Speedups of ergodic $\mathbb{Z}^d-$actions

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Some history

Theorem 1 *(Arnoux, Ornstein, Weiss 1985) Given any two ergodic measure-preserving transformations, there is a speedup of one which is isomorphic to the other.*

This result was a consequence of a theorem in the same paper explaining how arbitrary measure-preserving systems could be represented by models arising from cutting and stacking constructions.

Some terminology

Theorem 1 *Given any two ergodic measure-preserving transformations, there is a speedup of one which is isomorphic to the other.*

A *measure-preserving transformation (m.p.t.)* is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is measurable ($T^{-1}(A) \in \mathcal{X}$ for all $A \in \mathcal{X}$), measure-preserving ($\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{X}$), and $1 - 1$.

An m.p.t. is *ergodic* if its invariant sets all have zero or full measure.

Two m.p.t.s $(X, \mathcal{X}, \mu, T)$ and $(X', \mathcal{X}', \mu', T')$ are *isomorphic* if $\exists$ an isomorphism $\phi : (X, \mathcal{X}, \mu) \to (X', \mathcal{X}', \mu')$ satisfying $\phi \circ T = T' \circ \phi$ for $\mu-$a.e. $x \in X$.

Speedups

Theorem 1 *Given any two ergodic measure-preserving transformations, there is a speedup of one which is isomorphic to the other.*

Given m.p.t.s $(X, \mathcal{X}, \mu, T)$ and $(X, \mathcal{X}, \mu, \overline{T})$, we say $\overline{T}$ is a *speedup* of T if there exists a measurable function $v : X \to \{1, 2, 3, ...\}$ such that $\overline{T}(x) = T^{v(x)}(x)$ a.s.

$$\begin{tikzcd}
\cdots \arrow[r, "T"] & \bullet \arrow[r, "T"] \arrow[rrr, bend left, "\overline{T}"] & \bullet \arrow[r, "T"] \arrow[r, bend right, "\overline{T}"'] & \bullet \arrow[r, "T"] \arrow[rrrr, bend right, "\overline{T}"'] & \bullet \arrow[r, "T"] \arrow[rr, bend left, "\overline{T}"] & \bullet \arrow[r, "T"] & \bullet \arrow[r, "T"] \arrow[rr, bend left, "\overline{T}"] & \bullet \arrow[r, "T"] & \bullet \arrow[r, "T"] & \cdots
\end{tikzcd}$$

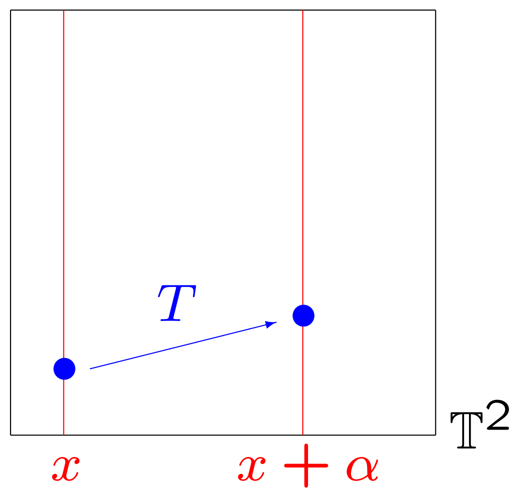
Remark: by definition, speedups are (μ−a.s.) defined on the entire space, preserve μ and are $1-1$.

A relative version of the AOW result

Theorem 2 *(Babichev, Burton, Fieldsteel 2011) Fix a 2nd ct-ble, locally cpct group G. Given any two ergodic group extensions by G, there is a relative speedup of one which is relatively isomorphic to the other.*

Application: Classification of $n-$point and certain countable extensions up to speedup equivalence.

