# Adding 1 in a mirror universe

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# Adding 1 in the regular universe

**First:** the "regular universe" in this talk is the set of integers, which is denoted by $\mathbb{Z}$.

To understand addition in a mirror universe, we must first understand addition in the regular universe.

**Rhetorical question**

When we add 1 to an integer $n$, what do we really do?

# Adding 1 in the regular universe

**Example 1**

$$8452 + 1 = 8453$$

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**Question**

What exactly do we mean when we write 8452?

# Adding 1 in the regular universe

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## Question

What exactly do we mean when we write 8452?

## Answer: place value notation

The number 8452 is shorthand for the following expression:

$$8452 = 8 \cdot 1000 + 4 \cdot 100 + 5 \cdot 10 + 2 \cdot 1$$

# Adding 1 in the regular universe

## Example 1

$$8452 + 1 = 8453$$

## Question

What exactly do we mean when we write 8452?

## Answer: place value notation

Rewritten:

$$8452 = 8 \cdot 10^3 + 4 \cdot 10^2 + 5 \cdot 10^1 + 2 \cdot 10^0$$

# Adding 1 in the regular universe

## Example 1

$$8452 + 1 = 8453$$

## Question

What exactly do we mean when we write 8452?

## Answer: place value notation

In the opposite order:

$$8452 = 2 \cdot 10^0 + 5 \cdot 10^1 + 4 \cdot 10^2 + 8 \cdot 10^3$$

# Adding 1 in the regular universe

## Example 1

$$8452 + 1 = 8453$$

## Question

What exactly do we mean when we write 8452?

## More generally

When we write an integer like $a_n a_{n-1} a_{n-2} \dots a_3 a_2 a_1 a_0$ where each $a_j$ is a **digit**, we are really referring to the sum

$$\sum_{j=0}^{n} a_j 10^j.$$

# Adding 1 in the regular universe

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When we write an integer like $a_n a_{n-1} a_{n-2} \ldots a_3 a_2 a_1 a_0$ where each $a_j$ is a **digit**, we are really referring to the sum

$$\sum_{j=0}^{n} a_j 10^j. \leftarrow$$

**Note:** this is a finite sum

# Adding 1 in the regular universe

## Example 1

$$8452 + 1 = 8453$$

## Question

What exactly do we mean when we write 8452?

## P.S.

For now, a **digit** is a symbol taken from the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

# Adding 1 in the regular universe

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## Next Question

When we add 1 to 8452, to which digit do we add the 1?

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$$8452 + 1 = 8453$$

## Next Question

When we add 1 to 8452, to which digit do we add the 1?

## Answer

We add in the **ones** place (i.e. the **right-most** place).

**Example 2**

$$2739 + 1 = ?$$

As before, we add 1 in the ones place. But this time, because there's no single digit we have to represent "10" in the ones place, we put a 0 in the ones place, and **carry to the left** to obtain the answer 2740.

**Example 2**

$$2739 + 1 = ?$$

As before, we add 1 in the ones place. But this time, because there's no single digit we have to represent "10" in the ones place, we put a 0 in the ones place, and **carry to the left** to obtain the answer 2740.

**Example 3**

$$2079999 + 1 = ?$$

Lots of carrying here, but we eventually get 2080000.

# Adding 1 in the regular universe

To summarize, here's how addition by 1 works in the normal universe $\mathbb{Z}$:

## How to add 1 to any integer

1. Add 1 in the ones place (as far to the **right** as possible).
2. If the ones place is less than 10, you're done.
3. If the ones place would be 10, make the ones place 0, **carry to the left** and add 1 in the tens place.
4. Keep going until you don't have to carry any more.

Since the sum representing the integer is finite, it is a sure thing that you won't have to carry forever (so the procedure stops).

**Question**

In the previous procedure, what is special about 10?

**Question**

In the previous procedure, what is special about 10?

**Answer**

Nothing, except

# Binary notation

Computers use binary (a.k.a. **base** 2) notation, which is like regular notation except that there are only two digits: 0 and 1. In binary notation, the number 2 plays the role that 10 normally does.

