Speedups of $\mathbb{Z}^d$-odometers

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This talk is about actions of $\mathbb{Z}^d$

Definition

A $\mathbb{Z}^d-$ **Cantor minimal system** ($\mathbb{Z}^d$**-C.m.s.**) is a pair $(X, \mathbf{T})$ where X is a Cantor space and $\mathbf{T} = \{\mathbf{T}^{\mathbf{v}} : \mathbf{v} \in \mathbb{Z}^d\}$ is a minimal action of $\mathbb{Z}^d$ on X by homeomorphisms.

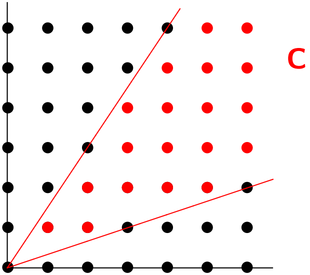
In this situation, we can write $\mathbf{T} = (T_1, ..., T_d)$ where T_j is shorthand for the action of standard basis vector $\mathbf{e}_j$.

If $d = 2$ and $\mathbf{T} = (T_1, T_2)$, I might call T_1 the **horizontal** direction and T_2 the **vertical** direction of the action.

Definition

A **cone** $\mathbf{C}$ is the intersection of $\mathbb{Z}^d - \{\mathbf{0}\}$ with any open, connected subset of $\mathbb{R}^d$ bounded by d distinct hyperplanes passing through the origin.

Example in $\mathbb{Z}^2$:



Definition

Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}$-C.m.s. and let $\mathbf{C} \subset \mathbb{Z}^{d_1}$ be a cone. A **$\mathbf{C}-$cocycle** is a function $\mathbf{p} : X \times \mathbb{Z}^{d_2} \to \mathbb{Z}^{d_1}$ such that for all $x \in X$,

1. $\mathbf{p}(x, \mathbf{0}) = \mathbf{0}$;
2. The cocycle equation
$$\mathbf{p}(x, \mathbf{v} + \mathbf{w}) = \mathbf{p}(x, \mathbf{v}) + \mathbf{p}(\mathbf{T}^{\mathbf{p}(x,\mathbf{v})}(x), \mathbf{w})$$
is satisfied for all $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^{d_2}$; and
3. $\mathbf{p}(x, \mathbf{e}_j) \in \mathbf{C}$ for all $j \in \{1, ..., d_2\}$.

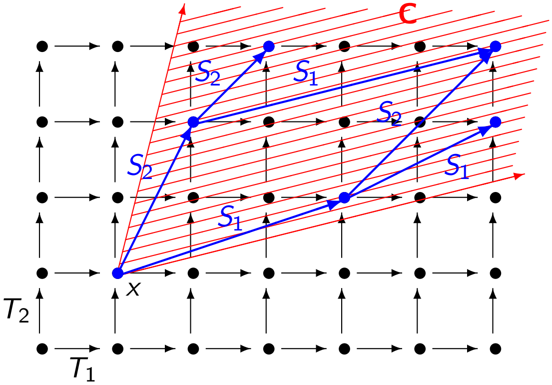
Definition

Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}$-C.m.s. and let $\mathbf{C} \subset \mathbb{Z}^{d_1}$ be a cone. A **$\mathbf{C}-$speedup** of $(X, \mathbf{T})$ is a $\mathbb{Z}^{d_2}-$C.s. $(X, \mathbf{S})$ where

$$\mathbf{S}^{\mathbf{v}}(x) = \mathbf{T}^{\mathbf{p}(x,\mathbf{v})}(x)$$

for some $\mathbf{C}$-cocycle $\mathbf{p}$.

A picture to explain ($d = 2$)



Here, $\mathbf{S} = (S_1, S_2)$ is a $\mathbf{C}-$speedup of $\mathbf{T} = (T_1, T_2)$.
In particular, for the indicated point x, we have

$$\mathbf{p}(x,(1,0)) = (3,1), \quad \mathbf{p}(x,(1,1)) = (5,3), \quad \text{etc.}$$

Why is this called a "speedup"?

