A tour of dynamical systems

David M. McClendon

Swarthmore College
Swarthmore, PA

Ferris State University
January 19, 2012

Some motivation

Consider the following questions, taken from math, physics and other areas:

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From numerical analysis

Determine which root (if any) of a function Newton's method converges to, given a particular "initial guess" of the root.

(Newton's method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$)

Some motivation

Consider the following questions, taken from math, physics and other areas:

From additive combinatorics

Prove that if you arbitrarily color the integers (using a finite set of crayons), then there must be a monochromatic arithmetic progression of arbitrarily long length.

An *arithmetic progression* is a list like $7, 11, 15, 19, 23, 27$ (this progression has length 6).

Some motivation

Consider the following questions, taken from math, physics and other areas:

From economics

Predict the price of a stock three weeks from now.

Some motivation

Consider the following questions, taken from math, physics and other areas:

From population biology

Given rates of reproduction and predation, describe fluctuations in the population of a species in a particular ecosystem as time passes.

Some motivation

Consider the following questions, taken from math, physics and other areas:

From physics

Explain ferromagnetism (how materials become magnets) via a mathematical model.

Some motivation

Consider the following questions, taken from math, physics and other areas:

All of these problems can be approached mathematically using techniques of *dynamical systems*.

Dynamical systems

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1. The phase space

The *phase space* X of a dynamical system is the set of all possible "positions" or "states" of the system.

For example, if the system is keeping track of the price of a stock, X is the set of all possible stock prices.

Loosely speaking, a *dynamical system* is anything quantifiable that changes over time. To formulate such an object mathematically, we need two things:

2. The evolution rule

The *evolution rule* T of a dynamical system is a function $T : X \to X$ that tells you, given your current state x, your state one unit of time from now.

For example, if the system is keeping track of a stock price, if the current price is 30, then $T(30)$ would be the price of the stock tomorrow (if time is measured in days).

Dynamical systems

Loosely speaking, a *dynamical system* is anything quantifiable that changes over time. To formulate such an object mathematically, we need two things:

Definition

A *(discrete) dynamical system* is be a pair (X, T) where X is some set and T is a function from X to itself.

(Usually one requires that X and T have some additional structure.)

Iterates

Given a dynamical system (X, T) and a point $x \in X$:

- $x =$ your present state
- $T(x) =$ your state one unit of time from now
- $T(T(x)) = T \circ T(x) =$ your state two units of time from now
- etc.

Definition

We define $T^n(x) = T \circ T \circ \cdots \circ T(x)$; therefore $T^n(x)$ is the state n units of time from now if x is your current state. T^n is called the n^{th} *iterate* of T.

Major problems in dynamical systems

Major problems in dynamical systems

Prediction problems

Given a dynamical system (X, T) and a point $x \in X$, predict $T^n(x)$ for large values of n.

- Do the numbers $x, T(x), T^2(x), T^3(x), ...$ follow a pattern?
- Do the numbers $T^n(x)$ have a limit as $n \to \infty$?
- If x is changed slightly, do the numbers $x, T(x), T^2(x), T^3(x), ...$ stay pretty much the same, or do they become drastically different?

Major problems in dynamical systems

Prediction problems

Frequently it is impossible to predict $T^n(x)$ for large n, in which case the question becomes one of explaining why such prediction is impossible (chaos theory).

Major problems in dynamical systems

Classification problems

- Given two dynamical systems, are they the same up to a change of language (i.e. isomorphic) or different?
- Are they same up to some weaker notion of equivalence?
- What are their commonalities?
- What are their differences?

Major problems in dynamical systems

Classification problems

- Given two dynamical systems, are they the same up to a change of language (i.e. isomorphic) or different?
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- What are their commonalities?
- What are their differences?

To approach this question, we invent useful vocabulary to describe various phenomena that might occur in a system.

Applications

Math, physics, biology, computer science, economics, etc.