### Decimal (a.k.a. base 10) notation:

$$10011 = 10011_{10} = 1 \cdot 10^4 + 0 \cdot 10^3 + 0 \cdot 10^2 + 1 \cdot 10^1 + 1 \cdot 10^0$$

### Binary notation:

$$10011 = 10011_2 = 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

**Remark:**

$$10011_2 = 19_{10}$$

because

$$\begin{aligned} {\color{red}1}{\color{blue}0}{\color{green}0}{\color{orange}1}{\color{violet}1}_2 &= {\color{red}1} \cdot 2^4 + {\color{blue}0} \cdot 2^3 + {\color{green}0} \cdot 2^2 + {\color{orange}1} \cdot 2^1 + {\color{violet}1} \cdot 2^0 \\ &= 16 + 0 + 0 + 2 + 1 \\ &= 19. \end{aligned}$$

**Does this make sense?**

If so, convert $1001010_2$ to base 10.

**Remark:**

$$10011_2 = 19_{10}$$

because

$$\begin{aligned} 10011_2 &= 1 \cdot 2^4 + 0 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 \\ &= 16 + 0 + 0 + 2 + 1 \\ &= 19. \end{aligned}$$

**Does this make sense?**

If so, convert $1001010_2$ to base 10.

$$\begin{aligned} 1001010_2 &= 1 \cdot 2^6 + 0 + 0 + 1 \cdot 2^3 + 0 + 1 \cdot 2^1 + 0 \\ &= 64 + 8 + 2 \\ &= 74. \end{aligned}$$

# Adding 1 in the regular universe (binary version)

Adding 1 in $\mathbb{Z}$ when the numbers are written in base 2 is similar to adding when the numbers are written in base 10:

## How to add 1 to any integer (base 2)

1. Add 1 in the ones place (as far to the **right** as possible).
2. If the ones place is less than 2, you're done.
3. If the ones place would be 2, make the ones place 0, **carry to the left** and add 1 in the tens place.
4. Keep going until you don't have to carry any more.

# Adding 1 in the regular universe (binary version)

## Some examples

$$101100 + 1 = 101101$$

$$10001 + 1 = 10010$$

$$1000100111 + 1 = 1000101000$$

# Adding 1 in the regular universe

**More generally**

Let $b$ be any natural number which is at least 2.

To add 1 to an integer in base $b$, add 1 in the ones place; if the ones place is less than the base, you're done; otherwise, make the ones place 0 and carry to the left as necessary.

If you really wanted to be fancy, you could assign different bases to each place (i.e. make the ones place base 2, the tens place base 5, the hundreds place base 16, etc.), but addition still works the same way.

**For the next few minutes:** assume everything is binary.

# The mirror universe

# Where do you put the mirror?

First, you need a decimal point (or is it "binary point"?) to mark the end of the number.

$$\begin{array}{r} 11101101. \\ +1\phantom{.} \\ \hline 11101110. \end{array}$$

# Where do you put the mirror?

First, you need a decimal point (or is it "binary point"?) to mark the end of the number.

Now for the mirror:

$$\begin{array}{r} 11101101. \\ +1\phantom{.} \\ \hline 11101110. \end{array}$$

# Where do you put the mirror?

First, you need a decimal point (or is it "binary point"?) to mark the end of the number.

Here's the mirror image:

$$\begin{array}{r|l} 11101101. & .10110111 \\ +1 & \quad\;\, 1+ \\ \hline 11101110. & .01110111 \end{array}$$

The right-hand side of this figure is the "mirror universe". What happens when we add 1 over there?

## In this mirror universe:

1. integers are represented by strings of digits written to the **right** of a decimal point (in this talk, nothing is allowed to the left of a decimal point in the mirror universe... otherwise it wouldn't be the mirror image of an integer);
2. to add 1, start by adding 1 in the **tenths** place (which as far to the left as you can start);
3. if you have to carry, you carry **to the right**.

## Does this make sense?

If so, compute $.1100101 + 1$ in the mirror universe.

# Addition in the mirror universe

## In this mirror universe:

1. integers are represented by strings of digits written to the **right** of a decimal point (in this talk, nothing is allowed to the left of a decimal point in the mirror universe... otherwise it wouldn't be the mirror image of an <u>integer</u>);
2. to add 1, start by adding 1 in the **tenths** place (which as far to the left as you can start);
3. if you have to carry, you carry **to the right**.

## Does this make sense?

If so, compute $.1100101 + 1$ in the mirror universe.

