3(T_3 = n) = \frac{1}{12} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{12} + \frac{1}{9} = \frac{7}{36}.$$

d) Let $\pi_2 = \left(\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 0\right)$; the distribution at time 3 is therefore $\pi_3 = \pi_2 P$ which is

$$\pi_3 = \pi_2 P = \left(\frac{1}{12}, \frac{1}{6}, \frac{5}{48}, \frac{97}{240}, \frac{29}{120}\right).$$

3. First, set up a transition matrix for the Markov chain: let the rows and columns correspond, respectively to "cold", "colder" and "coldest":

$$P = \begin{pmatrix} .2 & .6 & .2 \\ .4 & 0 & .6 \\ .5 & .5 & 0 \end{pmatrix}$$

a) This is

$$\begin{aligned} P(X_5 = 2 \,|\, X_1 = X_2 = X_3 = X_4 = 1) &= P(X_5 = 2 \,|\, X_4 = 1) \\ &\qquad \text{(by the Markov property)} \\ &= P(1,2) = .6. \end{aligned}$$

b) This is $P(X_2 = 2 \,|\, X_0 = 1) = P^2(1,2)$. The $(1,2)$ entry of matrix P^2 is

$$(.2)(.6) + (.6)(0) + (.2)(.5) = .12 + .1 = .22.$$

c) This is

$$\begin{aligned}
P(X_2 = 1 \mid X_1 = X_3 = 2) &= \frac{P(X_1 = 2, X_2 = 1, X_3 = 2)}{P(X_1 = 2, X_3 = 2)} \\
&= \frac{\pi_1(2)P(2,1)P(1,2)}{\pi_1(2)P^2(2,2)} \\
&= \frac{P(2,1)P(1,2)}{P^2(2,2)} \\
&= \frac{(.4)(.6)}{(.4)(.6) + (.6)(.5)} = \frac{.24}{.54} = \frac{4}{9}.
\end{aligned}$$

3.9 Spring 2016 Exam 2

1. Let $\{X_t\}$ be the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ whose transition matrix is

$$\begin{pmatrix} 0 & 0 & .4 & .6 & 0 & 0 \\ 0 & 0 & .8 & .2 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ 0 & 0 & 0 & 0 & .5 & .5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   a) (1.5) Find all stationary distributions of this Markov chain. If there are no stationary distributions, explain why no stationary distribution exists.

   b) (1.10) Find all steady-state distributions of this Markov chain. If there are no steady-state distributions, explain why no steady-state distribution exists.

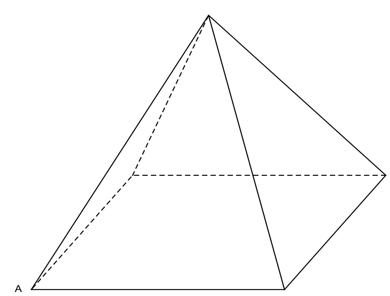
   c) (1.8) How many times in the first 2000 steps would you expect to be in state 4?

   d) (1.10) Find the Cesàro limit of $P^n(3, 3)$.

   e) (1.10) Estimate $P^{300}(3, 5)$.

   f) (1.10) Estimate $P^{400}(3, 5)$.

2. Here is a picture of a pyramid with a square base:



Let $\{X_t\}$ be a Markov chain described as follows: the states of the Markov chain are the vertices (corners) of the pyramid; to transition from one state to another, you uniformly choose an edge leaving your current state and travel along that edge to the vertex where it ends. That vertex is your next state.

   a) (1.10) Suppose you start in the lower left-hand corner of the base, at the point marked A in the picture above. How many steps, on the average, should you expect it to take you to return to A for the first time?

b) (1.10) Suppose you constructed the same type of Markov chain, but with a pyramid that has a base which is a polygon with n sides (in part (a), n was 4 since a square has 4 sides). For this Markov chain, answer the same question as was posed part (a).

3. Let $\{X_t\}$ be a Markov chain with state space $\{0, 1, 2, ...\}$ whose transition function is

$$P(x, y) = \begin{cases} \frac{2}{3}\left(\frac{1}{3}\right)^y & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x > 0 \text{ and } y = 2x \\ \frac{1}{3} & \text{if } x > 0 \text{ and } y = 4x \\ \frac{1}{3} & \text{if } x > 0 \text{ and } y = 0 \\ 0 & \text{else} \end{cases}$$

a) (1.10) Let π be the stationary distribution of this Markov chain. Find $\pi(0)$ and $\pi(4)$.

b) (1.10) Estimate $P^n(7, 0)$ for n large.

c) (1.10) Find the mean return time to state 99.

4. (1.10) Let $\{X_t\}$ be a Markov chain with state space $\{0, 1, 2, ...\}$ whose transition function is

$$P(x, y) = \begin{cases} \frac{C}{(y+1)^2} & \text{if } x = 0 \\ 1 & \text{if } x > 0 \text{ and } y = x - 1 \\ 0 & \text{if } x > 0 \text{ and } y \neq x - 1 \end{cases}$$

where C is a constant which makes the transition probabilities from state 0 add up to 1 (if you really care, $C = \frac{6}{\pi^2}$, but you can just call it C throughout this problem). Determine (with appropriate justification) whether $\{X_t\}$ is positive recurrent, null recurrent or transient.

Solutions

1. a) Write $\pi = (a, b, c, d, e, f)$ and set $\pi P = \pi$. This gives the equations

$$\begin{cases} a = e + .5f \\ b = .5f \\ c = .4a + .8b \\ d = .6a + .2b \\ e = .5c + .5d \\ f = .5c + .5d \end{cases}$$

Adding the first two equations together, we get $a + b = e + f$; adding the third and fourth equations together we get $c + d = a + b$; that means $a + b$, $c + d$ and $e + f$ are all equal; since all six numbers add to 1 that means

$$a + b = c + d = e + f = \frac{1}{3}.$$

Furthermore, from the last two equations $e = f$,so e and f must both be $\frac{1}{6}$. Substituting into the first two equations, we get $a = \frac{1}{4}$ and $b = \frac{1}{12}$. Using the third and fourth equations to solve for c, we get

$$\pi = \left(\frac{1}{4}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

b) Since $\{X_t\}$ has period 3, it has no steady-state distribution.

c) This is $2000\pi(4) = \frac{2000}{6}$.

d) This is $\pi(3) = \frac{1}{6}$.

e) $P^{300}(3, 5) = 0$ since you can only get from 3 to 5 in n steps if $n \equiv 1 \mod 3$.

f) Since $400 \mod 3 \equiv 1$, $P^{400}(3, 5) \approx d \cdot \pi(5) = 3\frac{1}{6} = \frac{1}{2}$.

2. a) Let the states of the Markov chain be $\{1, 2, 3, 4, 5\}$ where 5 is the top. Let $\pi = (a, b, c, d, e)$ be the stationary distribution; by symmetry, all the points on the bottom are the same so $\pi = (a, a, a, a, e)$. Thus $4a + e = 1$. Now, by the stationarity equation with $y = 5$, we have

$$\begin{aligned} \pi(5) &= \sum_{x \in \mathcal{S}} \pi(x) P(x, 5) \\ e &= aP(1, 5) + aP(2, 5) + aP(3, 5) + aP(4, 5) + eP(5, 5) \\ e &= a\frac{1}{3} + a\frac{1}{3} + a\frac{1}{3} + a\frac{1}{3} + e0 \\ e &= \frac{4}{3}a. \end{aligned}$$

Since $e = \frac{4}{3}a$ and $4a + e = 1$, we have $\left(4 + \frac{4}{3}\right) a = 1$, i.e. $a = \frac{3}{16}$. Thus the mean return time to any state other than 5 (such as A) is $\frac{1}{a} = \frac{16}{3}$.

b) This time, the stationary distribution would be of the form $\pi = (a, a, a, ..., a, b)$ where $na + b = 1$. By the stationarity equation with $y =$the top, we have

$$\begin{aligned}
\pi(\text{top}) &= \sum_{x \in \mathcal{S}} \pi(x) P(x, \text{top}) \\
b &= aP(1,5) + aP(2,5) + ... + aP(n, \text{top}) + bP(\text{top}, \text{top}) \\
b &= a\frac{1}{3} + a\frac{1}{3} + ... + a\frac{1}{3} + b0 \\
b &= \frac{n}{3}a.
\end{aligned}$$

Since $na + b = 1$ and $b = \frac{n}{3}a$, we have $\left(n + \frac{n}{3}\right) a = 1$ so $a = \frac{3}{4n}$. Thus the mean return time to any corner point other than the top is $\frac{4n}{3}$.

3. a) Define a factor $\{Z_t\}$ by setting $Z_t = 0$ if $X_t = 0$ and setting $Z_t = 1$ if $X_t \neq 0$. This makes $\{Z_t\}$ a Markov chain with transition matrix

$$P_Z = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix};$$

since this matrix is doubly stochastic we have $P_Z = \left(\frac{1}{2}, \frac{1}{2}\right)$. Thus $\pi(0) = \frac{1}{2}$ in $\{X_t\}$ since 0 was ungrouped when constructing $\{Z_t\}$.