When $d = 1$, there are two cones:

$$\mathbf{C}_+ = \{1, 2, 3, ...\} \text{ and } \mathbf{C}_- = \{-1, -2, -3, ...\}.$$

A $\mathbf{C}_+$−speedup looks like this:

$$\begin{tikzcd}
\cdots \arrow[r, "T"] & \bullet_x \arrow[r, "T"] \arrow[rrr, bend left, "S"] & \bullet \arrow[r, "T"] \arrow[r, bend right, "S"'] & \bullet \arrow[r, "T"] \arrow[rrrr, bend right, "S"'] & \bullet \arrow[r, "T"] \arrow[rr, bend left, "S"] & \bullet \arrow[r, "T"] & \bullet \arrow[r, "T"] \arrow[rr, bend left, "S"] & \bullet \arrow[r, "T"] & \cdots
\end{tikzcd}$$

Here "$p(x)$" $= p(x, 1) = 3,\ p(x, 2) = 5$, etc.

The big picture

Question

Given a $\mathbb{Z}^d$-C.m.s. $(X, \mathbf{T})$, how "similar" does a $\mathbf{C}$-speedup $(X, \mathbf{S})$ have to be to $(X, \mathbf{T})$?

Restated

Given $(X, \mathbf{T})$ and $(Y, \mathbf{S})$, is there a $\mathbf{C}$-speedup of $(X, \mathbf{T})$ which is the "same" as $(Y, \mathbf{S})$?

Notions of "sameness"

Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}$-C.m.s. and let $(Y, \mathbf{S})$ be a $\mathbb{Z}^{d_2}$-C.m.s.

Definition

$(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are **conjugate** if $d_1 = d_2$ and there is a homeomorphism $\Phi : X \to Y$ such that

$$\mathbf{S}^{\mathbf{v}}(\phi(x)) = \Phi(\mathbf{T}^{\mathbf{v}}(x))$$

for all $x \in X$ and all $\mathbf{v} \in \mathbb{Z}^{d_1}$.

Concept: same action, but different labels on the phase space.

Notions of "sameness"

Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}$-C.m.s. and let $(Y, \mathbf{S})$ be a $\mathbb{Z}^{d_2}$-C.m.s.

Definition

$(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are **isomorphic** if $d_1 = d_2$, there is a homeomorphism $\Phi : X \to Y$ and a group isomorphism $\vartheta : \mathbb{Z}^{d_1} \to \mathbb{Z}^{d_1}$ such that

$$\mathbf{S}^{\mathbf{v}}(\Phi(x)) = \Phi(\mathbf{T}^{\vartheta(\mathbf{v})}(x))$$

for all $x \in X$ and all $\mathbf{v} \in \mathbb{Z}^{d_1}$.

Concept: same action, but different labels on the phase space and different labels on the group elements.

Notions of "sameness"

Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}$-C.m.s. and let $(Y, \mathbf{S})$ be a $\mathbb{Z}^{d_2}$-C.m.s.

Definition

$(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are **orbit equivalent** if there is a homeomorphism $\Phi : X \to Y$ such that for every $x \in X$,

$$\Phi\left(\bigcup_{\mathbf{v}\in\mathbb{Z}^{d_1}} \mathbf{T}^{\mathbf{v}}(x)\right) = \bigcup_{\mathbf{v}\in\mathbb{Z}^{d_2}} \mathbf{S}^{\mathbf{v}}(\Phi(x)).$$

Concept: same orbit relation.