**Answer:**

$$.1100101 + 1 = .0010101$$

Consider this (base 2) object:

$$.1100101$$

**Recall**

In the regular universe (base 2),

$$\begin{aligned} 1101 &= 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 \end{aligned}$$

and more generally

$$a_n a_{n-1} a_{n-2} \cdots a_2 a_1 a_0 = \sum_{j=0}^{n} a_j 2^j.$$

# What are the objects in this mirror universe?

Consider this (base 2) object:

$$.1100101$$

**Recall also**

In base 10, numbers after a decimal point like .752 mean

$$.7 + .05 + .002 = 7 \cdot 10^{-1} + 5 \cdot 10^{-2} + 2 \cdot 10^{-3}.$$

More generally,

$$.a_1 a_2 a_3 \cdots a_n = \sum_{j=1}^{n} a_j 10^{-j}.$$

# What are the objects in this mirror universe?

Consider this (base 2) object:

$$.1100101$$

So .1100101 could be interpreted as

$$1 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 0 \cdot 2^{-4} + 1 \cdot 2^{-5} + 0 \cdot 2^{-6} + 1 \cdot 2^{-7}$$
$$= \frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{0}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128}$$
$$= \frac{101}{128}.$$

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# What are the objects in this mirror universe?

Consider this (base 2) object:

$$.1100101$$

So .1100101 could be interpreted as

$$\begin{aligned}
&1 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 0 \cdot 2^{-4} + 1 \cdot 2^{-5} + 0 \cdot 2^{-6} + 1 \cdot 2^{-7} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{0}{8} + \frac{0}{16} + \frac{1}{32} + \frac{0}{64} + \frac{1}{128} \\
&= \frac{101}{128}.
\end{aligned}$$

This means $\frac{101}{128}$ **is an integer** in the mirror universe!

# What are the objects in the mirror universe?

**Does this make sense?**

If so, tell me what number is represented by .111 in the mirror universe.

# What are the objects in the mirror universe?

**Does this make sense?**

If so, tell me what number is represented by .111 in the mirror universe.

**Answer:**

$$.111 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

# Now, let's add 1

**Example 1**

$$.001 + 1 = ?$$

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According to our rules for adding 1,

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Let's translate this into numbers:

$$.001 = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} = \frac{1}{8}.$$

$$.101 = \frac{1}{2} + \frac{0}{4} + \frac{1}{8} = \frac{5}{8}.$$

**Example 1**

$$.001 + 1 = ?$$

According to our rules for adding 1,

$$.001 + 1 = .101. \tag{1}$$

So equation (1) above is just a fancy of stating the obvious (?) fact that

$$\frac{1}{8} + 1 = \frac{5}{8}.$$

# Now, let's add 1

**Example 1**

$$.001 + 1 = ?$$

According to our rules for adding 1,

$$.001 + 1 = .101. \tag{1}$$

So equation (1) above is just a fancy of stating the **obvious** fact that

$$\frac{1}{8} + 1 = \frac{5}{8}$$

(**in the mirror universe**).

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Since $\frac{1}{8} + 1 = \frac{5}{8}$, it stands to reason that $1 = \frac{1}{2}$, right?

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## Example 2

$$.1010 + 1 = ?$$

# Now, let's add 1

Since $\frac{1}{8} + 1 = \frac{5}{8}$, it stands to reason that $1 = \frac{1}{2}$, right?

**Example 2**

$$.1010 + 1 = ?$$

$$.1010 + 1 = .0110$$

which translates to

$$\frac{5}{8} + 1 = \frac{3}{8}.$$

Therefore, in the mirror universe,

$$1 = -\frac{2}{8} = -\frac{1}{4}.$$

# Adding 1 in the mirror universe is not adding a constant

## Recall from the preceding slides

- to "mirror" add 1 to $\frac{1}{8}$, we "regular" add $\frac{1}{2}$ because $\frac{1}{8} + 1 = \frac{5}{8}$.
- to "mirror" add 1 to $\frac{5}{8}$, we "regular" add $\frac{-1}{4}$ because $\frac{5}{8} + 1 = \frac{3}{8}$.

# Adding 1 in the mirror universe is not adding a constant

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## Epiphany

Maybe mirror addition by 1 **corresponds to regular addition by a variable quantity** when we think of the "mirror integers" as fractions.

# Adding 1 in the mirror universe is not adding a constant

## Recall from the preceding slides

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## Epiphany

Maybe mirror addition by 1 **corresponds to regular addition by a variable quantity** when we think of the “mirror integers” as fractions.