Some examples

Example 1

Let $X = \mathbb{R}$ and let $T(x) = -x$.
Then $T^2(x) = T(T(x)) = -(-x) = x$, and similarly

$$T^n(x) = \begin{cases} x & \text{if } n \text{ is even} \\ -x & \text{if } n \text{ is odd} \end{cases}$$

Some examples

Example 1

In terms of "arrows", we see this dynamics:

$$\ldots \to x \to -x \to x \to -x \to x \to -x \to x \to \ldots$$

where T takes each point to the right by one arrow, and moving by n arrows corresponds to the passage of n units of time.

Some examples

Example 1

So it is easy to describe the behavior of x as time passes.

Example 2

Let $X = \mathbb{R}$ and let $T(x) = \frac{1}{2}x$.
Then $T^2(x) = \frac{1}{4}x$ and similarly $T^n(x) = \frac{1}{2^n}x$ for all x and n, and we see

$$\lim_{n\to\infty} T^n(x) = 0$$

no matter what x is.

In particular, changing the value of x a little bit doesn't affect the values of $T^n(x)$ much (they are approaching 0).

Some examples

Example 3

Let $X = [0,1]$ and let $T(x) = 4x(1-x)$.

Some examples

Example 3

Let $X = [0, 1]$ and let $T(x) = 4x(1 - x)$.

Let $x = .345$. Then the iterates of x are...

Some examples

Example 3

$\{0.345, 0.9039, 0.347459, 0.906925, 0.337648, 0.894567, 0.377268,$
$0.939747, 0.226489, 0.700766, 0.838772, 0.540934, 0.993298,$
$0.0266299, 0.103683, 0.371731, 0.934188, 0.245922, 0.741777,$
$0.766176, 0.716602, 0.812334, 0.60979, 0.951784, 0.183564, 0.59947,$
$0.960421, 0.152052, 0.515728, 0.999011, 0.00395398, 0.0157534,$
$0.0620209, 0.232697, 0.714197, 0.816479, 0.599364, 0.960507,$
$0.151732, 0.514838, 0.999119, 0.00351956, 0.0140287, 0.0553275,$
$0.209065, 0.661428, 0.895764, 0.373485, 0.935976, 0.2397, 0.728977,$
$0.790279, 0.662953, 0.893786, 0.379731, 0.942142, 0.218042, 0.682,$
$0.867505, 0.459761, 0.993523, 0.0257389, 0.100306, 0.360978, \ldots\}$

Example 3

In particular, the numbers have no discernable pattern.

What's more, is that if you change x from .345 to something like .346, the iterates you obtain from the new x look nothing like the iterates you obtain from the old x.

What can you do with dynamics?

Direct applications of the prediction problem

1. Predict prices of stocks (up to a point)
2. Predict the paths of hurricanes (up to a point)
3. Predict the outcome of Newton's method
4. Explain ferromagnetism (via the Ising model)
5. Model sports, including American football (my former undergraduate student K. Goldner)

What can you do with dynamics?

Other things

1. Study tilings of the plane (related: crystals and quasicrystals)
2. Explain the recurrence of particular geometric patterns in Islamic architecture
3. Find patterns in certain sets of numbers (like arithmetic progressions)
4. Solve Diophantine approximation problems (Oppenheim conjecture)
5. Draw cool pictures of fractals (Mandelbrot and Julia sets)

What can you do with dynamics?

Solve famous math problems

The Poincaré conjecture, which states that every simply connected, closed, 3−dimensional manifold is homeomorphic to a sphere, was solved by Perelman (2006) by studying the properties of a dynamical system called the *Ricci flow*.

What can you do with dynamics?

Solve famous problems and make money

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The Poincaré conjecture is one of the "Millenium Prize Problems"; solving the Poincaré conjecture made Perelman eligible for a $\$1,000,000$ prize.

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Perelman turned down the prize money.

What can you do with dynamics?

Make money

The search engine Google ranks pages using a mechanism coming from a specific kind of dynamical system called a *Markov chain*.

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Sergey Brin and Larry Page, the inventors of Google, have a combined personal wealth of $33 billion as of 2011.

