Discrete Dynamical Systems and Chaos

<u>Definition:</u>

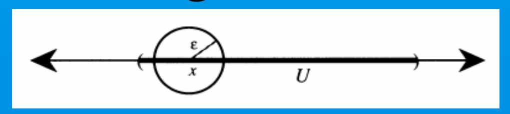
- A **discrete dynamical system** is a function $f\colon X\to X$ that is composed with itself over and over again. The function denoted by f^n is the composition: $f^n = f\circ f\circ \ldots \circ f$

- Example 1:
- *Let* $f\colon\mathbb{R}\to\mathbb{R}$ *be defined as* $f(x)=x$
  - $f(1)=1$
  - $f^2(1)=f(f(1))=f(1)=1$
  - In general $f^n(x_0)=x_0,\ \forall x_o\in\mathbb{R}$

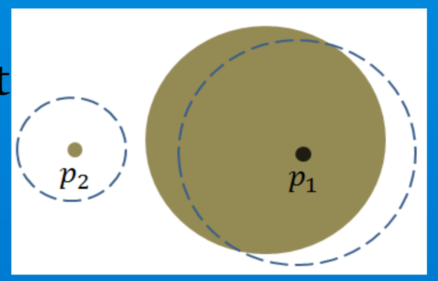
- Example 2: Address Scheme
- *Let* $f\colon\mathbb{R}\to\mathbb{R}$ *be defined as* $f(x)=-x^3$
  - $f(1)=-1$
  - $f^2(1)=f(f(1))=f(-1)=1$
  - $f^3(1/2)=f(f(f(1/2)))=f(f(-1/8))=f(1/512)\approx 0$
  - $f^2(0)=f(f(0))=f(0)=0$
  - $f^3(2)=f(f(f(2)))=f(f(-8))=f(512)=134217728$
  - In general $f^n(x_0)=(-1)^n(x_0)^{3n},\forall x_0\in\mathbb{R}$

- The logistic function
  - $h_r\colon\mathbb{R}\to\mathbb{R}$ *defined as* $h_r(x)=rx(1-x)$ *where* $r>0$
    - r is known as the **growth rate**
    - x is the percent of **carrying capacity** the population is currently at.
      - Note: $0\le x\le 1$ where 0 is extinction and 1 is capacity.
  - Used by biologists to model population growth.

- Neighborhoods
  - A **neighborhood** of x can be thought of as all the points which are "near" x.