By the stationarity equation (in $\{X_t\}$ with $y = 1$ we get

$$\pi(1) = \sum_x \pi(x) P(x, 1) = \pi(0) P(0, 1) = \frac{1}{2} \cdot \frac{2}{9} = \frac{1}{9}.$$

Repeating this with $y = 2$, we get

$$\pi(2) = \sum_x \pi(x) P(x, 2) = \pi(0) P(0, 2) + \pi(1) P(1, 2) = \frac{1}{3} \cdot \frac{2}{27} + \frac{1}{9} \cdot \frac{1}{3} = \frac{5}{81}.$$

Last, with $y = 4$ we get

$$\begin{aligned}
\pi(4) &= \sum_x \pi(x) P(x, 4) = \pi(0) P(0, 4) + \pi(1) P(1, 4) + \pi(2) P(2, 4) \\
&= \frac{1}{3} \cdot \frac{2}{81} + \frac{1}{9} \cdot \frac{1}{3} + \frac{5}{81} \cdot \frac{1}{3} \\
&= \frac{16}{243}.
\end{aligned}$$

b) Since the chain is aperiodic (you can get from 0 to itself in 1 step), $P^n(7, 0) \approx \pi(0) = \frac{1}{6}$ for n large.

c) By the stationarity equation with $y = 99$,

$$\pi(99) = \sum_x \pi(x)P(x, 99) = \pi(0)P(0, 99) = \frac{1}{3} \cdot \frac{2}{3}\left(\frac{1}{3}\right)^{99} = \frac{2}{3^{101}}.$$

That means the mean return time to 99 is $[\pi(99)]^{-1} = \frac{1}{2}(3^{101})$.

4. Notice that 0 returns to itself for the first time on the n^{th} step precisely when the first step from 0 is to step $n-1$. So

$$P_0(T_0 = n) = P(0, n-1) = \frac{C}{n^2}.$$

Therefore

$$f_0 = \sum_{n=1}^{\infty} P_0(T_0 = n) = \sum_{n=1}^{\infty} \frac{C}{n^2} = \sum_{n=0}^{\infty} P(0, n) = 1$$

so 0 is recurrent. At the same time,

$$m_0 = E_0(T_0) = \sum_{n=1}^{\infty} nP_0(T_0 = n) = \sum_{n=1}^{\infty} \frac{C}{n}$$

which diverges (since it is a harmonic series). Therefore $m_0 = \infty$ so 0 is not positive recurrent. Since 0 is neither transient nor positive recurrent, it is null recurrent. Since the chain is irreducible (you can get from any x to any y by going $x \to x-1 \to x-2 \to \cdots \to 1 \to 0 \to y$), the whole chain must be null recurrent.

3.10 Spring 2016 Exam 3

1. Let $\{X_t\}$ be the CTMC with state space $\{1, 2, 3\}$ whose time t transition matrix is

$$P(t) = \frac{1}{50} \begin{pmatrix} 18 + 4e^{-5t} + 28e^{-8t} & 17 + e^{-5t} - 18e^{-8t} & 15 - 5e^{-5t} - 10e^{-8t} \\ 18 + 24e^{-5t} - 42e^{-8t} & 17 + 6e^{-5t} + 27e^{-8t} & 15 - 30e^{-5t} + 15e^{-8t} \\ 18 - 32e^{-5t} + 14e^{-8t} & f(t) & 15 + 40e^{-5t} - 5e^{-8t} \end{pmatrix}$$

and whose infinitesimal matrix is

$$Q = \frac{1}{50} \begin{pmatrix} -244 & 139 & 105 \\ 216 & -246 & 30 \\ 48 & 112 & -160 \end{pmatrix}$$

a) (3.2) Suppose you weren't told what Q was. Write down the formula which you would use to compute Q from $P(t)$.

b) (3.2) Find $f(t)$.

c) (3.4) Find the stationary distribution of $\{X_t\}$.

d) (3.2) Find the holding rate of state 1.

e) (3.4) Find the mean return time to state 1.

f) (3.2) Suppose that you start in state 2.

   i. Find the probability that after 4 units of time, you are in state 3.

   ii. Find the probability that you stay in state 2 for at least 10 units of time before you jump for the first time.

   iii. Find the probability that when you jump for the first time, you jump to state 1.

2. (2.6) Let $\{X_t\}$ be a discrete-time birth-death chain with state space $\{0, 1, 2, ...\}$ such that $p_0 = 1$, and for all $x \geq 1$,

$$p_x = \frac{1}{\sqrt{x+1}} \quad \text{and} \quad q_x = \frac{1}{\sqrt{x+2}}.$$

Classify this chain as recurrent or transient, justifying your reasoning.

3. Suppose that the price X_t of a stock at time t (in days) is modeled by a Brownian motion with drift, whose parameters are $\mu = 3$ and $\sigma^2 = 8$. Suppose further that you have rescaled the problem so that the current price of the stock is 0.

a) (4.1) Find the probability that 9 days from now, the stock price is at most 22.

b) (4.1) Find the probability that the price of the stock is greater 20 days from now than it will be 10 days from now.

c) (4.1) Find the mean of the stock price 6 days from now.

d) (4.1) Find the variance of the stock price 6 days from now.

e) (4.1, 4.4) Find the covariance between the price of the stock at time 4 and the price of the stock at time 5.

f) (4.1) What is the probability that the price of the stock hits 25 before it hits 18?

4. Let $\{W_t\}$ be a standard, 2-dimensional Brownian motion, starting at the point $(4, -3)$.

   a) (4.7) Find the probability that W_t hits the circle $x^2 + y^2 = 1$ before it hits the circle $x^2 + y^2 = 49$?

   b) (4.7) What is the probability that W_t hits the x-axis at some time $t > 0$? Explain your answer.

   c) (4.7) What is the probability that W_t hits the origin at some time $t > 0$? Explain your answer.

Solutions

1. a) $Q = P'(0)$.

   b) The third row must sum to 1, so

$$\begin{aligned} f(t) &= 50 - [18 - 32e^{-5t} + 14e^{-8t}] - [15 - 30e^{-5t} + 15e^{-8t}] \\ &= 17 - 8e^{-5t} - 9e^{-8t}. \end{aligned}$$

   c) Since the stationary distribution π is always steady-state in a CTMC, we know that for each state y, $\pi(y) = \lim_{t\to\infty} P_{xy}(t)$ so

$$\pi = \left(\frac{18}{50}, \frac{17}{50}, \frac{15}{50}\right).$$

   d) The holding rate is $q_1 = -q_{11} = \frac{244}{50}$.

   e) We have $\pi(1) = \frac{1}{m_1 q_1}$; from parts (c) and (d) we have $\frac{18}{50} = \frac{1}{\frac{244}{50} m_1} = \frac{50}{244 m_1}$ so $m_1 = \frac{50\cdot 50}{18\cdot 244} = \frac{625}{1098}$.

   f) Suppose that you start in state 2.

      i. This is $P_{23}(4) = \frac{1}{50}\left(15 - 30e^{-20} + 15e^{-32}\right)$.

      ii. The waiting time at state 2 is $W_2 \sim Exp(q_2) = Exp(-q_{22}) = Exp(246/50)$, so this probability is $P(Exp(246/50) \geq 10) = e^{-2460/50} = e^{-246/5}$.

      iii. This is $\pi_{21} = \frac{q_{21}}{q_2} = \frac{216/50}{246/50} = \frac{36}{41}$.

2. Following the language in the notes, define

$$\gamma_y = \frac{q_y q_{y-1}\cdots q_1}{p_y p_{y-1}\cdots p_1} = \frac{\frac{1}{\sqrt{y+2}}\cdot\frac{1}{\sqrt{y+1}}\cdot\frac{1}{\sqrt{y}}\cdots\frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{4}}\cdot\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{y+1}}\cdot\frac{1}{\sqrt{y}}\cdot\frac{1}{\sqrt{y-1}}\cdots\frac{1}{\sqrt{4}}\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{y+2}}}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{y+2}}$$

   Now

$$\sum_y \gamma_y = \sum_{y=1}^{\infty} \frac{\sqrt{2}}{\sqrt{y+2}} = \sqrt{2}\sum_{y=3}^{\infty}\frac{1}{y^{1/2}}$$

   which diverges (since it is a p-series with $p = \frac{1}{2} \leq 1$). Therefore by the theorem on recurrence/transience of birth-death chains, this Markov chain is **recurrent**.

3. a) $X_9 = X_9 - 0 = X_9 - X_0$ is normal with mean $3(9-0) = 27$ and variance $8(9-0) = 72$. So $P(X_9 \leq 22) = P(n(27,72) \leq 22) = \Phi\left(\frac{22-27}{\sqrt{72}}\right) = \Phi\left(\frac{-5}{\sqrt{72}}\right)$.

   b) $X_{20} - X_{10}$ is normal with mean $3(20-10) = 30$ and variance $8(20-10) = 80$. So $P(X_{20} - X_{10} \geq 0) = P(n(30,80) \geq 0) = 1 - \Phi\left(\frac{0-30}{\sqrt{80}}\right) = \Phi\left(\frac{30}{\sqrt{80}}\right)$.

   c) $X_6 = X_6 - 0 = X_6 - X_0$ is normal with mean $3(6-0) = 18$.

d) $X_6 = X_6 - 0 = X_6 - X_0$ is normal with variance $8(6-0) = 48$.

e) By definition, $X_t = W_t + \mu t$ where $\{W_t\}$ is a BM with $\sigma^2 = 8$. Therefore

$$\begin{aligned} Cov(X_4, X_5) &= Cov(W_4 + 4\mu, W_5 + 5\mu) \\ &= Cov(W_4, W_5) + Cov(4\mu, W_5) + Cov(W_4, 5\mu) + Cov(4\mu, 5\mu) \\ &= \sigma^2 \min(4,5) + 0 + 0 + 0 \\ &= 8 \cdot 4 = 32. \end{aligned}$$

In this computation, we use the property that the covariance of any constant with any other random variable is zero (because constants are independent of any other r.v.s).

f) This is **0** (Brownian paths are continuous with probability one, and you can't get from 0 to 25 without first passing 18 by the Intermediate Value Theorem.)