What do I study?

With my colleague Aimee S.A. Johnson, I am studying a modified version of the classification problem:

Motivating question

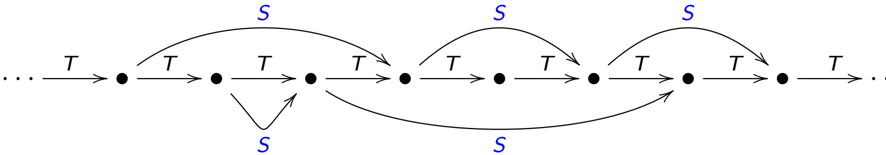
Suppose you are given two dynamical systems. Can you modify system number 1 into an isomorphic copy of system number 2, only by very slightly changing the evolution rule of system number 1?

Speedups

Let (X, T) be a dynamical system. To each point in X, assign a number in $\{1, 2, 3, 4, ...\}$; call that number $v(x)$. Create a new dynamical system (X, S) by defining

$$S(x) = T^{v(x)}(x).$$

Such an S is called a *speedup* of T:



Notice that whenever $v(x) = 1$, the function S coincides with T.

Speedups

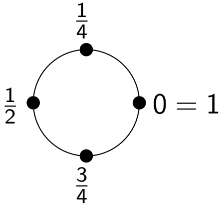
Question

Suppose you are given two dynamical systems. Is there a speedup of one system which is an isomorphic copy of the other?

Two systems that are very different

Example 1: Circle rotation

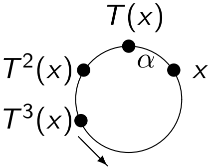
Let $X = S^1$ (a circle), where points are labeled by their angle measure in "units", where 1 unit corresponds to 2π radians or 360°. (In other words, $X = \mathbb{R}/\mathbb{Z}$.)



Example 1: Circle rotation

Define $T : X \to X$ by $T(x) = x + \alpha$ where $\alpha \in (0,1)$ is irrational.

(If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then $T^q(x) = x + p = x$ for all x, so the dynamics are "trivial".)



Two systems that are very different

Example 1: Circle rotation

Suppose we take two small arcs $A, B \subseteq X$. I want to describe whether A "mixes" with B as time passes. To do this, consider the size (arc length) of $T^n(A) \cap B$ as n increases:

Two systems that are very different

Example 1: Circle rotation

For the choices of α, A and B on the board, notice that

$$\lim_{n \to \infty} \mathit{length}(T^n(A) \cap B)$$

doesn't exist; indeed, the numbers $\mathit{length}(T^n(A) \cap B)$ are mostly zero but are occasionally close to the length of B.

Two systems that are very different

Example 1: Circle rotation

Because $T^n(A)$ doesn't consistently overlap with B for large choices of t, we say A and B are not *mixed* by T. In fact, circle rotations almost never mix any two sets.

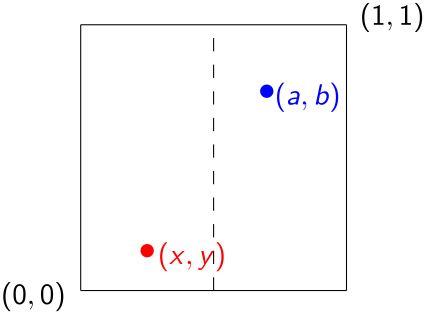
Example 2: Baker's transformation

Let $X = [0,1) \times [0,1)$ (a square) and define $T : X \to X$ by

$$T(x,y) = \begin{cases} (2x, \frac{1}{2}y) & \text{if } x < \frac{1}{2} \\ (2x - \frac{1}{2}, \frac{1}{2}y + \frac{1}{2}) & \text{if } x \geq \frac{1}{2} \end{cases}$$

This (X, T) is called the *baker's transformation*.

