Dynamical systems and van der Waerden's theorem

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Dynamical systems

Loosely speaking, a “dynamical system” is a mathematical model for anything that changes as time passes.

Things modeled by dynamical systems

1. (Economics) the value of a stock or commodity
2. (Biology) the deer population in western Michigan
3. (Meteorology) the temperature at a fixed spot
4. (Astronomy) the position of a comet
5. (Physics) the motion of a pendulum

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To formulate such an object mathematically, we need two things:

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To formulate such an object mathematically, we need two things:

1. The phase space

The **phase space** X of a dynamical system is the set of all possible "positions" or "states" of the system.

For example, if the system is keeping track of the price of a stock as time passes, X is the set of all possible stock prices.

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To formulate such an object mathematically, we need two things:

2. The evolution rule

The **evolution rule** or **transformation** T of a dynamical system is a function $T : X \to X$ that tells you, given your current state x, your state one unit of time from now.

For example, if the system is keeping track of a stock price, if the current price is 30, then $T(30)$ would be the price of the stock tomorrow (if time is measured in days).

Dynamical systems

Loosely speaking, a "dynamical system" is a mathematical model for anything that changes as time passes.

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A **(discrete) dynamical system** is a pair (X, T) where X is some set and T is a function from X to itself.

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Unfortunately, this is too general a situation to say much mathematically, so usually one requires that X and T have some additional "structure".

Additional structures

Each additional "structure" you might require on X and T gives rise to a different subfield of dynamical systems:

Subfields within dynamical systems

1. **One-dimensional dynamics:** $X \subseteq \mathbb{R}$ or S^1
2. **Smooth dynamics:** X is a manifold; T differentiable
3. **Complex dynamics:** $X = \mathbb{C}$; T rational map
4. **Ergodic theory:** X is a measure space; T is a measure-preserving transformation
5. **Algebraic dynamics:** X is a quotient of a Lie group; T is a translation
6. **Topological dynamics:** X is a compact metric space; T is continuous

Definition

A **topological dynamical system (t.d.s.)** is a pair (X, T) where X is a compact metric space and T is a continuous function from X to itself.

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A set X is a **metric space** if there is a function d which measures the distance between points in a reasonable way:

$$d(x, y) = \text{the distance between } x \text{ and } y$$

Definition

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I won't tell you exactly what **compact** means here, but think of a compact space as one that is "closed" (i.e. contains all its boundary points) and "bounded" (i.e. you can enclose the set in a circle/sphere of finite radius).

Definition

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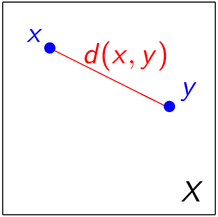

Definition

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A function $T : X \to X$ is called **continuous** if whenever points x and y are sufficiently close to one another, the points $T(x)$ and $T(y)$ can't be too far apart.

More precisely, this means that for every number $\epsilon > 0$, there is a corresponding number $\delta > 0$ such that whenever $d(x, y) < \delta$, it must be that $d(T(x), T(y)) < \epsilon$.

Topological dynamical systems

Definition

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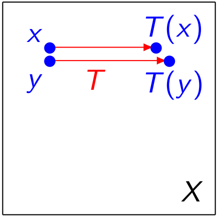


Given a dynamical system (X, T) and a point $x \in X$:

- $x =$ your present state
- $T(x) =$ your state one unit of time from now
- $T(T(x)) = T \circ T(x) =$ your state two units of time from now
- $T(T(T(x))) = T \circ T \circ T(x) = T^3(x)$
- etc.

Definition

We define $T^n(x) = T \circ T \circ \cdots \circ T(x)$; therefore $T^n(x)$ is the state n units of time from now if x is your current state. T^n is called the n^{th} **iterate** of T.

Major problems in dynamical systems

Major problems in dynamical systems

Prediction problems

Given a dynamical system (X, T) and a point $x \in X$, predict $T^n(x)$ for large values of n.