If the problem had a -18 instead of an 18, the solution would be

$$\begin{aligned} P_0(T_{25} < T_{-18}) &= \frac{1 - \exp\left(\frac{-2\mu(25)}{\sigma^2}\right)}{\exp\left(\frac{-2\mu(25)}{8}\right) - \exp\left(\frac{-2\mu(-18)}{\sigma^2}\right)} \\ &= \frac{1 - \exp\left(\frac{-75}{4}\right)}{\exp\left(\frac{-75}{4}\right) - \exp\left(\frac{27}{2}\right)}. \end{aligned}$$

4. a) Let $\mathbf{x} = (4, -3)$; we see that $||\mathbf{x}|| = \sqrt{4^2 + (-3)^2} = 5$. We want the probability that given that you start at a vector of magnitude 5, you hit a circle of radius $R_1 = 1$ before you hit a circle of radius $R_2 = 7$. That probability is

$$P_{\mathbf{x}}(T_{R_1} < T_{R_9}) = 1 - \frac{\ln 5 - \ln 1}{\ln 7 - \ln 1} = 1 - \frac{\ln 5}{\ln 7}.$$

b) The y-coordinate of $\{W_t\}$ is a standard, 1-dimensional BM. By recurrence and irreducibility, this one-dimensional BM hits every possible value with probability 1, so this probability is **1**.

c) The x- and y-coordinates of $\{W_t\}$ are independent, continuous r.v.s. The probability that they take the same value at the same time is therefore **0** (because the probability that any two independent continuous r.v.s take the same value is always zero).

Chapter 4

Exams from 2017 to present

4.1 Spring 2017 Exam 1

1. The daily price of a share of McClendonCorp stock is modeled by a simple random walk, where each day, the stock price increases by $\$1$ with probability $\frac{3}{5}$ and decreases by $\$1$ with probability $\frac{2}{5}$.

   a) (2.5) Suppose you buy a share of this stock for $\$12$, planning to sell whenever the share hits a value of either $\$15$ or $\$10$. What is the probability that when you sell, your share is worth $\$15$?

   b) (2.5) Suppose you buy a share of this stock for $\$8$. What is the probability that this share is eventually worthless?

2. Two computers, Hydra and Fritz, play a series of chess games against each other, until one of them has won four more games than the other, at which time that computer is crowned the "champion". The computers are equally good, so they both win 15% of the time (i.e. most of the time, they play to a draw).

   a) (2.5) How many games should they expect to play before a champion is crowned?

   b) (2.5) Suppose that in the first ten games, Hydra wins twice, Fritz wins once and there are seven draws. Given this, what is the probability that Hydra ends up being the champion?

   c) (2.5) Suppose that in the first ten games, Hydra wins twice, Fritz wins once and there are seven draws. Given this, how many more games would you expect it would take to crown a champion?

3. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/8 & 3/8 \\ 0 & 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

   a) (1.7) Classify the states of this Markov chain as recurrent or transient.
   b) (1.3) Find $P(X_5 = 3 \,|\, X_4 = 4)$.
   c) (1.3) Find $P(X_5 = 3 \,|\, X_4 = 4, X_3 = 3, X_2 = 4)$.
   d) (1.3) If the distribution at time 0 is $(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$, find the distribution at time 2.
   e) (1.6) Find $P_4(T_3 = 7)$.
   f) (1.6) Find $f_{1,1}$, $f_{1,2}$, $f_{4,1}$, $f_{4,2}$ and $f_{4,3}$.

4. (2.6) Let $\{X_t\}$ be an irreducible birth-death chain with state space $\{0, 1, 2, 3, ...\}$ such that $p_0 = 1$ and such that for all $x \geq 1$, $p_x = \frac{1}{4x}$ and $q_x = \frac{1}{8x}$. Classify this chain as recurrent or transient, justifying your reasoning.

Solutions

1. Let X_t be the price of the share after t days. $\{X_t\}$ is a simple random walk with $p = \frac{3}{5}$ and $q = \frac{2}{5}$.

   a) Use the escape probabilities for a biased random walk: $P_{12}(T_{15} < T_{10}) = \frac{1-\left(\frac{q}{p}\right)^{12-10}}{1-\left(\frac{q}{p}\right)^{15-10}} = \frac{1-\left(\frac{2}{3}\right)^2}{1-\left(\frac{2}{3}\right)^5} = \frac{\frac{5}{9}}{\frac{211}{243}} = \frac{135}{211}$.

   b) Since the walk is positively biased, by the Gambler's Ruin theorem, $P_8(T_0 < \infty) = f_{8,0} = \left(\frac{p}{q}\right)^{8-0} = \left(\frac{2}{3}\right)^8$.

2. Let X_t be the number of games Hydra is ahead after t total games; this makes $\{X_t\}$ a simple random walk with $p = q = .15$ (so the walk is unbiased). Let $T = T_{4,-4}$, so that $a = -4$ and $b = 4$.

   a) We assume $X_0 = x = 0$ and compute, using the formula coming from Wald's Second Identity, $ET = \frac{(x-a)(b-x)}{p+q} = \frac{(4-0)(0-(-4))}{.15+.15} = \frac{16}{.3} = \frac{160}{3}$.

   b) Use escape probabilities: $P_1(T_4 < T_{-4}) = \frac{1-(-4)}{4-(-4)} = \frac{5}{8}$.

   c) We assume $X_0 = x = 1$ and compute, using the formula coming from Wald's Second Identity, $ET = \frac{(x-a)(b-x)}{p+q} = \frac{(4-1)(1-(-4))}{.15+.15} = \frac{15}{.3} = 50$.

3. a) States 1 and 2 are absorbing, hence recurrent. 3 and 4 both lead to 1, but 1 does not lead back to either 3 or 4, so 3 and 4 are transient.

   b) This is $P(4,3) = \frac{1}{2}$.

   c) By the Markov property, this is $P(X_5 = 3 \,|\, X_4 = 4) = P(4,3) = \frac{1}{2}$.

   d) Let $\pi_0 = \left(\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2}\right)$. Then the distribution at time 2 is $\pi_2 = \pi_0 P^2 = \left(\frac{5}{16}, \frac{15}{32}, \frac{3}{32}, \frac{1}{8}\right)$.

   e) Suppose $X_0 = 4$. To hit 3 for the first time on the seventh step, you must have gone around the loop at state 4 six times, then go to step 3. Thus $P_4(T_3 = 7) = P(4,4)^6 P(4,3) = \left(\frac{1}{4}\right)^6 \frac{1}{2}$.

   f) $f_{1,1} = 1$ since 1 is recurrent.

   $f_{1,2} = 0$ since 1 and 2 belong to different communicating classes.

   To find $f_{4,1}$, use methods for absorption probabilities to obtain the system
   $$\begin{cases} f_{3,1} = \frac{1}{4} + \frac{1}{8} f_{3,1} + \frac{3}{8} f_{4,1} \\ f_{4,1} = \frac{1}{2} f_{3,1} + \frac{1}{4} f_{4,1} \end{cases}$$
   which has solution $f_{3,1} = \frac{2}{5}$ and more importantly, $f_{4,1} = \frac{4}{15}$.

   Since there are only finitely many transient states, $f_{4,1} + f_{4,2} = 1$ so $f_{4,2} = \frac{11}{15}$.

Last, compute $f_{4,3}$ directly using the definition. Observe that by repeating the reasoning of part (e) of this question, $P_4(T_3 = n) = \left(\frac{1}{4}\right)^{n-1}\frac{1}{2}$. Thus

$$f_{4,3} = \sum_{n=1}^{\infty} P_4(T_3 = n) = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n-1}\frac{1}{2} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n} = \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}} = \frac{2}{3}.$$

4. First, for each $y \geq 1$,

$$\gamma_y = \frac{q_y q_{y-1}\cdots q_2 q_1}{p_y p_{y-1}\cdots p_2 p_1} = \frac{\frac{1}{8y}\cdot\frac{1}{8(y-1)}\cdots\frac{1}{16}\cdot\frac{1}{8}}{\frac{1}{4y}\cdot\frac{1}{4(y-1)}\cdots\frac{1}{8}\frac{1}{4}} = \frac{\frac{1}{8^y y!}}{\frac{1}{4^y y!}} = \frac{4^y}{8^y} = \left(\frac{1}{2}\right)^y.$$

Now $\sum_y \gamma_y = \sum_y \left(\frac{1}{2}\right)^y$ converges (it is a geometric series with common ratio $\frac{1}{2}$), so $\{X_t\}$ is **transient** by the theorem governing recurrence and transience for birth-death chains.

4.2 Spring 2017 Exam 2

1. Consider a Markov chain $\{X_t\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

   a) (1.5) Compute the stationary distribution of this Markov chain.
   b) (1.8) Find the mean return time to state 3.
   c) (1.8) Suppose you start in state 1. Find the expected amount of steps it would take for you to return to state 1 for the eighth time.
   d) (1.8) Suppose you start in state 2. Find the expected number of visits to state 1 in the first 1000 units of time.
   e) (1.10) Suppose the initial distribution is $\left(\frac{5}{17}, \frac{3}{17}, \frac{9}{17}\right)$. Estimate the time 1000000 distribution.
   f) (1.10) Estimate $P^n(2, 1)$ for large n.