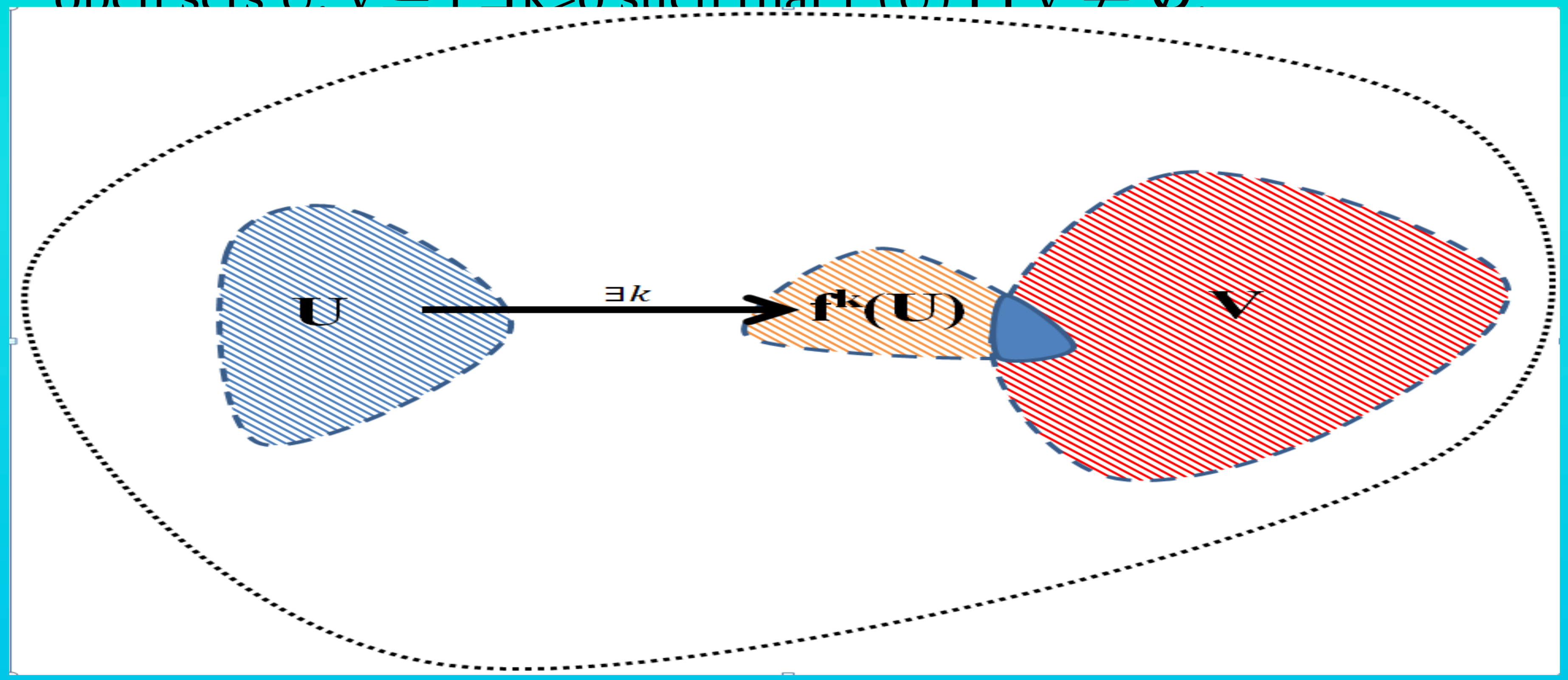


- Accumulation Points
  - A point is an **accumulation point** if every neighborhood of that point contains a different point in the set.
    - p_1 is an accumulation point
    - p_2 is not
  - Note that the point need not be an element of the set.



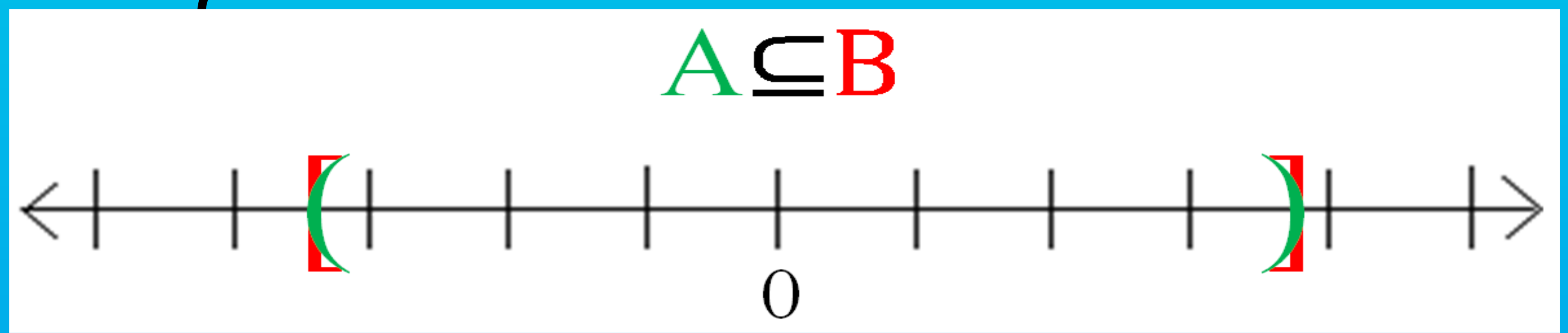
<u>Definition:</u>

f: X→X is said to be **topologically transitive** if for any pair of open sets U, V⊆ I ∃k>0 such that $f^k(U)\cap V\neq\emptyset$.