- Do the numbers $x, T(x), T^2(x), T^3(x), \ldots$ follow a pattern?
- Do the numbers $T^n(x)$ have a limit as $n \to \infty$?
- If x is changed slightly, do the numbers $x, T(x), T^2(x), T^3(x), \ldots$ stay pretty much the same, or do they become drastically different?

Prediction problems

Frequently it is impossible to predict $T^n(x)$ for large n, in which case the question becomes one of explaining why such prediction is impossible (chaos theory).

Prediction problems have applications in math, physics, biology, computer science, economics, etc.

An example where prediction is easy

Let $X = [0, \infty)$ and let $T(x) = x^2$. Then:

- If $x = 1$, then $T^n(x) = 1$ for all n.
- If $x < 1$, then $T^n(x) \to 0$ as $n \to \infty$.
- If $x > 1$, then $T^n(x) \to \infty$ as $n \to \infty$.

Major problems in dynamical systems

An example where prediction is hard

Let $X = [0, 1]$ and let $T(x) = 4x(1 - x)$. Then if $x = .345$, the iterates of x are ...

An example where prediction is hard

$\{0.345, 0.9039, 0.347459, 0.906925, 0.337648, 0.894567, 0.377268,$
$0.939747, 0.226489, 0.700766, 0.838772, 0.540934, 0.993298,$
$0.0266299, 0.103683, 0.371731, 0.934188, 0.245922, 0.741777,$
$0.766176, 0.716602, 0.812334, 0.60979, 0.951784, 0.183564, 0.59947,$
$0.960421, 0.152052, 0.515728, 0.999011, 0.00395398, 0.0157534,$
$0.0620209, 0.232697, 0.714197, 0.816479, 0.599364, 0.960507,$
$0.151732, 0.514838, 0.999119, 0.00351956, 0.0140287, 0.0553275,$
$0.209065, 0.661428, 0.895764, 0.373485, 0.935976, 0.2397, 0.728977,$
$0.790279, 0.662953, 0.893786, 0.379731, 0.942142, 0.218042, 0.682,$
$0.867505, 0.459761, 0.993523, 0.0257389, 0.100306, 0.360978, \ldots\}$

An example where prediction is hard

So if $X = [0,1]$, $T(x) = 4x(1-x)$ and $x = .345$, the iterates of x have no discernable pattern.

What's more, is that if you change x from .345 to something like .346, the iterates you obtain from the new x look nothing like the iterates you obtain from the old x.

Major problems in dynamical systems

Classification problems

- Given two dynamical systems, are they the same up to a change of language (i.e. isomorphic) or different?
- Are they same up to some weaker notion of equivalence?
- What are their commonalities?
- What are their differences?

Major problems in dynamical systems

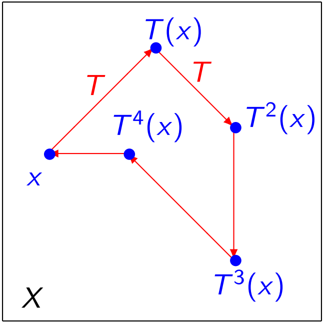
Classification problems

- Given two dynamical systems, are they the same up to a change of language (i.e. isomorphic) or different?
- Are they same up to some weaker notion of equivalence?
- What are their commonalities?
- What are their differences?

To approach this question, we invent useful vocabulary to describe various phenomena that might occur in a system.

Definition

Let (X, T) be a dynamical system. A point $x \in X$ is called **periodic** if $T^n(x) = x$ for some $n \geq 1$.



$x = T^5(x)$

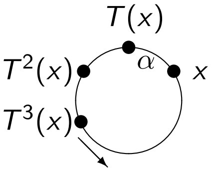
An example of vocabulary

Definition

Let (X, T) be a dynamical system. A point $x \in X$ is called **periodic** if $T^n(x) = x$ for some $n \geq 1$.

Example: circle rotation

Let X be a circle (label points by their angle measure in degrees) and let $T(x) = x + \alpha$.