2. Dysfunctional family members Al, Bal, Cal and Dal continue to have trouble passing the salt around the dinner table. Assume:
   - Al only passes the salt to Bal;
   - Bal only passes the salt to Cal or Al, and passes to them with equal probability;
   - Cal only passes the salt to Dal or Bal, and is three times as likely to pass the salt to Dal as he is to Bal;
   - Dal only passes the salt to Cal or Al, and is three times as likely to pass the salt to Cal as he is to Al.

   a) (1.10) Suppose Al has the salt now. Find the probability that Cal has the salt after 50000 passes.
   b) (1.10) Suppose Bal has the salt now. Find the probability that Cal has the salt after 50000 passes.

3. (1.10) Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{0, 1, 2, 3, ...\}$ and transition function

$$P(x, y) = \begin{cases} \frac{1}{2^{y+1}} & \text{if } x = 0 \\ \frac{3}{4} & \text{if } x > 0 \text{ and } y = x + 1 \\ \frac{1}{4} & \text{if } x > 0 \text{ and } y = 0 \\ 0 & \text{else} \end{cases}$$

   Find the mean return time to state 2.

4. Suppose that $\{X_t\}$ is a birth-death chain with state space $\{0, 1, 2, 3, ...\}$ where $p_0 = 1$ and for all $x \geq 1$, $p_x = q_x = \frac{1}{x+1}$.

   a) (2.6) Show that this chain has no stationary distributions.

   b) **(Bonus)** (2.6) Determine, with justification, whether the chain is positive recurrent, null recurrent, or transient.

Solutions

1. a) Let the stationary distribution be $\pi = (a, b, c)$; set $\pi = \pi P$ to get

$$\begin{cases} \frac{2}{3}a & +\frac{1}{3}b & +\frac{1}{3}c & = a \\ \frac{1}{3}a & & +\frac{1}{3}c & = b \\ & \frac{2}{3}b & +\frac{1}{3}c & = c \\ a & +b & +c & = 1 \end{cases}$$

   From the third equation, $b = c$. Substituting into the second equation, we get $a = 2b$ so from the last equation, we have $2b + b + b = 1$, i.e. $b = \frac{1}{4}$. Thus $\pi = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

   b) $m_3 = \frac{1}{\pi(3)} = 4$.

   c) This is $8m_1 = 8 \cdot \frac{1}{\pi(1)} = 8 \cdot 2 = 16$.

   d) By the ergodic theorem, this is $E_2(V_{1,1000}) \approx 1000 \cdot \pi(1) = 500$.

   e) Since the period of this irreducible, positive recurrent Markov chain is 1, the stationary distribution is steady-state, so $\pi_{1000000} \approx \pi = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

   f) $P^n(2, 1) \approx \pi(1) = \frac{1}{2}$.

2. To get started, write the transition matrix for this Markov chain (let the state space be $\{A, B, C, D\}$):

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \end{pmatrix}.$$

   Writing $\pi = (a, b, c, d)$ and solving $\pi P = \pi$ to find the stationary distribution by methods similar to Problem 1, we obtain $\pi = \left(\frac{5}{26}, \frac{7}{26}, \frac{4}{13}, \frac{3}{13}\right)$.

   Next, by looking at the directed graph of the Markov chain, we can see that the period of this chain is $d = 2$, and that the states alternate as follows: $\{A, C\} \leftrightarrow \{B, D\}$.

   a) It is possible for Al to pass the salt to Cal in an even number of passes, so we have $P^{50000}(A, C) \approx d \cdot \pi(C) = 2 \cdot \frac{4}{13} = \frac{8}{13}$.

b) Bal can only pass the salt to Cal in an odd number of passes, $P^{50000}(B,C) = 0$.

3. Define a factor $\{Z_t\}$ of this Markov chain as follows: let $Z_t = 0$ if $X_t = 0$ and let $Z_t = 1$ if $X_t \neq 0$. This makes $\{Z_t\}$ a Markov chain with transition matrix

$$P_Z = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$

The stationary distribution of $\{Z_t\}$ (found by the same method as in Problem 1) is $\pi_Z = \left(\frac{1}{3}, \frac{2}{3}\right)$. That means the stationary distribution π of $\{X_t\}$ satisifes $\pi(0) = \frac{1}{3}$ (since 0 was ungrouped when constructing the factor $\{Z_t\}$.

By the stationarity equation, we know that for any state y, $\pi(y) = \sum_x \pi(x)P(x,y)$. Substituting in $y = 1$ and observing that $P(x,1) = 0$ unless $x = 0$, this means $\pi(1) = \pi(0)P(0,1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. Now plug in $y = 2$ to the stationarity equation and observe that $P(x,2) = 0$ unless $y = 0$ or 1. So we obtain

$$\begin{aligned} \pi(2) &= \sum_x \pi(x)P(x,2) \\ &= \pi(0)P(0,2) + \pi(1)P(1,2) \\ &= \frac{1}{3} \cdot \frac{1}{8} + \frac{1}{12} \cdot \frac{3}{4} \\ &= \frac{1}{24} + \frac{1}{16} \\ &= \frac{5}{48}. \end{aligned}$$

Finally, $m_2 = \frac{1}{\pi(2)} = \frac{48}{5}$.

4. a) We will apply Theorem 3.7 (which tells you when a birth-death chain has a stationary distribution). First, compute ζ_y:

$$\zeta_y = \frac{p_0 p_1 \cdots p_{y-1}}{q_1 q_2 \cdots q_y} = \frac{1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{y}}{\frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{y+1}} = \frac{\frac{1}{y!}}{\frac{1}{(y+1)!}} = \frac{(y+1)!}{y!} = y + 1.$$

Now $\sum_y \zeta_y = \sum_y (y+1) = 1 + 2 + 3 + 4 + ... = \infty$, so the birth-death chain has no stationary distributions.

b) From part (a), we know the chain cannot be positive recurrent (otherwise, it would have a stationary distribution). To classify it as recurrent or transient, go back to the method of Chapter 2, and compute γ_y:

$$\gamma_y = \frac{q_1 q_2 q_3 \cdots q_y}{p_1 p_2 p_3 \cdots p_y} = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{(y+1)!}}{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{(y+1)!}} = 1.$$

Now $\sum_y \gamma_y = \sum_y 1 = 1 + 1 + 1 + 1 + ... = \infty$, so by the theorem on recurrence/transience of birth-death chains from Chapter 2, $\{X_t\}$ is recurrent.

Since the chain is recurrent but not positive recurrent, it must be **null recurrent**.

4.3 Spring 2017 Exam 3

1. Consider a CTMC $\{X_t\}$ with state space $\{1, 2, 3\}$ and transition function

$$P(t) = \begin{pmatrix} \frac{25}{63} - \frac{1}{9}e^{-9t} + \frac{5}{7}e^{-7t} & \frac{8}{21} + \frac{1}{3}e^{-9t} - \frac{5}{7}e^{-7t} & \frac{2}{9} - \frac{2}{9}e^{-9t} \\ \frac{25}{63} - \frac{1}{9}e^{-9t} - \frac{2}{7}e^{-7t} & \frac{8}{21} + \frac{1}{3}e^{-9t} + \frac{2}{7}e^{-7t} & g(t) \\ \frac{25}{63} + \frac{7}{18}e^{-9t} - \frac{11}{14}e^{-7t} & \frac{8}{21} - \frac{7}{6}e^{-9t} + \frac{11}{14}e^{-7t} & \frac{2}{9} + \frac{7}{9}e^{-9t} \end{pmatrix}.$$

   a) (3.2) Find $g(t)$.
   b) (3.2) Find $P(X_4 = 2 \,|\, X_0 = 2)$.
   c) (3.2) Find $P(X_t = 2 \text{ for all } t < 4 \,|\, X_0 = 2)$.
   d) (3.2) Find $P(X_{10} = 3 \,|\, X_8 = 1, X_5 = 2, X_1 = 1)$.
   e) (3.2) Find the jump probability $\pi_{3,1}$.

2. Dysfunctional family members Al, Bal, Cal, Dal and Eal still have problems passing the salt at the dinner table. Assume that, by letting X_t be the person who has the salt at time t, we can model $\{X_t\}$ by a CTMC with state space $\{A, B, C, D, E\}$ and infinitesimal matrix

$$Q = \begin{pmatrix} -4 & 3 & 0 & 0 & 1 \\ 4 & -8 & 4 & 0 & 0 \\ 0 & 1 & -4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

   a) (3.4) Find the stationary distribution of this CTMC.
   b) (3.2) Suppose that Cal has the salt now. What is the probability that the salt gets passed before time 8?
   c) (3.2) Suppose that Cal has the salt now. What is the probability that when Cal passes the salt, he passes to Al?
   d) (3.2) Suppose that Cal has the salt now. What is the probability that when Cal passes the salt, he passes to Dal?

e) (3.4) Suppose that Cal has the salt now. How long would you expect it to take for the salt to be passed back to Cal?

f) (3.4) Suppose that Cal has the salt now. What is the expected total length of time Cal should have the salt in the next 610 units of time?

g) (3.4) Of the five members of the family, who (on the average) waits the longest to pass the salt? Explain.

3. Consider a pure birth process on $\mathcal{S} = \{0, 1, 2, 3, ...\}$ where for each x, $\lambda_x = 2^x$.

   a) (3.5) Is this process transient, null recurrent, or positive recurrent? Explain.

   b) (3.5) What is the holding rate of state 6?

   c) (3.5) Find $P_{33}(t)$.

   d) (3.5) Write the forward equation for the process.

4. Suppose that the value of a commodity is modeled by a Brownian motion with $\sigma^2 = 12$.

   a) (4.1) What is the variance of the value of the commodity at time 4?

   b) (4.1) What is the probability that at time 3, the value of the commodity is at least 4?

   c) (4.1) Suppose the commodity's value was 7 at time 2 and 10 at time 3. Given this, what is the probability that the value of the commodity is at most 12 at time 9?

   d) (4.2) What is the probability that the commodity's value reaches 6 at some time before $t = 10$?