## Question

What is that variable quantity?

# An advantage of the mirror universe

Recall that a regular integer can only consist of **finitely** many digits to the left of the decimal point. For example,

$$...101010101010101010.$$

is not an integer, because the corresponding infinite series

$$... + 2^{11} + 2^9 + 2^7 + 2^5 + 2^3 + 2^1 = \sum_{j=0}^{\infty} 2^{2j+1}$$

diverges.

**But,** in the mirror universe, we can put **infinitely** many digits to the right of the decimal point.

**Example**

$.0101010101010101.... = ?$

**But,** in the mirror universe, we can put **infinitely** many digits to the right of the decimal point.

**Example**

$.0101010101010101.... = ?$

**Answer:**

$$\begin{aligned} .0101010.... &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \frac{1}{2^8} + ... \\ &= \sum_{j=1}^{\infty} \frac{1}{2^{2j}} \\ &= \frac{1}{4} \sum_{j=0}^{\infty} \left( \frac{1}{4} \right)^j \end{aligned}$$

**But,** in the mirror universe, we can put **infinitely** many digits to the right of the decimal point without a problem.

**Example**

$.0101010101010101.... = ?$

**Answer:**

$$\begin{aligned} .0101010.... &= \frac{1}{4}\sum_{j=0}^{\infty}\left(\frac{1}{4}\right)^j \\ &= \frac{1}{4}\left(\frac{1}{1-\frac{1}{4}}\right) \\ &= \frac{1}{3}. \end{aligned}$$

# An advantage of the mirror universe

In fact, **any** infinite sequence of binary digits

$$.a_1 a_2 a_3 a_4 a_5 \ldots$$

corresponds to a **convergent** infinite series

$$\sum_{j=1}^{\infty} a_j 2^{-j} = \sum_{j=1}^{\infty} \frac{a_j}{2^j}.$$

This series converges by the Comparison Test (hooray for Calc 2!), because it is positive and is at most

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = \frac{1}{2}\left(\frac{1}{1-\frac{1}{2}}\right) = 1.$$

**The punch line**

The elements of the mirror universe are real numbers in the interval $[0, 1]$.

# Representing a real number in $[0, 1]$ in the mirror universe

**Question**

Given a real number $x \in [0, 1]$, what does $x$ look like in the mirror universe?

**Short answer**

$$x = .a_1 a_2 a_3 a_4 \ldots.$$

where each $a_j$ is a 0 or 1.

**Follow-up question**

Given $x \in [0, 1]$ as above, is $a_1$ equal to 0 or 1?

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## Follow-up question

Given $x \in [0, 1]$ as above, is $a_1$ equal to 0 or 1?

## Notice

If $a_1 = 0$, then $x \leq 0 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + ... = \frac{1}{2}$.

If $a_1 = 1$, then $x \geq \frac{1}{2} + 0 + 0 + 0 + 0 + ... = \frac{1}{2}$.

# Representing a real number in $[0,1]$ in the mirror universe

## Follow-up question

Given $x \in [0,1]$ as above, is $a_1$ equal to 0 or 1?

## Notice

If $a_1 = 0$, then $x \leq 0 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \ldots = \frac{1}{2}$.

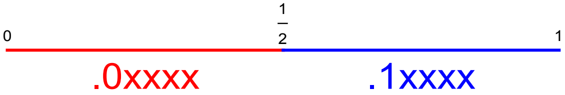
If $a_1 = 1$, then $x \geq \frac{1}{2} + 0 + 0 + 0 + 0 + \ldots = \frac{1}{2}$.

## Conclusion

- if $x < \frac{1}{2}$, then we can set the first digit $a_1 = 0$.
- If $x \geq \frac{1}{2}$, then we can set the first digit $a_1 = 1$.