History of speedups: ergodic theory

Neveu (1969): characterized functions $p(x, 1)$ which can generate a speedup cocycle for a measure-preserving $\mathbb{Z}$-action

Arnoux-Ornstein-Weiss (1985): given any two ergodic Lebesgue measure-preserving $\mathbb{Z}$-actions (X, μ, T) and (Y, ν, S), there is a $\mathbf{C}_+$-speedup of one which is (measurably) conjugate to the other (ancestor: Dye's Theorem)

Babichev-Burton-Fieldsteel (2013): relative versions of Arnoux-Ornstein-Weiss (ancestor: Rudolph's relative orbit equivalence theory)

Johnson-M (2014, 2018): versions of AOW and BBF for measure-preserving $\mathbb{Z}^d$-actions (ancestor: Connes-Feldman-Weiss's classification of hyperfinite equivalence relations)

History of speedups: topological dynamics

Ash (2014): gave necessary and sufficient conditions on $\mathbb{Z}$-C.m.s. (X, T) and (Y, S) so that there is a $\mathbf{C}_+$-speedup of (X, T) conjugate to (Y, S) (ancestor: Giordano-Putnam-Skau's classification of $\mathbb{Z}$-C.m.s. up to orbit equivalence)

Johnson-M: generalized (most of) Ash's work to $\mathbb{Z}^d$-C.m.s. (ancestor: Giordano-Matui-Putnam-Skau)

Alvin-Ash-Ormes (2018): studied the structure of <u>bounded</u> speedups of $\mathbb{Z}$-C.m.s., with particular emphasis on odometer actions and substitutions.

Bounded speedups

Definition

A speedup given by $\mathbf{C}$-cocycle $\mathbf{p}$ is called **bounded** if $\{\mathbf{p}(x, \mathbf{e}_j) : x \in X\}$ is bounded for each $j \in \{1, ..., d\}$.

Note: A speedup is bounded if and only if $\mathbf{p} : X \times \mathbb{Z}^{d_2} \to \mathbb{Z}^{d_1}$ is continuous.

In the rest of this talk

We will discuss bounded speedups of $\mathbb{Z}^d$-odometers (with the aim of generalizing AAO).

$\mathbb{Z}^d$-odometers were introduced by Cortez in 2004. They are defined as follows:

The phase space

Let

$$\mathbb{Z}^d \geq G_1 \geq G_2 \geq G_3 \geq G_4 \geq \cdots$$

be a decreasing sequence of subgroups of $\mathbb{Z}^d$, each of which have finite index in $\mathbb{Z}^d$, such that $\bigcap_{j=1}^{\infty} G_j = \{\mathbf{0}\}$. Let X be the inverse limit

$$X = \varprojlim (\mathbb{Z}^d / G_j).$$

$\mathbb{Z}^d$-odometers

$\mathbb{Z}^d$-odometers were introduced by Cortez in 2004. They are defined as follows:

The phase space

Each element $\mathbf{x}$ of X is formally an infinite sequence of cosets, i.e. something like

$$\mathbf{x} = (\mathbf{x}_1 + G_1, \mathbf{x}_2 + G_2, \mathbf{x}_3 + G_3, \ldots)$$

where the $\mathbf{x}_j$ are "commensurate", i.e. since $G_j \geq G_{j+1}$, there is a natural map

$$\pi_j : \mathbb{Z}^d / G_{j+1} \to \mathbb{Z}^d / G_j;$$

for such a sequence to be in X we require that, for all j,

$$\pi_j(\mathbf{x}_{j+1} + G_{j+1}) = \mathbf{x}_j + G_j.$$

$\mathbb{Z}^d$-odometers

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The action

X is a Cantor space, and also a topological group with addition defined coordinate-wise, where the addition in the j^{th} coordinate is the usual (vector) addition in the quotient group $\mathbb{Z}^d/G_j$.

Given any $\mathbf{v} \in \mathbb{Z}^d$, we can "convert" $\mathbf{v}$ into an element of X by setting

$$\tau(\mathbf{v}) = (\mathbf{v} + G_1, \mathbf{v} + G_2, \mathbf{v} + G_3, \ldots)$$

Define the action $\mathbf{T}$ of $\mathbb{Z}^d$ on X by $\mathbf{T}^{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \tau(\mathbf{v})$. Any $\mathbb{Z}^d$-C.m.s. conjugate to such an $(X, \mathbf{T})$ is called a **$\mathbb{Z}^d$-odometer**.