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Example: circle rotation

Let X be a circle (label points by their angle measure in degrees) and let $T(x) = x + \alpha$.

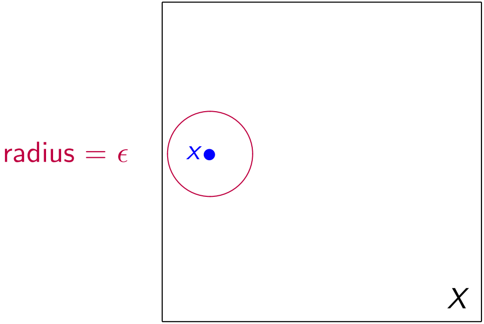
Exercise: Show that if $\alpha \in \mathbb{Q}$, every point $x \in X$ is periodic, but if $\alpha \notin \mathbb{Q}$, no points in X are periodic.

Consequence: Rotations by irrational angles are very different than rotations by rational angles.

Another example of vocabulary

Definition

Let (X, T) be a t.d.s. A point $x \in X$ is called **recurrent** if for every $\epsilon > 0$, there is $n > 0$ such that $d(T^n(x), x) < \epsilon$.



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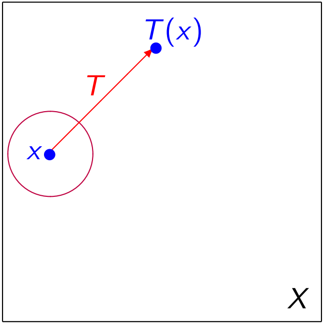
radius $= \epsilon$



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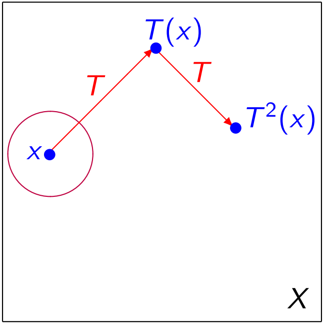

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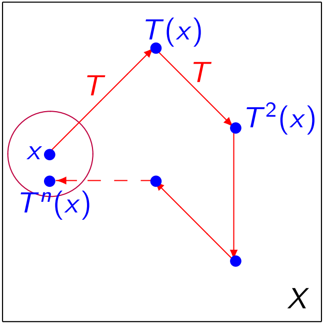

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Exercise: Prove that in a circle rotation, every point is recurrent (irrespective of whether you rotate by a rational or irrational number of degrees).

The notion of recurrence, for example, distinguishes circle rotations from dynamical systems which have points which are not recurrent.

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The notion of recurrence, for example, distinguishes circle rotations from dynamical systems which have points which are not recurrent.

Question

Is there a t.d.s. with no recurrent points?

Theorem

Let (X, T) be a t.d.s. Then there is a recurrent point x.

Proof sketch (for experts only): Consider the family $\mathcal{F}$ of closed, nonempty subsets Y of X satisfying $T(Y) \subseteq Y$. Partially order the sets in $\mathcal{F}$ by inclusion; by Zorn's Lemma $\mathcal{F}$ has a minimal element, say Y_0. For every $y \in Y_0$, we have

$$\overline{\bigcup_{j=0}^{\infty} T(y)} = Y_0$$

(otherwise minimality of Y_0 is violated) and it follows that every $y \in Y_0$ is recurrent.

Remark: There are other (longer) proofs of this that do not use Zorn's Lemma or any other form of the Axiom of Choice.

Review and preview

Review

- A dynamical system is a mathematical model for a quantity that changes as time passes;
- usually one is interested in prediction and classification problems related to these systems;
- a t.d.s. is a pair (X, T) where X is compact metric and T is continuous;
- $T^n(x)$ means $T \circ T \circ \cdots \circ T(x)$;
- x is **recurrent** if for every $\epsilon > 0$, there is $n \geq 1$ such that $d(T^n(x), x) < \epsilon$;
- every t.d.s. has at least one recurrent point.