# Representing a real number in $[0, 1]$ in the mirror universe

A picture:



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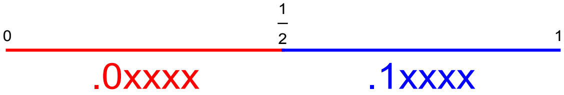
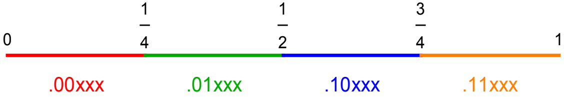



By the same logic, we can continue to find the subsequent digits:



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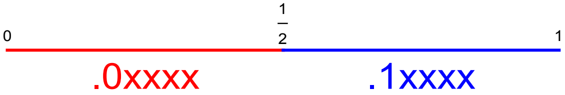


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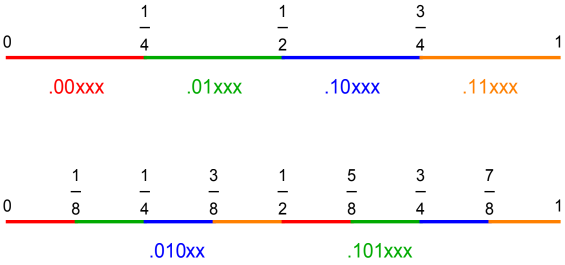


# Back to adding 1

**Recall**

Earlier, I said that to "mirror" add 1, you have to add a variable amount to a number in $[0, 1]$.

Let's figure out what that variable amount is.

**Case 1: $x < \frac{1}{2}$**

In this case, $x = .0a_2a_3a_4...$ so

$$x + 1 = .1a_2a_3a_4...$$

The amount we have added is $.1 - .0 = \frac{1}{2}$.

**Recall**

Earlier, I said that to "mirror" add 1, you have to add a variable amount to a number in $[0,1]$.

Let's figure out what that variable amount is.

**Case 2: $\frac{1}{2} \leq x < \frac{3}{4}$**

In this case, $x = .10a_3a_4a_5...$ so

$$x + 1 = .01a_3a_4a_5...$$

The amount we have added is $.01 - .10 = -\frac{1}{4}$.

# Back to adding 1

## Recall

Earlier, I said that to "mirror" add 1, you have to add a variable amount to a number in $[0, 1]$.

Let's figure out what that variable amount is.

## Case 3: $\frac{3}{4} \leq x < \frac{7}{8}$

In this case, $x = .110a_4a_5...$ so

$$x + 1 = .001a_4a_5...$$

The amount we have added is $.001 - .110 = -\frac{5}{8}$.

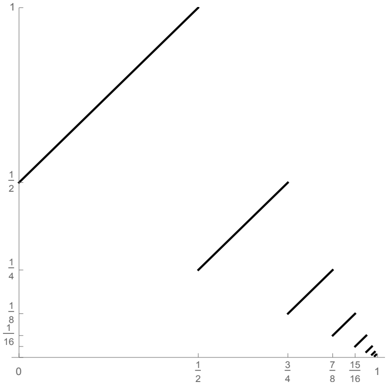
# Putting this all together

Given $x \in [0,1]$, define $T(x)$ to be what we get when we "mirror" add 1 to $x$. Then $T$ has this piecewise-defined formula:

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{2}) \\ x - \frac{1}{4} & \text{if } x \in [\frac{1}{2}, \frac{3}{4}) \\ x - \frac{5}{8} & \text{if } x \in [\frac{3}{4}, \frac{7}{8}) \\ x - \frac{13}{16} & \text{if } x \in [\frac{7}{8}, \frac{15}{16}) \\ \vdots & \vdots \\ x - \frac{2^{n+1}-3}{2^{n+1}} & \text{if } x \in [1 - 2^{-n}, 1 - 2^{-n-1}) \\ \vdots & \vdots \\ 0 & \text{if } x = 1 \end{cases}$$

Rather than remember this formula, it's easier to show you a graph of $T$:

Given $x \in [0,1]$, define $T(x) = x$"$+1$" where the addition is in the mirror universe. Then the graph of $T$ is



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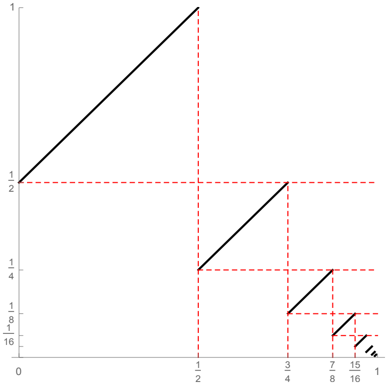

# Who cares?

I study dynamical systems. In dynamical systems, you take a set $X$ and a function $T : X \to X$ and you **iterate** $T$ (that means you apply $T$ repeatedly to each input).

## Notation

Suppose $T : X \to X$. If $x$ and $y$ are elements of $X$, then the notation

$$x \to y$$

means that $T(x) = y$.