$\mathbb{Z}^d$-odometers

As an example, the dyadic odometer comes from the sequence of groups $G_j = 2^j\mathbb{Z}$, i.e.

$$2\mathbb{Z} \geq 4\mathbb{Z} \geq 8\mathbb{Z} \geq 16\mathbb{Z} \geq \cdots \geq 2^j\mathbb{Z} \geq \cdots$$

For $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ...) \in X$, the coset $\mathbf{x}_j$ labels the level to which $\mathbf{x}$ belongs at the j^{th} stage when one does the traditional cutting and stacking construction:

$\mathbb{Z}^d$-odometers

$$\mathbf{x} = \quad (\mathbf{x}_1, \quad \mathbf{x}_2, \quad \mathbf{x}_3, \quad \ldots$$

$$\begin{tikzcd}
\boxed{1+2\mathbb{Z}} & \boxed{3+4\mathbb{Z}} & \boxed{7+8\mathbb{Z}} \\
\boxed{0+2\mathbb{Z}} \arrow[u, "T"] & \boxed{2+4\mathbb{Z}} \arrow[u, "T"] & \vdots \arrow[u, "T"] \\
 & \boxed{1+4\mathbb{Z}} \arrow[u, "T"] & \vdots \arrow[u, "T"] \\
 & \boxed{0+4\mathbb{Z}} \arrow[u, "T"] & \boxed{0+8\mathbb{Z}} \arrow[u, "T"]
\end{tikzcd}$$

$\mathbb{Z}^d$-odometers

If $d > 1$, $\mathbb{Z}^d$-odometers can be more complicated: as an example, consider the $\mathbb{Z}^2$-odometer given by

$$G_j = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^j \mathbb{Z}^2$$

After the first iteration of "cutting and stacking", we obtain this picture of how the action acts on the $\mathbf{x}_1$ coordinate:

$$\boxed{\begin{tikzcd} \boxed{(0,1) + G_1} \arrow[r, "T_1"] & \boxed{(1,1) + G_1} \\ \boxed{(0,0) + G_1} \arrow[u, "T_2"] \arrow[r, "T_1"] & \boxed{(1,0) + G_1} \arrow[u, "T_2"'] \end{tikzcd}}$$

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Here, there is "skewing" when T_2 sends cosets in the top row back to the bottom, since $(0,2) \equiv (1,0) \mod G_1$.

Theorem (Alvin-Ash-Ormes 2018)

Let (X, T) be a $\mathbb{Z}$-odometer, and suppose (X, S) is a bounded $\mathbf{C}_+$-speedup of (X, T). If S is minimal, then (X, S) is a $\mathbb{Z}$-odometer.

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An identical result holds in higher dimensions:

Theorem (Johnson-M)

Let $\mathbf{C} \subseteq \mathbb{Z}^{d_1}$ be any cone. Let $(X, \mathbf{T})$ be a $\mathbb{Z}^{d_1}-$odometer, and suppose $(X, \mathbf{S})$ is a bounded $\mathbf{C}$-speedup of $(X, \mathbf{T})$. If $\mathbf{S}$ is minimal, then $\mathbf{S}$ is a $\mathbb{Z}^{d_2}-$odometer.

Theorem (Alvin-Ash-Ormes 2018)

Let (X, T) be a $\mathbb{Z}$-odometer, and suppose (X, S) is a bounded $\mathbf{C}_+$-speedup of (X, T). If S is minimal, then (X, S) is a $\mathbb{Z}$-odometer which is topologically conjugate to (X, T).

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This is too much to hope for in $\mathbb{Z}^d$ when $d \geq 2$.