Solutions

1. a) The second row of $P(t)$ must sum to 1, so
$$g(t) = 1 - P_{21}(t) - P_{22}(t) = \frac{2}{9} - \frac{2}{9}e^{-9t}.$$

   b) $P(X_4 = 2 \,|\, X_0 = 2) = P_{22}(4) = \frac{8}{21} + \frac{1}{3}e^{-18} + \frac{2}{7}e^{-14}$.

   c) First, $q_2 = -q_{22} = -P'_{22}(0) = -(-3-2) = 5$, so $W_2 \sim Exp(5)$. Therefore $P(X_t = 2 \text{ for all } t < 4 \,|\, X_0 = 2) = P(W_2 \geq 4) = P(Exp(5) \geq 4) = e^{-5 \cdot 4} = e^{-20}$.

   d) $P(X_{10} = 3 \,|\, X_8 = 1, X_5 = 2, X_1 = 1) = P(X_{10} = 3 \,|\, X_8 = 1) = P_{13}(10 - 8) = \frac{2}{9} - \frac{2}{9}e^{-18}$.

   e) $\pi_{3,1} = \frac{q_{31}}{q_3} = \frac{P'_{31}(0)}{-P'_{33}(0)} = \frac{2}{7}$.

2. a) Let $\pi = (a, b, c, d, e)$ be the stationary distribution; setting $\pi Q = \mathbf{0}$ we obtain
$$\begin{cases} -4a + 4b &= 0 \\ 3a - 8b + c &= 0 \\ 4b - 4c + d &= 0 \\ 2c - 3d + e &= 0 \\ a + c + 2d - e &= 0 \end{cases}$$
From the first equation, $a = b$. Substituting into the second equation, we get $c = 5b$. From the third equation, that means $d = 16b$. Then, from the fourth equation, we get $e = 38b$. Since $a + b + c + d + e = 1$, we have $b + b + 5b + 16b + 38b = 1$ so $b = \frac{1}{61}$. It follows that $\pi = \left(\frac{1}{61}, \frac{1}{61}, \frac{5}{61}, \frac{16}{61}, \frac{38}{61}\right)$.

   b) This is $P(W_C \leq 8) = P(Exp(q_C) \leq 8) = P(Exp(4) \leq 8) = 1 - e^{-4 \cdot 8} = 1 - e^{-32}$.

   c) $\pi_{CA} = \frac{q_{CA}}{q_C} = \frac{0}{4} = 0$.

   d) $\pi_{CD} = \frac{q_{CD}}{q_C} = \frac{2}{4} = \frac{1}{2}$.

   e) This question is asking for the mean return time to state C, i.e. m_C. We know $\pi(C) = \frac{1}{m_C q_C}$ so $\frac{5}{61} = \frac{1}{4m_C}$. Solving for m_C, we get $m_C = \frac{61}{20}$.

   f) From the ergodic theorem, this is $610\pi(C) = 50$ units of time.

   g) The time a family member waits to pass the salt is exponential with parameter q_x. Since the expected value of an exponential r.v. with parameter q_x is $\frac{1}{q_x}$, the family member who waits the longest to pass the salt is the family member with the smallest holding rate. This is Eal, whose holding rate is 1.

3. a) This is a pure birth process. So once you leave a state (via a birth), you cannot come back (since there are no deaths). Thus the process is transient.

b) $q_6 = (\lambda_6 + \mu_6) = 2^6 + 0 = 64$.

c) $P_{33}(t) = P(X_t = 3 \,|\, X_0 = 3) = P(W_3 \geq t) = P(Exp(2^3) \geq t) = e^{-8t}$.

d) The generic forward equation is $P'_{x,y}(t) = \sum_z P_{x,z}(t) q_{zy}$. The only z for which $q_{zy} \neq 0$ are $z = y - 1$ (in which case $q_{z,y} = q_{y-1,y} = \lambda_{y-1} = 2^{y-1}$) and $z = y$ (in which case $q_{yy} = -q_y = -\lambda_y = -2^y$), so the forward equation reduces to

$$P'_{x,y}(t) = 2^{y-1} P_{x,y-1}(t) - 2^y P_{x,y}(t).$$

4. Throughout this solution, let W_t denote the value of the commodity at time t.

a) $Var(W_4) = Var(n(0, 4 \cdot 12)) = Var(n(0, 48)) = 48$.

b) $P(W_3 \geq 4) = P(n(0, 3 \cdot 12) \geq 4) = 1 - \Phi\left(\frac{4}{\sqrt{36}}\right) = 1 - \Phi\left(\frac{2}{3}\right)$.

c) By the Markov property, this is

$$P(W_9 \leq 12 \,|\, W_3 = 10) = P(W_9 - W_3 \leq 2) = P(n(0, 6 \cdot 12) \leq 2) = \Phi\left(\frac{2}{\sqrt{72}}\right).$$

d) By the Reflection Principle, this is

$$P(T_6 \leq 10) = 2 - 2\Phi\left(\frac{6}{\sqrt{12}\sqrt{10}}\right) = 2 - 2\Phi\left(\frac{6}{\sqrt{120}}\right).$$

4.4 Spring 2021 Exam 1

1. Let $\{X_t\}$ be the Markov chain with state space $\mathcal{S} = \{1,2,3\}$, initial distribution $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, and transition matrix

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} \end{pmatrix}.$$

   a) (1.3) Compute the time 2 transition matrix.
   b) (1.3) Compute $P(X_1 = 1)$.
   c) (1.3) Compute $P(X_3 = 1 \,|\, X_1 = 2, X_0 = 1)$.
   d) (1.3) Compute

$$P(X_{13} = 2, X_{12} = 2, X_{11} = 3, X_7 = 1, X_4 = 2 \,|\, X_3 = 1, X_1 = 1, X_0 = 2).$$

   e) (1.5) Compute the stationary distribution of $\{X_t\}$.

2. Let $\{X_t\}$ be a Markov chain with state space $\mathcal{S} = \{1,2,3,4,5,6,7\}$ and transition matrix

$$P = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & \frac{1}{7} & \frac{3}{7} & 0 & 0 & 0 & \frac{2}{7} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ \frac{2}{5} & 0 & \frac{1}{5} & \frac{1}{5} & 0 & \frac{1}{5} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \end{pmatrix}.$$

   a) (1.7) Classify the states of this Markov chain as recurrent or transient.
   b) (1.7) Compute $P_5(T_6 = 10)$.
   c) (1.7) Compute $f_{2,1}$, $f_{1,5}$, and $f_{3,1}$.

3. (1.7) Let $\{X_t\}$ be a Markov chain with state space $\{0,1,2,3,...\}$ and transition function

$$P(x,y) = \begin{cases} \left(\frac{1}{2}\right)^y & \text{if } x = 0, y \geq 1 \\ \frac{1}{5} & \text{if } x \geq 1, y = 0 \\ \frac{4}{5} & \text{if } x \geq 1, y = x+1 \\ 0 & \text{else} \end{cases}.$$

   Determine, with justification, whether this chain is recurrent or transient.

4. (Group presentations) Choose four of the five parts (a), (b), (c), (d) and (e).

   a) Let $\{X_t\}$ be an Ehrenfest chain with $d = 5$.

      i. What is the period of this chain?

      ii. Compute $P(X_4 = 2 \,|\, X_1 = 5)$.

   b) Let $\{(B_t, R_t)\}$ be the Pólya urn, where initially the urn contains 1 blue and 1 red marble (i.e. $(B_0, R_0) = (1, 1)$).

      i. What is the probability that $(B_t, R_t) = (5, 5)$ for some $t \geq 0$?

      ii. What is the expected number of blue marbles in the urn at time 3?

   c) Consider a Galton-Watson chain, where each male has two offspring with probability q and zero offspring with probability $1 - q$. If the extinction probability $f_{1,0}$ is $\frac{4}{5}$, what is the value of q?

   d) Consider a discrete queuing chain where the number of arrivals to the queue during each time period is uniform on $\{0, 1, 2, 3\}$.

      i. Compute the transition probability $P(2, 0)$.

      ii. Is this chain recurrent or transient? Justify your answer.

   e) Let $\{Z_t\}$ be a Wright-Fisher chain with $d = 10$.

      i. What is the transition probability $P(6, 7)$? (This answer doesn't need to be simplified.)

      ii. Compute $P_7(T_0 < \infty)$.

      iii. Compute $E_3(Z_8)$.

Solutions

1. a) $P^2 = \boxed{\begin{pmatrix} \frac{17}{36} & \frac{5}{12} & \frac{1}{9} \\ \frac{5}{18} & \frac{11}{18} & \frac{1}{9} \\ \frac{5}{18} & \frac{5}{12} & \frac{11}{36} \end{pmatrix}}$.

   b) $\pi_1(1) = \sum_{x \in \mathcal{S}} \pi_0(x) P(x,1) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{2}{3}\right) + \frac{1}{4}(0) = \frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$.

   c) By the Markov property,
   $$P(X_3 = 1 \,|\, X_1 = 2, X_0 = 1) = P(X_3 = 1 \,|\, X_1 = 2) = P^2(2,1).$$
   From part (a), this is $\boxed{\frac{5}{18}}$.

   d) Note $P(X_{12} = 2 \,|\, X_{11} = 3) = 0$, so this entire probability is $\boxed{0}$.

   e) Write $\pi = (a,b,c)$ and solve $\pi P = \pi$ (together with $a+b+c=1$) to get $\pi = \boxed{\left(\frac{10}{29}, \frac{15}{29}, \frac{4}{29}\right)}$.