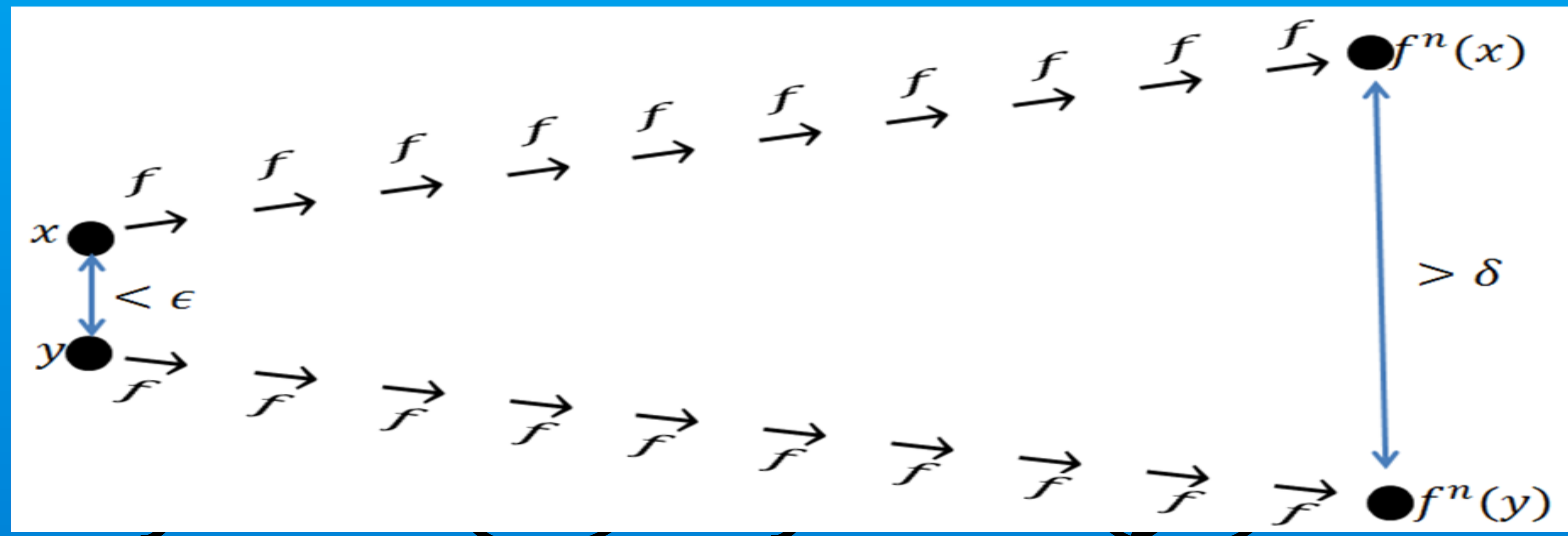
<u>Definition:</u>

Let A ⊆ B. Then **A is dense in B** if for all x∈*B*,



<u>Definition:</u>

The function f: X ⟶ X exhibits **sensitive dependence on initial**



"Practically speaking, sensitive dependence implies that if we are using an iterated function to model long-term behavior (such as population growth, weather, or economic performance) and the function exhibits sensitive dependence, then any error in measurement of the initial conditions may result in large differences between the predicted behavior and the actual behavior of the system we are modeling."