Preview

In the remainder of the talk, I want to show you how the ideas of dynamical systems can be used to prove a theorem which seems to have nothing to do with dynamics, given how it is stated.

Definition

A **coloring** of a set S is a function from S to a finite set. The elements of the range of the function are called **colors**.

Example

Let $S = \mathbb{N} = \{0, 1, 2, 3...\}$. Let

$$f(x) = \begin{cases} \text{red} & \text{if } x \text{ is a multiple of } 3 \\ \text{blue} & \text{otherwise} \end{cases}$$

This produces the coloring

$$\color{red}{0} \quad \color{blue}{1} \quad \color{blue}{2} \quad \color{red}{3} \quad \color{blue}{4} \quad \color{blue}{5} \quad \color{red}{6} \quad \color{blue}{7} \quad \color{blue}{8} \quad \color{red}{9} \quad \color{blue}{10} \quad \color{blue}{11} \quad \color{red}{12} \quad \color{blue}{13} \quad \color{blue}{14} \quad \color{red}{15} \cdots$$

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which can also be thought of as

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Let $S = \mathbb{N} = \{0, 1, 2, 3...\}$. Let

$$f(x) = \begin{cases} \text{red} & \text{if } x \text{ is a multiple of } 3 \\ \text{blue} & \text{otherwise} \end{cases}$$

and also written as

$$R, B, B, R, B, B, R, B, B, R, B, B, ...$$

Arithmetic progressions

Definition

An **arithmetic progression (AP)** is a finite subset of the natural numbers of the form

$$\{n, n+g, n+2g, n+3g, n+4g, ..., n+(d-1)g\}$$

where $n, g \in \mathbb{N}$. g is called the **gap size** of the AP; d is called the **length** of the AP.

Example

$\{7, 12, 17, 22, 27, 32, 37, 42\}$ is an AP of length 8 and gap size 5.

Theorem (van der Waerden, 1927)

Given any coloring of the natural numbers and given any d, there is a monochromatic AP of length d.

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Given any coloring of the natural numbers and given any d, there is a monochromatic AP of length d.

van der Waerden proved this theorem using purely combinatorial methods (sieving). Today, there are many proofs known (one using graph theory, one using harmonic analysis, one using complex analysis).

Theorem (van der Waerden, 1927)

Given any coloring of the natural numbers and given any d, there is a monochromatic AP of length d.

In 1977 Furstenberg gave a proof of this theorem using topological dynamics!

Interestingly, the dynamical proof of van der Waerden's theorem can be adapted to prove lots of similar results (about the existence of monochromatic patterns) which as of today have no other method of proof.

van der Waerden's Theorem

Theorem (van der Waerden, 1927)

Given any coloring of the natural numbers and given any d, there is a monochromatic AP of length d.

To prove this statement using topological dynamics, you need a topological dynamical system (X, T).

Question

What is the X, and what is the T?

What is X?

To start with, let the set of colors be called C. C is a finite set.

Example

$C = \{\text{red, green, blue}\} = \{R, G, B\}$.

What is X?

Next, let X be the set of infinite sequences where each element in the sequence is an element of C.

Example

If $C = \{\text{red, green, blue}\} = \{R, G, B\}$, one element of X might be

$$x = R, G, G, B, R, G, B, R, R, G, B, ...$$

Note that each coloring of $\mathbb{N}$ gives rise to a single point in X. For example, the above point x would come from the coloring (just as well, x "is" the coloring)

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad \cdots$$

What is X?

Recall: X is the set of colorings (i.e. infinite sequences of colors). Now we define the distance between two colorings:

Definition

Given $x, x' \in X$, set

$$d(x, x') = \frac{1}{2^n} \Leftrightarrow x \text{ and } x' \text{ disagree at position } n$$
$$\text{but agree at positions } 0, 1, 2, ..., n-1$$

Example

Let $x = R, G, G, V, R, G, O, R, R, V, G, ...$ and
let $x' = R, G, G, V, V, O, R, G, G, ...$
Then $d(x, x') = \frac{1}{2^4}$ (they disagree in the fourth position).