## Example 1

$X = \mathbb{Z}$; $T(x) = x + 1$ (regular addition).

$$0 \to 1 \to 2 \to 3 \to 4 \to \cdots \to n \to n+1 \to \cdots \to \infty.$$

# Who cares?

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**Example 2**

$X = [0, 1]$; $T(x) =$ mirror addition of 1.

$$\frac{1}{3} \to \frac{5}{6} \to \frac{5}{24} \to \frac{17}{24} \to \frac{11}{24} \to \frac{23}{24} \to \frac{5}{96} \to \frac{53}{96} \to \cdots$$

$.0101010101.... \to .1101010101... \to .0011010101... \to$

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## Example 2

$X = [0, 1]$; $T(x) =$ mirror addition of 1. This dynamical system $(X, T)$ is called the **dyadic odometer**.

# Who cares?

I study dynamical systems. In dynamical systems, you take a set $X$ and a function $T : X \to X$ and you **iterate** $T$ (that means you apply $T$ repeatedly to each input).

**Example 2 with different input**

$X = [0, 1]$; $T(x) =$ mirror addition of 1.

$$\begin{aligned} \pi - 3 \approx .141592 \to .641593 \to .391593 \to .891593 \\ \to .0790927 \to .5790927 \to .329093 \\ \cdots \to \cdots .118155 \cdots \to \cdots .332999 \cdots \to \cdots .575186 \cdots \end{aligned}$$

The numbers look somewhat random, but they aren't.

# Who cares?

I study dynamical systems. In dynamical systems, you take a set $X$ and a function $T : X \to X$ and you **iterate** $T$ (that means you apply $T$ repeatedly to each input).

## Example 3

Let $X = [0, 1]$, where numbers are represented by strings like

$$.a_1 a_2 a_3 a_4 \ldots$$

where each place is kept in its own base (maybe the tenths place is base 5, the hundredths place is base 13, etc.).

Defining $T(x) = x + 1$ by addition on the left with carry to the right, we get a dynamical system called an **odometer**.

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## Example 3

As a specific example, suppose the tenths place is base 2, the hundredths place is base 6, the thousandths place is base 5, and all the other places are base 3. Then

$$.04411... \to .14411... \to .05411... \to .15411 \to .00021... \to .10021... \\ \to .01021... \to .11021... \to .02021... \to \cdots$$

# Who cares?

Odometers are fundamental examples in the study of dynamical systems, in part because **every (interesting) dynamical system has an odometer hidden inside of it**.

## Theorems

Dye: Every free ergodic, measure-preserving dynamical system is orbit equivalent to the dyadic odometer.

Jewett-Krieger: Every free ergodic, measure-preserving dynamical system has a topological model which is, at worst, a slight generalization of an odometer.

Ellis-Gottschalk: Every minimal topological dynamical system contains a factor isomorphic to an odometer.

Coven-Yassawi: Every cellular automaton contains an embedded odometer.

# More on cellular automata

The dyadic odometer is the simplest interesting (matter of opinion) example of something called a **cellular automaton**.

To visualize a cellular automaton, think of an infinite ticker tape with symbols (like 0 or 1) on it. Here's an example of a ticker tape:

$$0100101110101110101000101001001\cdots$$

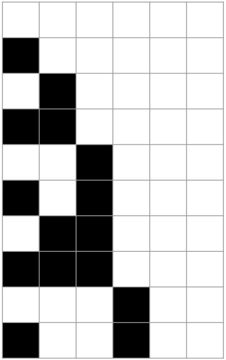
$\cdots$

A **cellular automaton (c.a.)** is a rule that could be used to create another ticker tape from another, with the property that:

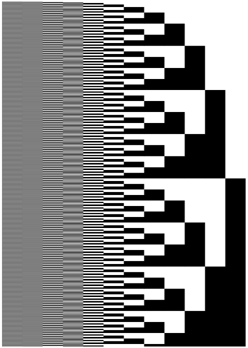
if you first delete the left-most $n$ symbols from the ticker tape, then do the c.a., you get the same thing as if you first do the c.a., then delete the left-most $n$ symbols of what you get.