Richard A. Holmgren

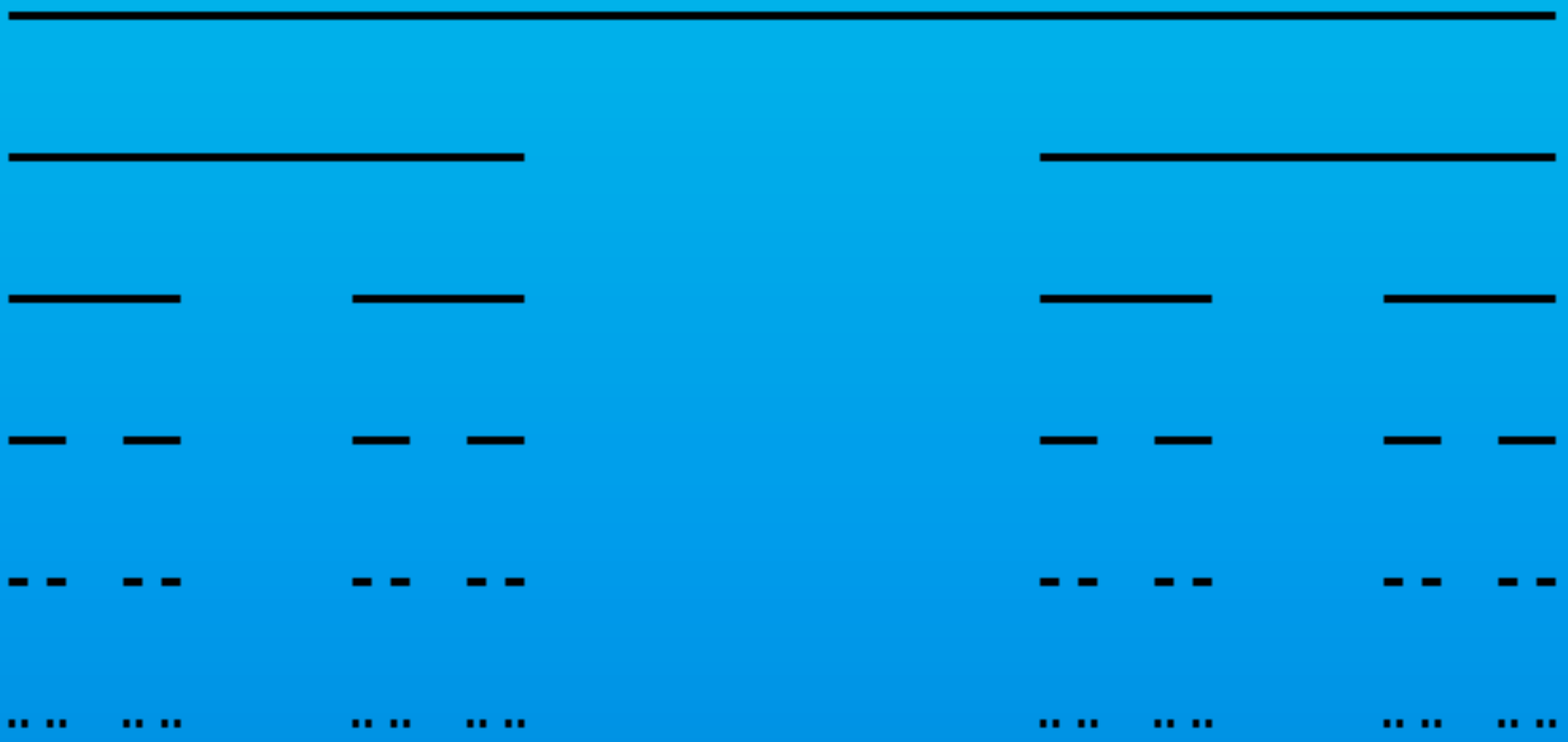
<u>Definition:</u>

A nonempty set Γ ⊆ ℝ is called a **Cantor set** if

- Γ is closed and bounded. (Sets of real numbers with these characteristics are called **compact sets**.)
- Γ contains no intervals of positive length. (Sets of this nature are called **totally disconnected**.)
- Every point in Γ is an accumulation point of Γ. (When closed, such sets are called **perfect sets**.)

Note: Cantor sets do not have to be subsets of ℝ.

Cantor middle 3rd set.



<u>Definition:</u>

The function $h\colon\Lambda\to\Lambda$ is **chaotic** if

a) the periodic points of h are dense in Λ,