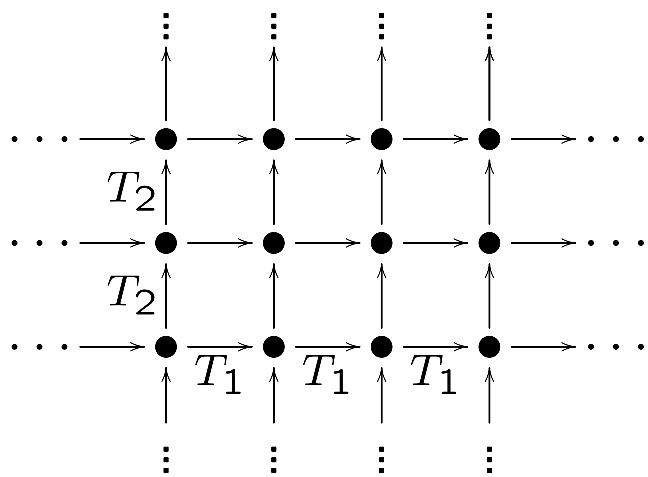
Example of a group extension: $T : \mathbb{T}^2 \to \mathbb{T}^2$ defined by $T(x, y) = (x + \alpha, y + x)$:


T
$\mathbb{T}^2$
x
x + α

What about $\mathbb{Z}^2$ (or $\mathbb{Z}^d$) actions?

Two commuting m.p. transformations T_1 and T_2 on the same space comprise a $\mathbb{Z}^2$–*action* $\mathbf{T}$, where $\mathbf{t} = (t_1, t_2) \in \mathbb{Z}^2$ acts on X by

$$\mathbf{T}_{\mathbf{t}}(x) = T_1^{t_1} T_2^{t_2}(x).$$



Question: What is a “speedup” of such an action?

$\mathbb{Z}^2-$speedups

Definition: A *cone* $\mathbf{C}$ is the intersection of $\mathbb{Z}^2 - \{\mathbf{0}\}$ with any open, connected subset of $\mathbb{R}^2$ bounded by two distinct rays emanating from the origin.

Definition: A $\mathbf{C}-$*speedup* of $\mathbb{Z}^2-$system $\mathbf{T} = (T_1, T_2)$ is another $\mathbb{Z}^2-$system $\overline{\mathbf{T}} = (\overline{T}_1, \overline{T}_2)$ (defined on the same space as $\mathbf{T}$) such that

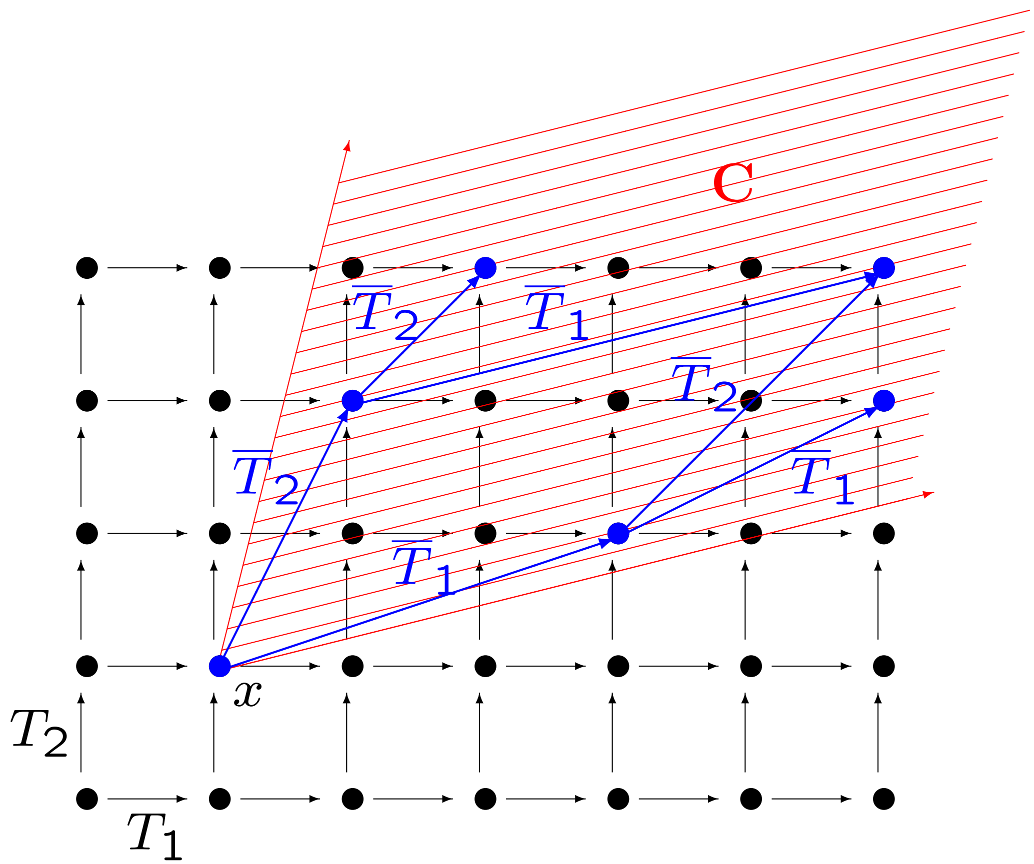
$$\overline{T}_1(x) = T_1^{v_{11}(x)} \circ T_2^{v_{12}(x)}(x)$$