For instance, $\mathbf{S} = (T_2, T_1)$ is a speedup of $\mathbf{T} = (T_1, T_2)$ via the cocycle $\mathbf{p}(x, (v_1, v_2)) = (v_2, v_1)$, but such an $\mathbf{S}$ and $\mathbf{T}$ are, in general, not conjugate.

But these $\mathbf{S}$ and $\mathbf{T}$ are isomorphic. Must a minimal bounded speedup of $\mathbb{Z}^d$-odometer $(X, \mathbf{T})$ be isomorphic to $(X, \mathbf{T})$?

Bounded speedups of $\mathbb{Z}^d$−odometers

Theorem (Alvin-Ash-Ormes 2018)

Let (X, T) be a $\mathbb{Z}$-odometer, and suppose (X, S) is a bounded $\mathbf{C}_+$-speedup of (X, T). If S is minimal, then (X, S) is a $\mathbb{Z}$-odometer which is topologically conjugate to (X, T).

Theorem (Johnson-M)

Let $\mathbf{C} \subseteq \mathbb{Z}^2$ be any cone containing $(1, 0)$, $(0, 1)$ and $(1, 1)$. Then, there exist $\mathbb{Z}^2$-odometers $(X, \mathbf{T})$ and $(X, \mathbf{S})$ such that $(X, \mathbf{S})$ is a bounded $\mathbf{C}$-speedup of $(X, \mathbf{T})$, but $(X, \mathbf{S})$ and $(X, \mathbf{T})$ are not isomorphic.

Example of a bounded, non-isomorphic speedup

Let $(X, \mathbf{T})$ be the $\mathbb{Z}^2$-odometer associated to the sequence of groups $3^j\mathbb{Z} \times 2^j\mathbb{Z}$ (i.e. the product-type $\times 3$, $\times 2$-odometer).

To define the speedup cocycle $\mathbf{p}$:

- set $\mathbf{p}(\mathbf{x}, (1,0)) = (1,0)$ for all $\mathbf{x} \in X$;
- set

$$\mathbf{p}(\mathbf{x}, (0,1)) = \begin{cases} (0,1) & \text{if } \mathbf{x}_1 \equiv (0,0), (1,0) \text{ or } (2,0) \\ & \quad \text{mod } (3\mathbb{Z} \times 2\mathbb{Z}) \\ (1,1) & \text{if } \mathbf{x}_1 \equiv (0,1), (1,1) \text{ or } (2,1) \\ & \quad \text{mod } (3\mathbb{Z} \times 2\mathbb{Z}) \end{cases};$$

- extend $\mathbf{p}$ to a function on $X \times \mathbb{Z}^2$ using the cocycle equation.

Let $(X, \mathbf{S})$ be the $\mathbf{C}$-speedup of $(X, \mathbf{T})$ given by $\mathbf{p}$.

Example of a bounded, non-isomorphic speedup

The speedup **S** has skewing that wasn't present in **T**:

$$
\begin{tikzcd}
\boxed{(0,1)+G_1} \arrow[r, "T_1"] & \boxed{(1,1)+G_1} \arrow[r, "T_1"] & \boxed{(2,1)+G_1} \\
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\boxed{(0,0)+G_1} \arrow[u, "T_2"] \arrow[r, "T_1"] & \boxed{(1,0)+G_1} \arrow[u, "T_2"] \arrow[r, "T_1"] & \boxed{(2,0)+G_1} \arrow[u, "T_2"]
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$$