2. a) $\{1,2\}$ and $\{4\}$ are communicating classes, and since they are finite, their members are recurrent. Every other state leads to 4, but 4 does not lead back, so $\boxed{\mathcal{S}_R = \{1,2,4\}}$ and $\boxed{\mathcal{S}_T = \{3,5,6,7\}}$.

   b) Given $X_0 = 5$, in order for T_6 to equal 10, the trajectory must be
   $$5 \xrightarrow{2/7} 5 \xrightarrow{2/7} 5 \xrightarrow{2/7} 5 \xrightarrow{2/7} 5 \xrightarrow{2/7} \cdots \xrightarrow{2/7} 5 \xrightarrow{3/7} 6.$$
   This has probability $\boxed{\frac{3}{7}\left(\frac{2}{7}\right)^9}$.

   c) $\boxed{f_{2,1} = 1}$ because 1 and 2 belong to the same recurrent communicating class. $\boxed{f_{1,5} = 0}$ because 1 is recurrent and 5 is transient. For $f_{3,1}$, we use a system of equations:
   $$\begin{cases} f_{3,1} &= \frac{1}{7}(1) + \frac{1}{7}(1) + \frac{3}{7} f_{3,1} + \frac{2}{7} f_{7,1} \\ f_{5,1} &= \frac{2}{7}(0) + \frac{2}{7} f_{5,1} + \frac{3}{7} f_{6,1} \\ f_{6,1} &= \frac{2}{5}(1) + \frac{1}{5} f_{3,1} + \frac{1}{5}(0) + \frac{1}{5} f_{6,1} \\ f_{7,1} &= \frac{1}{2}(1) + \frac{1}{4} f_{3,1} + \frac{1}{4} f_{6,1} \end{cases}$$
   (It turns out that the second equation isn't needed.) Simplifying the remaining equations, we get
   $$\begin{cases} 4f_{3,1} &= 2 + 2f_{7,1} & \Rightarrow f_{7,1} = 2f_{3,1} - 1 \\ 4f_{6,1} &= 2 + f_{3,1} & \\ 4f_{7,1} &= 2 + f_{3,1} + f_{6,1} & \Rightarrow f_{6,1} = 4f_{7,1} - f_{3,1} - 2 \end{cases}$$

Plugging the third equation into the second gives

$$16f_{7,1} - 4f_{3,1} - 8 = 2 + f_{3,1} \;\Rightarrow\; 16f_{7,1} = 10 + 5f_{3,1}$$

and plugging the first equation into this gives

$$16(2f_{3,1} - 1) = 10 + 5f_{3,1} \;\Rightarrow\; \boxed{f_{3,1} = \frac{26}{27}}.$$

3. First, we classify state 0. Note that to return to 0 for the first time on the n^{th} step, n must be at least 2 and your path must be

$$0 \longrightarrow y \xrightarrow{4/5} y+1 \xrightarrow{4/5} y+2 \xrightarrow{4/5} \cdots \xrightarrow{4/5} y+n-2 \xrightarrow{1/5} 0$$

and since you have to go to *some* $y \neq 0$ on the first step, the first arrow can be labelled with a 1 (by adding up the probabilities you go to each particular y). That means

$$\begin{aligned} f_0 = P_0(T_0 < \infty) &= \sum_{n=2}^{\infty} P_0(T_0 = n) \\ &= \sum_{n=2}^{\infty} \left(\frac{4}{5}\right)^{n-2} \frac{1}{5} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^{n} = \frac{1}{5}\left(\frac{1}{1-\frac{4}{5}}\right) = 1. \end{aligned}$$

Therefore 0 is recurrent, and since 0 leads to every other state, the entire chain is **recurrent**.

4. a) i. The period of the Ehrenfest chain is $\boxed{2}$.

   ii. The only way to get from state 5 to state 2 in $4-1=3$ steps is the sequence

   $$5 \xrightarrow{1} 4 \xrightarrow{4/5} 3 \xrightarrow{3/5} 2.$$

   This has probability $1\left(\frac{4}{5}\right)\frac{3}{5} = \boxed{\frac{12}{25}}$.

   b) i. The only time t for which $(B_t, R_t) = (5,5)$ is $t = 8$, because 8 marbles have to have been added to the jar. We then know from the formulas for n-step transitions in the P/'olya urn, that $P^8((1,1),(b,r)) = \frac{1}{b+r-1} = \boxed{\frac{1}{9}}$.

   ii. At time 3, the distribution is uniform on the four possible states $\{(4,1),(3,2),(2,3),(1,4)\}$. So the expected number of blue marbles is $E[B_3] = \frac{1}{4}(4+3+2+1) = \boxed{\frac{5}{2}}$.

c) Letting W be the number of offspring, we have

$$G_W(t) = E[t^W] = \sum_{w=0}^{\infty} t^w f_W(w) = (1-q) + qt^2.$$

The extinction probability $\eta = f_{1,0}$ satisfies

$$\begin{aligned} \eta &= G_W(\eta) \\ \Rightarrow \frac{4}{5} &= G_W\left(\frac{4}{5}\right) \\ \Rightarrow \frac{4}{5} &= (1-q) + q\left(\frac{4}{5}\right)^2 \\ \Rightarrow \frac{1}{5} &= \frac{9}{25}q \\ \Rightarrow \boxed{\frac{5}{9}} &= q. \end{aligned}$$

d) i. $P(2,0) = \boxed{0}$ since the line can only shrink by one member per unit of time.

ii. The expected number of arrivals is $\frac{1}{4}(0+1+2+3) = \frac{3}{2}$. Since this number is greater than 1, the queue is **transient**.

e) i. In a general Wright-Fisher chain, $P(x,y) = \frac{\binom{2x}{y}\binom{2d-2x}{d-y}}{\binom{2d}{d}}$. Here, we have $P(6,7) = \frac{\binom{2(6)}{7}\binom{2(10-6)}{10-7}}{\binom{2(10)}{10}} = \boxed{\frac{\binom{12}{7}\binom{8}{3}}{\binom{20}{10}}}$.

ii. In the Wright-Fisher chain, $f_{x,0} = \frac{d-x}{d}$, which is in this problem $\frac{10-7}{10} = \boxed{\frac{3}{10}}$.

iii. The expected state at time 3 in the Wright-Fisher chain is the starting state, so $E_3(X_8) = \boxed{3}$.

4.5 Spring 2021 Exam 2

1. Let $\{X_t\}$ be a Markov chain with state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 \end{pmatrix}.$$

   a) (1.8) Compute the mean return time to state 1.

   b) (1.10) Estimate $P^{2021}(1, 2)$ and $P^{2021}(1, 4)$.

   c) (1.8) What is $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} P^k(2, 3)$?

   d) (1.10) For how many times $t \in [1, 3600]$ would you expect that $X_t = 3$?

2. Two hockey teams, the Whales and Dolphins, play a series of games against each other. In each game, the Whales win with probability $.2$, the Dolphins win with probability $.2$, and the teams tie with probability $.6$. The teams will play repeatedly until one team has won three more games than the other, at which point that team will be declared the winner of the series.

   a) (2.5) Under this format, what is the probability that the Whales win this series?

   b) (2.5) How many games should the teams expect to play until the series is decided?

   c) (2.5) If the Dolphins win the first two games, what is the probability the Whales come back to win the series?

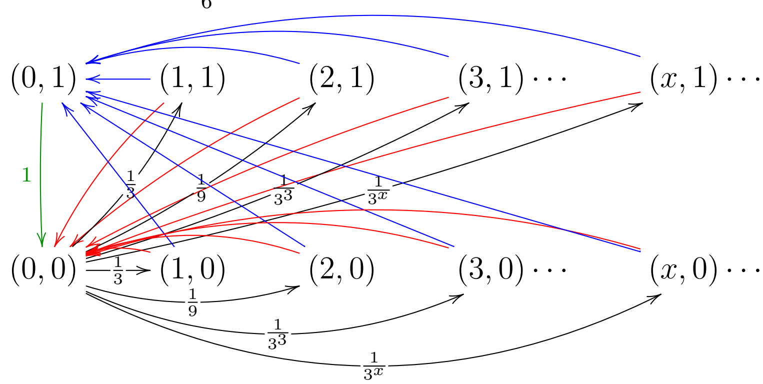
   d) (2.5) To reduce the expected number of games played, the teams agree to have a shootout in each game to break ties. However, the Dolphins happen to be really good at shootouts: assume they will win 80% of the games that are settled by a shootout (the Whales win the other 20%). In this new format, what is the probability the Dolphins win the series?

3. Let $\{X_t\}$ be a Markov chain with state space $\{0, 1, 2, ...\} \times \{0, 1\}$ (meaning the states are ordered pairs, where the first coordinate is a natural number and the second coordinate is either 0 or 1. The Markov chain has transition

probabilities defined by these formulas:

$$P((0,0),(x,y)) = \frac{1}{3^x} \text{ if } x > 0 \text{ and } y \in \{0,1\};$$
$$P((x,y),(0,0)) = \frac{1}{6} \text{ if } x > 0 \text{ and } y \in \{0,1\};$$
$$P((x,y),(0,1)) = \frac{5}{6} \text{ if } x > 0 \text{ and } y \in \{0,1\};$$
$$P((0,1),(0,0)) = 1.$$

In other words, $\{X_t\}$ has directed graph as shown below, where all the blue arrows are labelled with $\frac{5}{6}$ and all the red arrows are labelled with $\frac{1}{6}$:



a) (1.6) Explain why $\{X_t\}$ is irreducible.

b) (1.6) What is the period of $\{X_t\}$?

c) (1.10) Prove that $\{X_t\}$ is positive recurrent, and find the value of the stationary distribution π at the state $(0,0)$.

d) (1.10) Compute the mean return time to the state $(2021, 0)$.