$$\overline{T}_2(x) = T_1^{v_{21}(x)} \circ T_2^{v_{22}(x)}(x)$$

for some measurable function $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) = ((v_{11}, v_{12}), (v_{21}, v_{22})) : X \to \mathbf{C}^2$.

Remark: The $\mathbf{v}$ must be defined so that $\overline{T}_1$ and $\overline{T}_2$ commute (so one cannot simply speed up T_1 and T_2 independently to obtain a speedup of $\mathbf{T}$).

A picture to explain



C
$\overline{T}_2$
$\overline{T}_1$
$\overline{T}_2$
$\overline{T}_2$
$\overline{T}_1$
$\overline{T}_1$
x
T_2
T_1

Here, $\overline{\mathbf{T}} = (\overline{T}_1, \overline{T}_2)$ is a $\mathbf{C}-$speedup of $\mathbf{T} = (T_1, T_2)$. In particular, for the indicated point x, we have

$$\mathbf{v}(x) = ((3, 1), (1, 2)).$$

Group extensions of $\mathbb{Z}^d$ actions

A *cocycle* for $\mathbb{Z}^d$–action $(X, \mathcal{X}, \mu, \mathbf{T})$ is a measurable function $\sigma : X \times \mathbb{Z}^d \to G$ satisfying

$$\sigma_{\mathbf{v}}(\mathbf{T}_{\mathbf{w}}(x))\,\sigma_{\mathbf{w}}(x) = \sigma_{\mathbf{v}+\mathbf{w}}(x)$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^2$ and (almost) all $x \in X$.

(Here we denote $\sigma(x, \mathbf{v})$ by $\sigma_{\mathbf{v}}(x)$.)

Each cocycle σ generates a *G–extension* of $\mathbf{T}$, i.e. a $\mathbb{Z}^d$–action $(X \times G, \mathcal{X} \times \mathcal{G}, \mu \times Haar, \mathbf{T}^\sigma)$ defined by setting

$$\mathbf{T}^\sigma_{\mathbf{v}}(x, g) = (\mathbf{T}_{\mathbf{v}}(x), \sigma_{\mathbf{v}}(x)g)$$

for each $\mathbf{v} \in \mathbb{Z}^d$.

(Different σ may yield different G–extensions $\mathbf{T}^\sigma$ for the same "base action" $\mathbf{T}$.)

Our main result

Theorem 3 *(Johnson-M)* *Let G be a locally compact, second countable group. Given any two ergodic $\mathbb{Z}^d$−group extensions $\mathbf{T}^\sigma$ and $\mathbf{S}^\sigma$, and given any cone $\mathbf{C} \subseteq \mathbb{Z}^d$, there is a relative $\mathbf{C}$−speedup of $\mathbf{T}^\sigma$ which is relatively isomorphic to $\mathbf{S}^\sigma$.*

What follows is a sketch of the proof of this theorem when $d = 2$ and $G = \{e\}$ (with occasional brief remarks about what changes in the proof for more general G.)

We will refer to $\mathbf{T}^\sigma$ as the *bullet action* and $\mathbf{S}^\sigma$ as the *target action* . The goal will be **to speed up the bullet, so that it is isomorphic to the target.**