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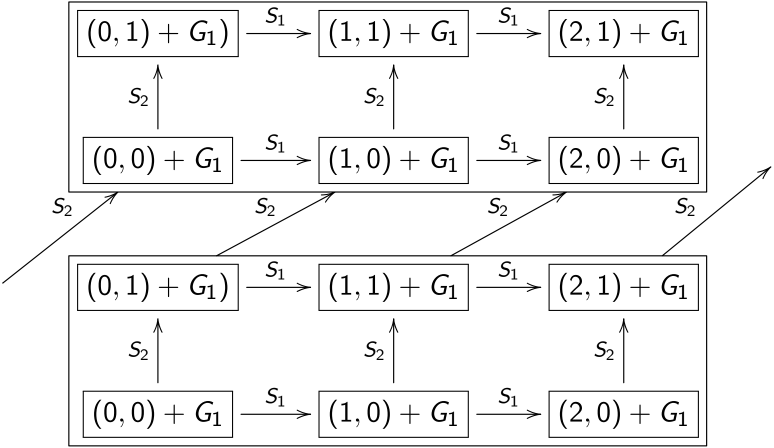

Example of a bounded, non-isomorphic speedup

To show $(X, \mathbf{T})$ and $(X, \mathbf{S})$ are not isomorphic, we use an alternate presentation of $\mathbb{Z}^d$-odometers found by Giordano, Putnam and Skau.

Using Pontryagin duality, they found that a $\mathbb{Z}^d$-odometer $(X, \mathbf{T})$ can be specified by a single subgroup $H(X, \mathbf{T})$ of $\mathbb{Q}^d$.

This group is related to the group cohomology of the action; in fact $H(X, \mathbf{T}) \cong H^1(X, \mathbf{T})$, where $H^1(X, \mathbf{T})$ is the first cohomology group of $\mathbb{Z}^d$ with coefficients in the module $C(X, \mathbb{Z})$.

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Using Pontryagin duality, they found that a $\mathbb{Z}^d$-odometer $(X, \mathbf{T})$ can be specified by a single subgroup $H(X, \mathbf{T})$ of $\mathbb{Q}^d$.

Furthermore, for a $\mathbb{Z}$-odometer (X, T), $H(X, T)$ is isomorphic to the dimension group $D(X, T)$ associated to the odometer.

Example of a bounded, non-isomorphic speedup

In our example, we can compute

$$H(X, \mathbf{T}) = \mathbb{Z}\left[\frac{1}{3}\right] \times \mathbb{Z}\left[\frac{1}{2}\right]$$

$$H(X, \mathbf{S}) = \left\{(x, y) \in \mathbb{Z}^2 : x \in \mathbb{Z}\left[\frac{1}{3}\right], y - \frac{1}{2}x \in \mathbb{Z}\left[\frac{1}{2}\right]\right\}.$$

These groups are isomorphic (which reflects the fact that $(X, \mathbf{T})$ and $(X, \mathbf{S})$ are continuously orbit equivalent).

But they aren't isomorphic in a good enough way...

Example of a bounded, non-isomorphic speedup

In our example, we can compute

$$H(X,\mathbf{T}) = \mathbb{Z}\left[\frac{1}{3}\right] \times \mathbb{Z}\left[\frac{1}{2}\right]$$

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If the odometers $(X,\mathbf{T})$ and $(X,\mathbf{S})$ were isomorphic, then by a theorem of Giordano, Putnam and Skau there would exist a matrix $A \in GL_2(\mathbb{Z})$ such that $A\,H(X,\mathbf{T}) = H(X,\mathbf{S})$.

But no such A exists (elementary linear algebra argument).

Speedups and orbit equivalence

While the $(X, \mathbf{S})$ and $(X, \mathbf{T})$ in the preceding example are not isomorphic, they are orbit equivalent. This holds in general:

Theorem (Johnson-M)

Let $\mathbf{C} \subseteq \mathbb{Z}^d$ be any cone. If $(X, \mathbf{S})$ is a bounded $\mathbf{C}$-speedup of $\mathbb{Z}^d$-odometer $(X, \mathbf{T})$, then $(X, \mathbf{S})$ and $(X, \mathbf{T})$ are orbit equivalent.

When $d = 1$, this implies AAO, because orbit equivalent $\mathbb{Z}$-odometers are automatically conjugate (follows from Boyle-Tomiyama).