4. Suppose you flip a fair coin over and over, independently, and record the result by setting
$$X_t = \begin{cases} 1 & \text{if the } t^{th} \text{ flip is heads} \\ 0 & \text{if the } t^{th} \text{ flip is tails} \end{cases}.$$

Let $\{\mathcal{F}_t\}$ be the natural filtration associated to the process $\{X_t\}$. Now, for each t, let $Y_t = \max\{X_1, X_2, X_3, ..., X_t\}$.

a) (2.3) Compute $E[Y_3|\mathcal{F}_1]$.

b) (2.4) Is $\{Y_t\}$ a martingale? Explain.

Solutions

1. a) Compute the stationary distribution of $\{X_t\}$ by setting $\pi P = \pi$ and $\pi(1)+\pi(2)+\pi(3)+\pi(4) = 1$ to get $\pi = \left(\frac{7}{36}, \frac{5}{18}, \frac{2}{9}, \frac{11}{36}\right)$. Then, $m_1 = \frac{1}{\pi(1)} = \boxed{\frac{36}{7}}$.

   b) The period of this Markov chain is 2, and the states move back and forth as follows: $\{1,4\} \leftrightarrow \{2,3\}$. Therefore, since you can go from 1 to 2 in an odd number of steps, $P^{2021}(1,2) \approx d\pi(2) = 2\left(\frac{5}{18}\right) = \boxed{\frac{5}{9}}$. However, $P^{2021}(1,4) = \boxed{0}$ since you can only go from 1 to 4 in an even number of steps.

   c) This Cesàro limit is $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} P^k(2,3) = \pi(3) = \boxed{\frac{2}{9}}$.

   d) By the ergodic theorem for Markov chains, this is $3600\pi(3) = 3600\left(\frac{2}{9}\right) = \boxed{800}$.

2. Throughout this problem, $\{X_t\}$ will record the number of games ahead or behind the Whales are. In parts (a), (b) and (c), $\{X_t\}$ is an irreducible simple random walk with $p = .2$, $q = .2$ and $x = 0$.

   a) This is $P_0(T_3 < T_{-3})$; since the walk is unbiased, this is $\frac{0-(-3)}{3-(-3)} = \frac{3}{6} = \boxed{\frac{1}{2}}$.

   b) We can apply the formula in the notes derived from Wald's Second Identity to get
   $$ET = \frac{Var(X_T)}{Var(S_j)} = \frac{(x-a)(b-x)}{p+q} = \frac{(0-(-3))(3-0)}{.2+.2} = \frac{9}{.4} = \frac{90}{4} = \boxed{22.5}.$$

   c) This is $P_{-2}(T_3 < T_{-3}) = \frac{-2-(-3)}{3-(-3)} = \boxed{\frac{1}{6}}$.

   d) In the new format, the probability the Whales win is $p = .2 + .2(.6) = .32$ and the probability the Dolphins win is $q = .2 + .8(.6) = .68$. So $\frac{q}{p} = \frac{.68}{.32} = \frac{17}{8}$. Now $\{X_t\}$ is a biased simple random walk, so
   $$P_0(T_{-3} < T_3) = \frac{\left(\frac{q}{p}\right)^3 - \left(\frac{q}{p}\right)^0}{\left(\frac{q}{p}\right)^3 - \left(\frac{q}{p}\right)^{-3}} = \boxed{\frac{\left(\frac{17}{8}\right)^3 - 1}{\left(\frac{17}{8}\right)^3 - \left(\frac{17}{8}\right)^{-3}}}.$$

3. a) Given states (x,y) and (a,b) where $x > 0$ and $y > 0$, you can go $(x,y) \to (0,1) \to (0,0) \to (a,b)$. So every state leads to every other state, so $\{X_t\}$ is irreducible.

b) Note $(0,0)$ can return to itself in two steps ($(0,0) \to (1,1) \to (0,0)$) or in three steps ($(0,0) \to (1,1) \to (0,1) \to (0,0)$), so the period of $(0,0)$ must divide both 2 and 3. Thus its period must be 1. So by irreducibility, $\{X_t\}$ has period $\boxed{1}$ (i.e. $\{X_t\}$ is aperiodic).

c) *Solution # 1:* Notice $(0,0)$ must return to itself in either 2 or 3 steps (because you go from $(0,0)$ to (x,y), then either to $(0,1)$ and on to $(0,0)$ or directly back to $(0,0)$). In particular,

$$P_{(0,0)}(T_{(0,0)} = 2) = \frac{1}{6} \quad \text{and} \quad P_{(0,0)}(T_{(0,0)} = 3) = \frac{5}{6}.$$

Therefore

$$m_{(0,0)} = E_{(0,0)}(T_{(0,0)}) = \frac{1}{6}(2) + \frac{5}{6}(3) = \frac{17}{6}.$$

Since this is finite, $(0,0)$ is positive recurrent, so by irreducibility we can conclude that $\boxed{\{X_t\} \text{ is positive recurrent}}$ and $\pi(0,0) = \left[m_{(0,0)}\right]^{-1} = \boxed{\frac{6}{17}}$.

Solution # 2: Define a factor $\{Z_t\}$ of $\{X_t\}$ by setting

$$Z_t = \begin{cases} 0 & \text{if } X_t = (0,0) \\ 1 & \text{if } X_t = (0,1) \\ 2 & \text{else} \end{cases}.$$

$\{Z_t\}$ is a Markov chain with transition matrix

$$P_Z = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ \frac{1}{6} & \frac{5}{6} & 0 \end{pmatrix};$$

By solving $\pi_Z P_Z = \pi_Z$ we find that the stationary distribution of $\{Z_t\}$ is

$$\pi_Z = \left(\frac{6}{17}, \frac{5}{17}, \frac{6}{17}\right).$$

Thus 0 has mean return time $\frac{17}{6}$ in $\{Z_t\}$. But this must equal the mean return time of $(0,0)$ in $\{X_t\}$ (since you return to $(0,0)$ in $\{X_t\}$ exactly when you return to 0 in $\{Z_t\}$), so $m_{(0,0)} = \frac{17}{6}$ in $\{X_t\}$. As this mean return time is finite, $(0,0)$ is positive recurrent, so by irreducibility $\boxed{\{X_t\} \text{ is positive recurrent}}$ and $\pi(0,0) = \left[m_{(0,0)}\right]^{-1} = \boxed{\frac{6}{17}}$.

d) Let π be the stationary distribution of $\{X_t\}$. From part (c), we have $\pi(0,0) = \left[m_{(0,0)}\right]^{-1} = \frac{6}{17}$. Now, observe that the only arrow entering

$(2021, 0)$ is the one coming from $(0, 0)$, labelled with $\frac{1}{3^{2021}}$. So by the stationarity equation, we see

$$\pi(2021, 0) = \sum_{(x,y)\in\mathcal{S}} \pi(x, y)P((x, y), (2021, 0)) = \pi(0, 0)\frac{1}{3^{2021}} = \frac{6}{17 \cdot 3^{2021}}.$$

Thus $m_{(2021,0)} = [\pi(2021, 0)]^{-1} = \boxed{\frac{17 \cdot 3^{2021}}{6}}$.

4. a) Letting ω denote a sample function for this process, we see that $\mathcal{F}_1$ is generated by the partition of Ω into sequences of flips starting with H and sequences of flips starting with T. If ω starts with H (i.e. the first flip is heads), then $X_1 = 1$ so $Y_3 = 1$ (i.e. Y_3 is constant) and therefore $E[Y_3|\mathcal{F}_1](\omega) = 1$.

   If ω starts with T (i.e. the first flip is tails), then if the next two flips are both tails (probability $\frac{1}{4}$) $Y_3 = 0$ but otherwise $Y_3 = 1$. So $E[Y_3|\mathcal{F}_1](\omega) = \frac{1}{4}(0) + \frac{3}{4}(1) = \frac{3}{4}$.

   To summarize, $\boxed{E[Y_3|\mathcal{F}_1](\omega) = \begin{cases} 1 & \text{if } \omega = \{H, ...\} \\ \frac{3}{4} & \text{if } \omega = \{T, ...\} \end{cases}}$

   b) Note Y_1 cannot take the value $\frac{3}{4}$, so it is impossible that $E[Y_3|\mathcal{F}_1] = Y_1$. Thus by definition, $\{Y_t\}$ is **not** a martingale.

4.6 Spring 2021 Exam 3

1. Let $\{X_t\}$ be a CTMC with state space $\{1, 2, 3, 4\}$ whose jump probabilities are given in the following matrix:

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}$$

   Suppose also that the holding rates of states $1, 2, 3$ and 4 are 1, $\frac{1}{2}$, 1 and 2, respectively.

   a) (3.2) Compute the infinitesimal matrix of $\{X_t\}$.

   b) (3.2) Suppose $X_3 = 2$. What is the probability that $X_t = 2$ for all $t \in [3, 8)$?

   c) (3.2) Use linear approximation to estimate $P_{23}(.01)$.

   d) (3.4) Compute the stationary distribution π of $\{X_t\}$.

   e) (3.4) Compute $\lim\limits_{t\to\infty} P_{13}(t)$.

   f) (3.4) Compute the mean return time to state 4.

   g) (3.4) Suppose $X_0 = 2$. Compute the expected amount of time spent in state 4 before the chain first returns to state 2.