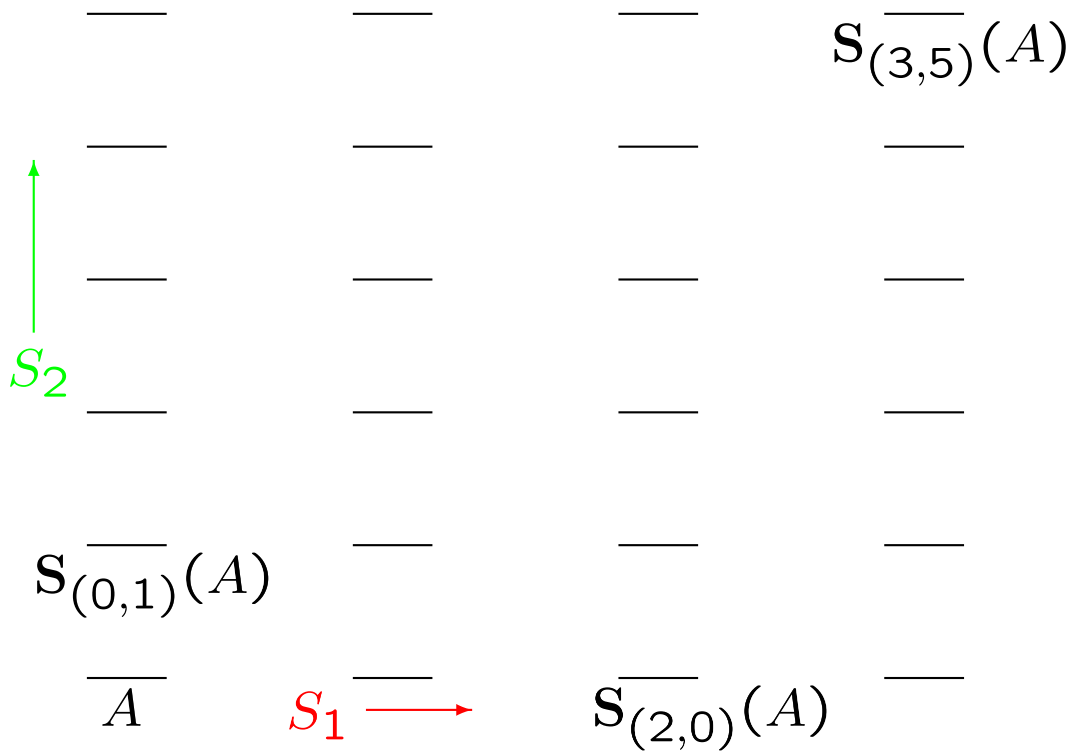
Preliminaries: Rohklin towers

A *Rohklin tower* τ for a m.p. $\mathbb{Z}^d-$action $(Y, \mathcal{Y}, \nu, \mathbf{S})$ is a collection of disjoint measurable sets of the form

$$\{\mathbf{S}_{(j_1, j_2, \ldots, j_d)}(A) : 0 \leq j_i < n_i \, \forall i\}$$

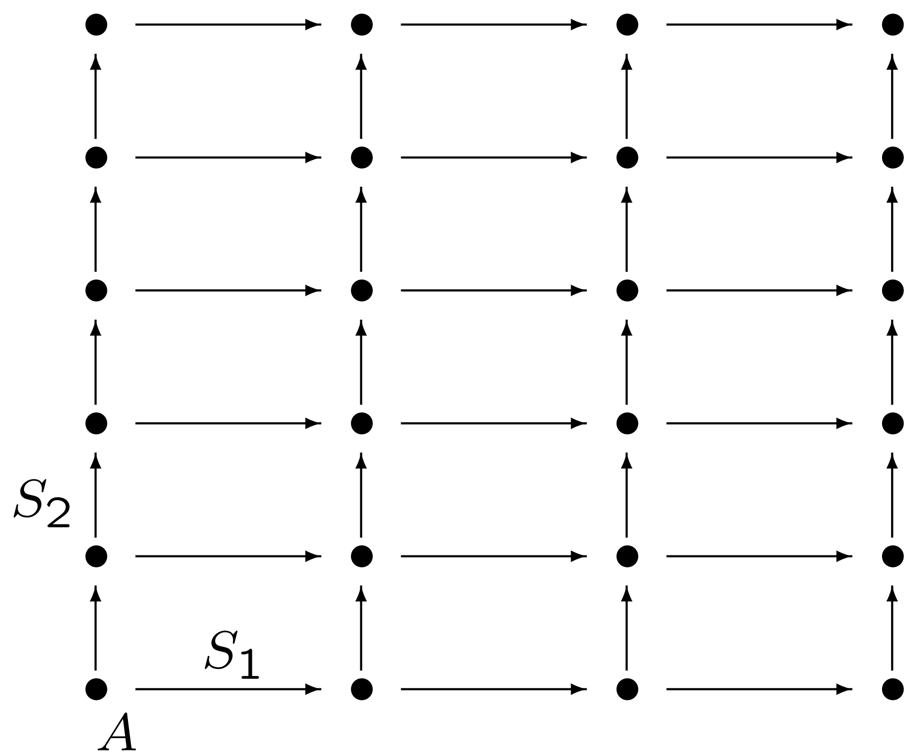
for some $A \in \mathcal{Y}$ with $\nu(A) > 0$. We refer to $\mathbf{n} = (n_1, ..., n_d)$ as the *size* of the Rohklin tower.