Consider what happens when we start with $x = 0$ and apply the dyadic odometer over and over:

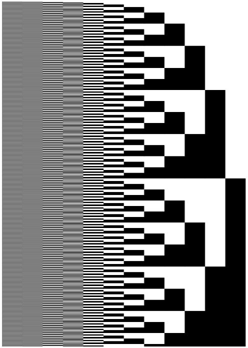
$.000000 \leftrightarrow$
$.100000 \leftrightarrow$
$.010000 \leftrightarrow$
$.110000 \leftrightarrow$
$.001000 \leftrightarrow$
$.101000 \leftrightarrow$
$.011000 \leftrightarrow$
$.111000 \leftrightarrow$
$.000100 \leftrightarrow$
$.100100 \leftrightarrow$



Continue this process to obtain the following picture (rescaled):



Continue this process to obtain the following picture (rescaled):



This generates a fractal!

## Other areas where cellular automata are used

1. Mathematics (fractals and chaos theory)
2. Complexity science and simulation theory
3. Musical composition
4. Models of traffic flow in urban areas
5. Cryptography (design of block ciphers)
6. Analysis of card shuffling

**Question**

What is adding "1" like if your objects are **vectors**, rather than numbers?

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What is adding "1" like if your objects are **vectors**, rather than numbers?

In this setting, the objects under consideration are "mirror images" of vectors in $\mathbb{Z}^d$ (if $d = 2$, a vector is an ordered pair like $(x, y)$ where $x$ and $y$ are integers).

We have to consider an **action** which is generated by two transformations (in $d = 2$): one which "mirror" adds the vector $(1, 0)$ and one which "mirror" adds the vector $(0, 1)$.

# My current research

**Question**

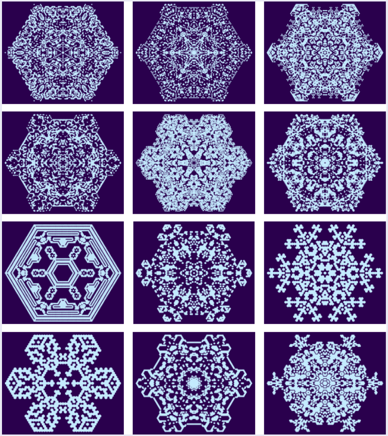
What is adding "1" like if your objects are **vectors**, rather than numbers?

It turns out that there are dynamical systems called $\mathbb{Z}^d$-**odometers** you can build by doing this (by choosing different bases in each place, and even more interesting things).

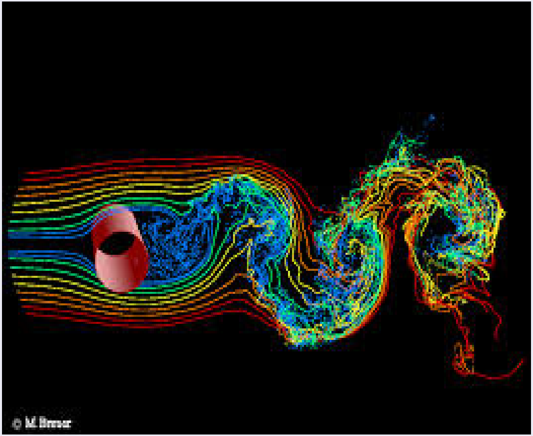
With my collaborator Aimee S.A. Johnson (Swarthmore), I'm working on classifying $\mathbb{Z}^d$-odometers up to various kinds of equivalence.

This will help us (someone interested in applications, not me) determine the mathematical similarities and differences between $\mathbb{Z}^d$-odometer and/or c.a. models used in many settings, such as...

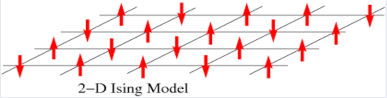
## Example 1: snowflake formation

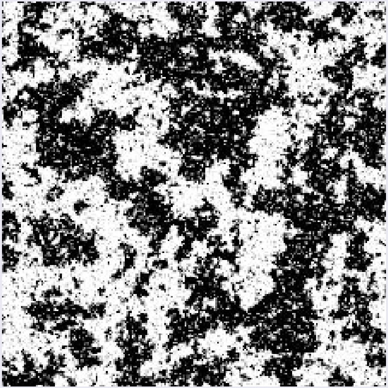


## Example 2: fluid dynamics and turbulence



## Example 3: Ising model of ferromagnetism



## Example 4: patterns in seashells

## Other examples

**Chemistry:** simulation of the Belousov-Zhabotinsky reaction

**Neuroscience:** simulation of neuron activity

**Engineering:** structural design of load-bearing beams

**Astronomy:** cosmological models for the universe

# The End