2. Suppose that the temperature, measured in degrees Fahrenheit, in Anytown U.S.A. at time t, measured in days, is modeled by a Brownian motion with parameter $\sigma^2 = 3$. Suppose that the temperature is currently 45° F.

   a) (4.1) Describe the temperature in Anytown one week from now as a common r.v. In particular, what class of r.v. is it, and what are the values of any of its parameters?

   b) (4.1) What is the probability that the temperature in Anytown two days from now is between 40° F and 50° F?

   c) (4.2) What is the probability that the temperature in Anytown reaches 70° F within the next twelve days?

   d) (4.3) What is the probability that the temperature in Anytown reaches 60° F before it reaches 40° F?

   e) (4.2) What is the probability that the temperature in Anytown two weeks from today is greater than the temperature in Anytown one week from today?

f) (4.2) Is Anytown a safe place for a human being to live (for the long term)? Explain.

3. (3.5) Let $\{Y_t\}$ be a birth-death CTMC with state space $\{0, 1, 2, ...\}$, with birth rates $\lambda_x = \sqrt{x+1}$ and death rates $\mu_x = \sqrt{x}$. Determine, with justification, whether or not this CTMC is positive recurrent, null recurrent, or transient.

4. An ant is crawling on a piece of concrete. Assume that the position of the ant (in inches) is modeled by a standard 2-dimensional Brownian motion, and that the ant's initial position is at the origin.

   a) (4.7) What is the probability that the ant returns to the origin at some time in the future?

   b) (4.7) Suppose the ant has been crawling around for a while, so that its position is now $(-3, 4)$. What is the probability that, starting from $(-3, 4)$, the ant crawls to within 2 inches of the origin before it crawls 8 inches away from the origin?

   c) **(Bonus)** Without using a calculator or computer to get a decimal approximation, determine (with justification) if your answer to part (b) is equal to $\frac{1}{2}$, greater than $\frac{1}{2}$, or less than $\frac{1}{2}$.

Solutions

1. a) Using the formulas $q_{xy} = q_x \pi_{xy}$ and $q_{xx} = -\sum_{y \neq x} q_{xy}$, we get

$$Q = (q_{xy}) = \boxed{\begin{pmatrix} -1 & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -1 & \frac{3}{4} \\ \frac{3}{2} & 0 & \frac{1}{2} & -2 \end{pmatrix}}.$$

   b) $P(X_t = 2 \text{ for all } t \in [3, 8) \,|\, X_3 = 2) = P_2(W_2 > 5) = P(Exp(-\frac{1}{2}) > 5) = \boxed{e^{-5/2}}$.

   c) $P_{23}(.01) \approx \delta_{23} + P'_{23}(.01) = 0 + q_{23}(.01) = \frac{1}{4}(.01) = \boxed{\frac{1}{400}} = \boxed{.0025}$.

   d) Let $\pi = (a, b, c, d)$ and solve $\pi Q = \mathbf{0}$ together with $a + b + c + d = 1$ to get:

$$\begin{cases} -a + \frac{1}{4}b + \frac{3}{2}d = 0 \\ a - \frac{1}{2}b + \frac{1}{4}c = 0 \\ \frac{1}{4}b - c + \frac{1}{2}d = 0 \\ \frac{3}{4}c - 2d = 0 \end{cases} \Rightarrow \begin{cases} -4a + b + 6d = 0 \\ 4a - 2b + c = 0 \\ b - 4c + 2d = 0 \\ 3c - 8d = 0 \end{cases}.$$

From the last equation, $d = \frac{3}{8}c$. Substitute into the third equation to get $b - 4c + \frac{3}{4}c = 0$, i.e. $b = \frac{13}{4}c$. Substitute into the second equation to get $4a = 2b - c = \frac{11}{2}c$, so $a = \frac{11}{8}c$.
Now, since $a + b + c + d = 1$, we have $\frac{11}{8}c + \frac{13}{4}c + c + \frac{3}{8}c = 1$, i.e. $c = \frac{1}{6}$.
This makes $d = \frac{1}{16}$, $a = \frac{11}{48}$ and $b = \frac{13}{24}$. So $\pi = \boxed{\left(\frac{11}{48}, \frac{13}{24}, \frac{1}{6}, \frac{1}{16}\right)}$.

e) Since $\{X_t\}$ has finite state space, it is positive recurrent. As $\{X_t\}$ is irreducible, we have $\lim_{t\to\infty} P_{13}(t) = \pi(3) = \boxed{\frac{1}{6}}$.

f) We know $\pi(4) = \frac{1}{m_4 q_4}$, so $m_4 = \frac{1}{q_4 \pi(4)} = \frac{1}{2 \cdot \frac{1}{16}} = \boxed{8}$.

g) This is $\tau_2(4) = m_2 \pi(4) = \frac{1}{q_2 \pi(2)} \pi(4) = \frac{1}{\frac{1}{2} \cdot \frac{13}{24}} \frac{1}{16} = \boxed{\frac{3}{13}}$.

2. We denote the temperature in Anytown t days from now by W_t. $\{W_t\}$ is a BM starting at 45 with $\sigma^2 = 3$.

a) $W_t - W_0 \sim n(0, \sigma^2 t)$ so $W_7 - 45 \sim n(0, 3 \cdot 7) = n(0, 21)$. Therefore $W_7 \sim 45 + n(0, 3 \cdot 7) = \boxed{n(45, 21)}$.

b) $P(40 < W_2 < 50) = P(40 < n(45, 3 \cdot 2) < 50) = P(40, n(45, 6) < 50) = P\left(\frac{40-45}{\sqrt{6}} < n(0,1) < \frac{50-45}{\sqrt{6}}\right) = \Phi\left(\frac{5}{\sqrt{6}}\right) - \Phi\left(\frac{-5}{\sqrt{6}}\right) = \boxed{2\Phi\left(\frac{5}{\sqrt{6}}\right) - 1}$.

c) By the Reflection Principle, this is

$$P_{45}(T_{70} < 12) = P_0(T_{25} < 12) = 2 - 2\Phi\left(\frac{25}{\sqrt{3}\sqrt{12}}\right) = \boxed{2 - 2\Phi\left(\frac{25}{6}\right)}.$$

d) By the escape time probability formulas, this is

$$P_{45}(T_{60} < T_{40}) = \frac{45 - 40}{60 - 40} = \frac{5}{20} = \boxed{\frac{1}{4}}.$$

e) This is $P(W_{14} > W_7) = P(W_{14} - W_7 > 0) = P(n(0, 3 \cdot 7) > 0 = P(n(0, 21) > 0) = 1 - \Phi(0) = \boxed{\frac{1}{2}}$.

f) $\{W_t\}$ is irreducible, meaning that eventually the temperature will reach every real number. So at some point in the future the temperature will be $1000000°$ F, at which point humans cannot survive. So Anytown is $\boxed{\textbf{NOT}}$ a safe place for a human to live for the long haul.

3. First, we determine whether $\{Y_t\}$ is recurrent or transient.

$$\sum_{y=0}^{\infty} \frac{\mu_1 \cdots \mu_y}{\lambda_1 \cdots \lambda_y} = \sum_{y=0}^{\infty} \frac{\sqrt{1}\sqrt{2}\cdots\sqrt{y}}{\sqrt{2}\sqrt{3}\cdots\sqrt{y+1}} = \sum_{y=0}^{\infty} \frac{1}{\sqrt{y+1}}.$$

This series diverges (it is a p-series with $p = \frac{1}{2} \leq 1$), so $\{X_t\}$ is recurrent.

Next, we determine the type of recurrence. We have, for all y,

$$\phi_y = \frac{\lambda_0 \lambda_1 \cdots \lambda_{y-1}}{\mu_1 \cdots \mu_y} = \frac{\sqrt{1}\sqrt{2}\cdots\sqrt{y}}{\sqrt{1}\sqrt{2}\cdots\sqrt{y}} = 1.$$

Since $\sum_{y=0}^{\infty} \phi_y = \sum_{y=0}^{\infty} 1 = \infty$, $\{Y_t\}$ is not positive recurrent. Therefore $\{Y_t\}$ is $\boxed{\text{null recurrent}}$.

4. a) This probability is $\boxed{0}$ (proven when we showed that 2-dimensional BM is point transient).

   b) Let $\mathbf{x} = (-3, 4)$; note $||\mathbf{x}|| = \sqrt{(-3)^2 + 4^2} = 5$. Denoting by A_r the circle of radius r centered at the origin, we are asked to compute

$$\begin{aligned} P_{\mathbf{x}}(T_{A_2} < T_{A_8}) &= 1 - P_{\mathbf{x}}(T_{A_8} < T_{A_2}) \\ &= 1 - \frac{\ln ||\mathbf{x}|| - \ln r}{\ln R - \ln r} \\ &= 1 - \frac{\ln 5 - \ln 2}{\ln 8 - \ln 2} \\ &= \boxed{\frac{\ln 8 - \ln 5}{\ln 8 - \ln 2}}. \end{aligned}$$

   c) By a log rule (specifically, the difference of logs is the log of the quotient), we can rewrite the answer to part (b) as $\frac{\ln \frac{8}{5}}{\ln 4}$ and by the change of base rule for logarithms, this is $\log_4 \frac{8}{5}$. Now since $4^{1/2} = 2 > \frac{8}{5}$, $\log_4 \frac{8}{5}$ is $\boxed{\text{less than } \frac{1}{2}}$.

   Remark: This might surprise you, because the ant is on a circle of radius 5 and the question asks for the probability it hits the circle of radius 2 before the circle of radius 8. Since 5 is halfway in between 2 and 8, you might think the answer is $\frac{1}{2}$. But that's only true in dimension 1.