Here is a tower (in $d = 2$) of height $(4, 6)$:


$\mathbf{S}_{(3,5)}(A)$
S_2
$\mathbf{S}_{(0,1)}(A)$
A
S_1
$\mathbf{S}_{(2,0)}(A)$

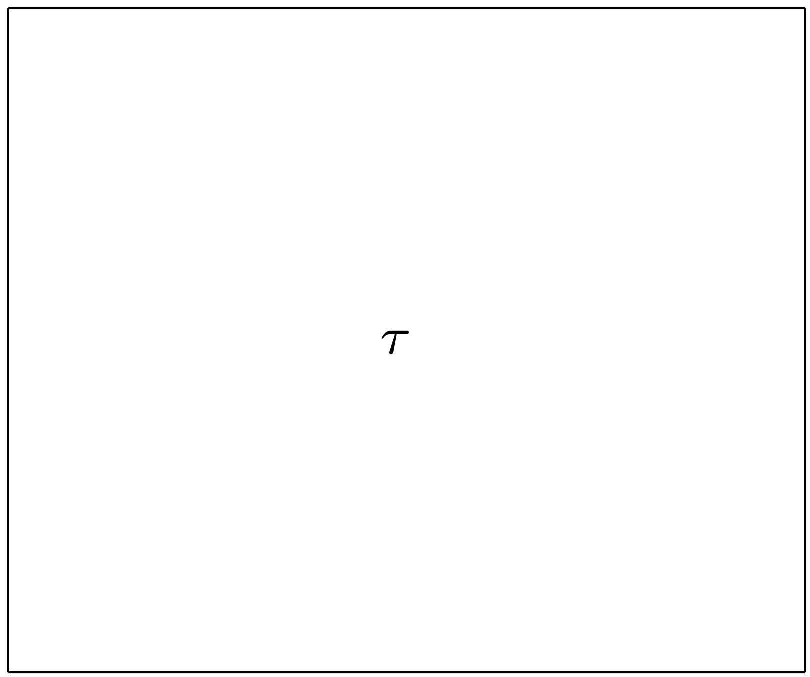
Preliminaries: Rohklin towers

Let's represent the same tower this way (each dot represents a set):


S_2
S_1
A

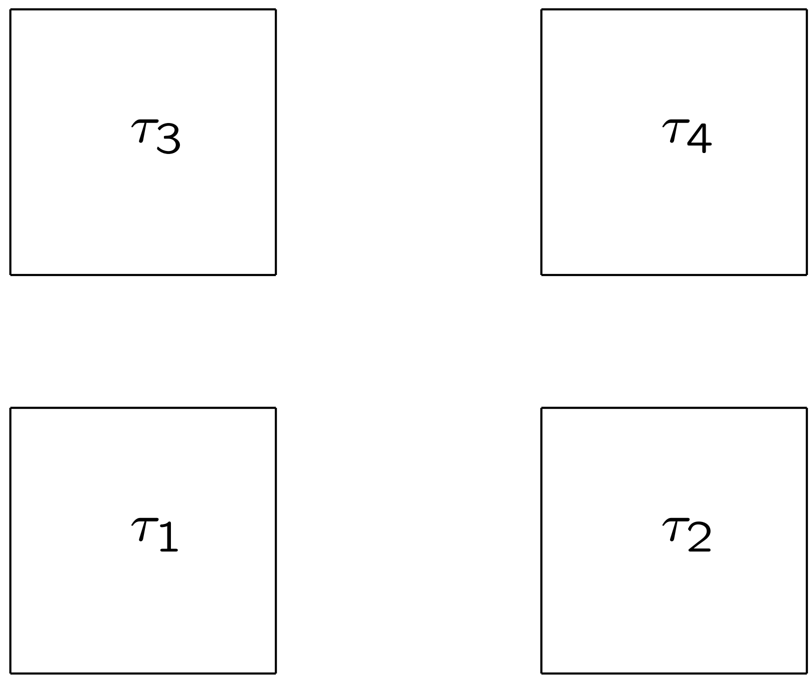
Preliminaries: Rohklin towers

Even better, let's just think of a tower as a picture like this (in reality, this rectangle is an array of sets mapped to each other by $\mathbf{S}$):