Example 2: Baker's transformation



Two systems that are very different

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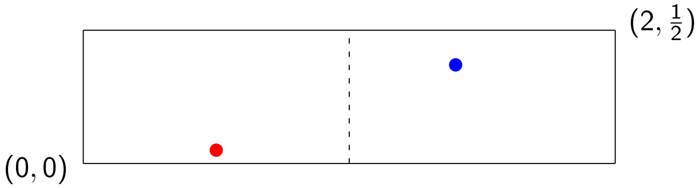

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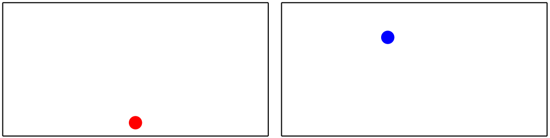

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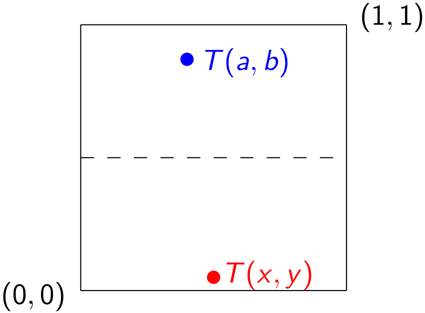
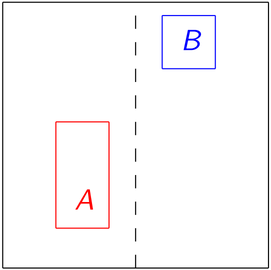


Two systems that are very different

Example 2: Baker's transformation

Let's see what happens to a set A under iteration by the baker's transformation.

Example 2: Baker's transformation



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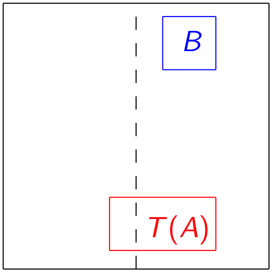

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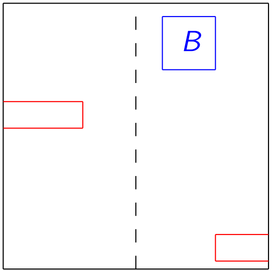
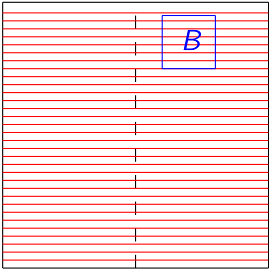


$T^2(A)$

Two systems that are very different

Example 2: Baker's transformation



$T^n(A)$, n large

Example 2: Baker's transformation

In particular, given any two sets A and B, you can show that

$$\lim_{n\to\infty} area(T^n(A)\cap B) = area(A)\cdot area(B).$$

Probabilistically, this means the probability you are eventually in B is independent of whether or not your present state is in A.

We say the baker's transformation is *mixing*.

Two systems that are very different

Circle rotations (*extremely* non-mixing) and the baker's map (*extremely* mixing) could not be more different.

How could you possibly speed up a rotation so that the sped-up rotation mixes sets?

Speedup isomorphism theorem

Theorem (AOW 1985; BBF 2011)

Given two ergodic dynamical systems, there is a speedup of one which is isomorphic to the other.

Moreover, for any $\epsilon > 0$ the function v defining the speedup can be constructed so that it takes the value 1 except on a set of size at most ϵ.

(and important but technical strengthenings)

Definition

To say a dynamical system is *ergodic* means it cannot be decomposed into two pieces, each of positive size, which do not interact with one another under iteration.

Definition

The collective action of d commuting functions on the same space is called a $\mathbb{Z}^d-$dynamical system.

If we call the generators of the action $T_1, ..., T_d$, then $\mathbf{t} = (t_1, ..., t_d)$ acts on X by $T^{\mathbf{t}}(x) = T_1^{t_1} \circ \cdots \circ T_d^{t_d}(x)$.

These systems are studied in the context of tilings, the Ising model of ferromagnetism, applications to additive combinatorics and graph theory, simultaneous Diophantine approximation, etc.

Theorem (Johnson-M)

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$\mathbb{Z}^d$ −dynamical systems

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