τ

Preliminaries: Castles

A *castle* $\mathcal{C}$ for a m.p. $\mathbb{Z}^d-$action $(Y, \mathcal{Y}, \nu, \mathbf{S})$ is a collection of finitely many disjoint Rohklin towers:


τ_3
τ_4
τ_1
τ_2

Step 1: generate the target action via cutting and stacking of castles

Lemma 1 *(essentially AOW) Let* $\mathbf{S}$ *be a* $\mathbb{Z}^d-$ *action. Then there is a sequence* $\{\mathcal{C}_k\}_{k=1}^{\infty}$ *of castles for* $\mathbf{S}$ *with the following properties:*

1. *For each* k*, all the towers comprising* $\mathcal{C}_k$ *have the same height.*
2. *Each* $\mathcal{C}_{k+1}$ *is obtained from* $\mathcal{C}_k$ *via cutting and stacking (thus* $\mathcal{C}_k \subseteq \mathcal{C}_{k+1}$*);*
3. $\nu\left(\bigcup_{k=1}^{\infty} \mathcal{C}_k\right) = 1$*;*
4. *The levels of the towers of all of the* $\mathcal{C}_k$ *generate* $\mathcal{Y}$*.*

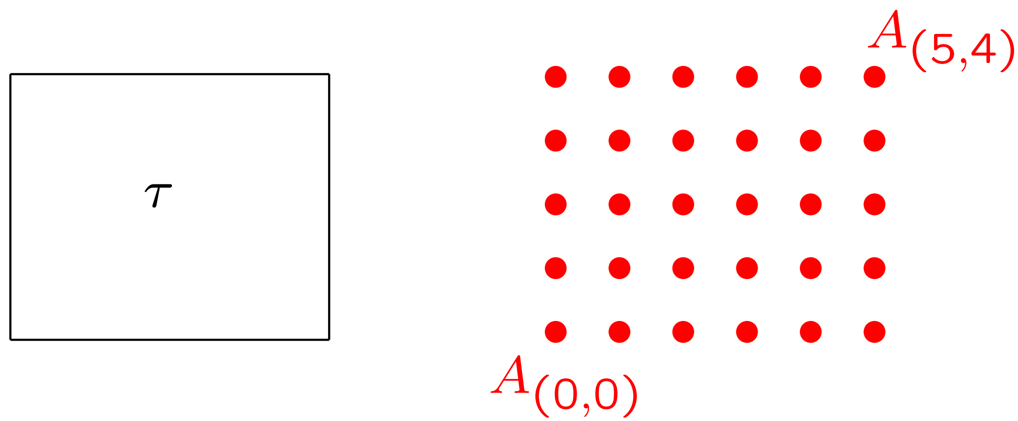
(We actually require a bit more than this if $G \neq \{e\}$.)

Step 2: choose sets in the bullet action to mimic the first castle

Start with castle $\mathcal{C}_1$ for $(Y, \mathcal{Y}, \nu, \mathbf{S})$. For each level L of each tower in $\mathcal{C}_1$, choose a measurable set of X with measure equal to the measure of L. Choose these sets so that they are all disjoint, and index them in the same way the levels of $\mathcal{C}_1$ are arranged.

Given tower $\tau \in \mathcal{C}_1 \subseteq Y \ldots$

$\ldots$ choose sets $A_{\mathbf{v}} \subseteq X$



Step 3: arrange the sets so that they form orbits of a partially defined speedup of the bullet action

Lemma 2 *Given disjoint, measurable subsets* $\{A_{(j_1,j_2)}\}_{0\le j_1<n_1, 0\le j_2<n_2}$ *of* X*, each having the same measure, one can* build a partial speedup of $\mathbf{T}$ on the sets*, i.e. construct measurable functions* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *taking values in* $\mathbf{C}$ *so that:*

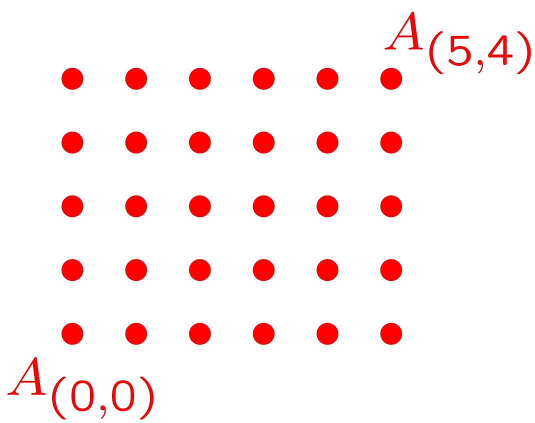
1. $\mathbf{T}_{\mathbf{v}_1}(A_{(j_1,j_2)}) = A_{(j_1+1,j_2)}$ *(a.s.);*
2. $\mathbf{T}_{\mathbf{v}_2}(A_{(j_1,j_2)}) = A_{(j_1,j_2+1)}$ *(a.s.);*
3. $\mathbf{T}_{\mathbf{v}_1} \circ \mathbf{T}_{\mathbf{v}_2} = \mathbf{T}_{\mathbf{v}_2} \circ \mathbf{T}_{\mathbf{v}_1}$.
4. *(Also, extra stuff if* $G \neq \{e\}$*.)*

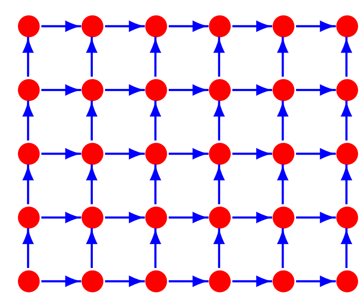
Given sets

$A_{\mathbf{v}} \subseteq X \ldots$

... construct

$\overline{\mathbf{T}}_1 = (\mathbf{T}_{\mathbf{v}_1}, \mathbf{T}_{\mathbf{v}_2})$

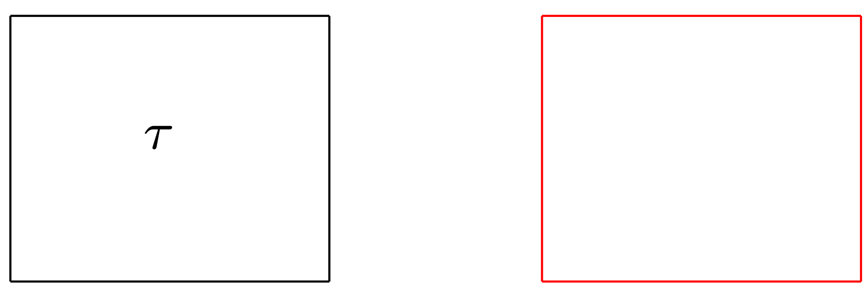




Step 3 continued

After repeating steps 1 and 2 for each tower in $\mathcal{C}_1$, we get a partially defined speedup $\overline{\mathbf{T}}_1$ of $\mathbf{T}$ which is "level-wise isomorphic" to the action of $\mathbf{S}$ on its castle $\mathcal{C}_1$.

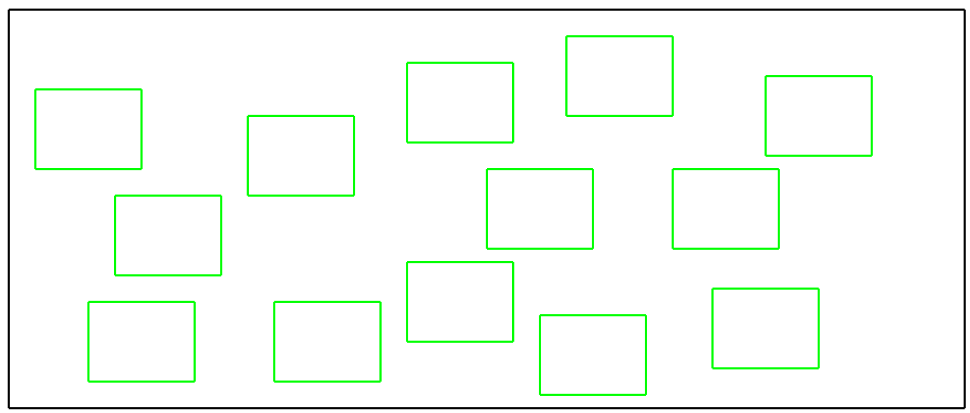
Given each tower for $\mathbf{S}$...	... we get a tower $\{A_{\mathbf{v}}\}$ for $\overline{\mathbf{T}}_1$



Step 4: from one castle to the next

Suppose we have produced a partially defined speedup $\overline{\mathbf{T}}_k$ of $\mathbf{T}$ which is isomorphic to $\mathbf{S}$ on the levels of the towers of some castle $\mathcal{C}_k$.

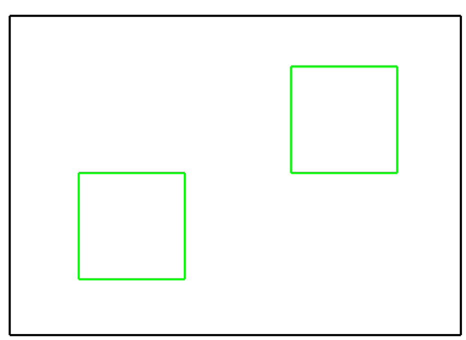
Recall that each $\mathcal{C}_{k+1}$ is obtained from $\mathcal{C}_k$ by cutting and stacking. Thus we can view $\mathcal{C}_{k+1}$ as a collection of towers that look like this, where the green towers are towers in $\mathcal{C}_k$:



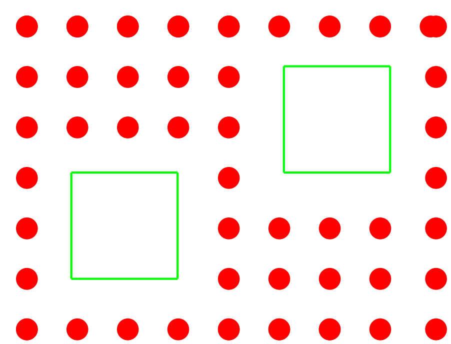
Step 4: from one castle to the next

Pick measurable sets of X (disjoint from each other and from the previously chosen sets) corresponding to the levels of these towers which were not in the previous tower (i.e. weren't green).

Given this tower in $\mathcal{C}_{k+1}$...



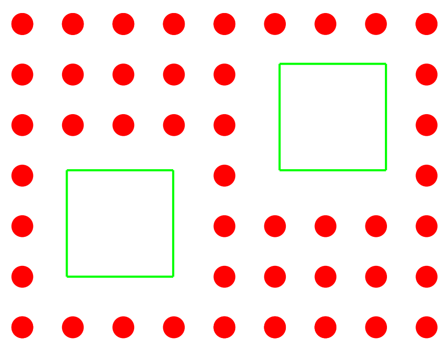
...choose sets in X indicated by dots



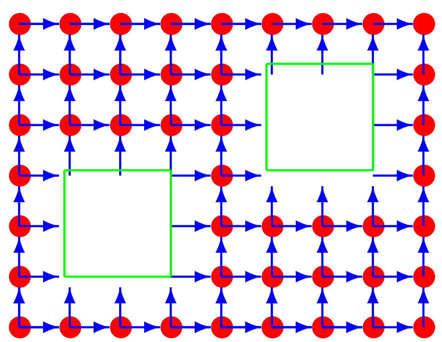
Step 4: from one castle to the next

Theorem 4 (Quilting Theorem) *(J-M) Given the picture described on the previous slide, one can build a partial* $\mathrm{C}-$*speedup on all the subsets of* X *which extends all the partial speedups already constructed on the green "patches".*

Given this...



...build $\overline{\mathbf{T}}_{k+1}$:



This produces a partially-defined $\mathbf{C}-$speedup $\overline{\mathbf{T}}_{k+1}$ of $\mathbf{T}$ extending $\overline{\mathbf{T}}_k$, which is "level-wise isomorphic" to the action of $\mathbf{S}$ on its castle $\mathcal{C}_{k+1}$.

Step 5: repeat procedure of step 4 indefinitely

This produces a sequence of partially-defined speedups $\overline{\mathbf{T}}_k$ of $\mathbf{T}$, defined on more and more of X. Since the union of the castles $\mathcal{C}_k$ has full measure, we obtain a speedup

$$\overline{\mathbf{T}} = \lim_{k\to\infty} \overline{\mathbf{T}}_k$$

which is defined a.e. on X.

Since $\overline{\mathbf{T}}_k$ is level-wise isomorphic to the action of $\mathbf{S}$ on the levels of $\mathcal{C}_k$, and the levels of the castles generate the full $\sigma-$algebra $\mathcal{Y}$, we obtain $\overline{\mathbf{T}} \cong \mathbf{S}$.