What is X?

Fact

X, with the distance function d, is a compact metric space.

Reason (for experts only): Put the discrete topology on the set C of colors (C is finite, hence compact); the metric described earlier makes X homeomorphic to $C^{\mathbb{N}}$ with the product topology (which is compact by Tychonoff's theorem).

What is X?

Recall (from two slides ago)

$$d(x,x') = \frac{1}{2^n} \Leftrightarrow x \text{ and } x' \text{ disagree at position } n$$
$$\text{but agree at positions } 0,1,2,\dots,n-1$$

Observation

$$d(x,x') < 1 \Leftrightarrow \; x \text{ and } x' \text{ start with the same symbol}$$
$$\Leftrightarrow \; \text{the colorings } x \text{ and } x' \text{ give } 0 \text{ the same color}$$

What is T?

Recall: X is the set of sequences of colors (compact metric space).
Now for our transformation T:

Definition

Let X be the set of colorings (i.e. sequences of colors). Let $T : X \to X$ be the **shift map**, which takes an element of X and erases the first symbol in the sequence. Symbolically, if

$$x = c_0, c_1, c_2, c_3, c_4, c_5, c_6, \dots$$

then

$$T(x) = c_1, c_2, c_3, c_4, c_5, c_6, \dots$$

This T is continuous (if $d(x, x') < \frac{1}{2^n}$, then x and x' agree in the first n positions, so $T(x)$ and $T(x')$ agree in the first $n-1$ positions, so $d(T(x), T(x')) < \frac{1}{2^{n-1}}$).

What is T?

Recall: X is the set of sequences of colors (compact metric space). Now for our transformation T:

Definition

Let X be the set of colorings (i.e. sequences of colors). Let $T : X \to X$ be the **shift map**, which takes an element of X and erases the first symbol in the sequence. In terms of colorings, if

$$x = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad \cdots$$

then

$$T(x) = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cdots$$

What is T?

Example

If $x = R, G, G, B, R, G, B, R, R, B, G, \ldots$ then

$$\begin{aligned} T(x) &= G, G, B, R, G, B, R, R, B, G, \ldots \\ T^2(x) = T(T(x)) &= G, B, R, G, B, R, R, B, G, \ldots \\ T^3(x) &= B, R, G, B, R, R, B, G, \ldots \end{aligned}$$

In general, $T^n : X \to X$ forgets the first n entries of the sequence x.

What is T?

Example

If $x = R, G, G, B, R, G, B, R, R, B, G, \ldots$ then

$$T(x) = G, G, B, R, G, B, R, R, B, G, \ldots$$
$$T^2(x) = T(T(x)) = G, B, R, G, B, R, R, B, G, \ldots$$
$$T^3(x) = B, R, G, B, R, R, B, G, \ldots$$

Note:

The element at position 0 of $T^n(x)$ is the same as the element at position n of x.
More generally, the element at position m of $T^n(x)$ is the same as the element at position $m + n$ of x.

What is T?

Example

If $x = R, G, G, B, R, G, B, R, R, B, G, \ldots$ then

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As a consequence...

$$\begin{aligned} d(T^n(x), T^{n+g}(x)) < 1 &\Leftrightarrow T^n(x) \text{ and } T^{n+g}(x) \text{ have the same} \\ &\qquad \text{color at position } 0 \\ &\Leftrightarrow x \text{ has the same color at positions} \\ &\qquad n \text{ and } n+g. \end{aligned}$$

What is T?

Recall (from the previous slide)

$d(T^n(x), T^{n+g}(x)) < 1$ if and only if x has the same color at positions n and $n+g$.

What is T?

Recall (from the previous slide)

$d(T^n(x), T^{n+g}(x)) < 1$ if and only if x has the same color at positions n and $n+g$.

Similarly...

$d(T^n(x), T^{n+g}(x)) < 1$ and $d(T^n(x), T^{n+2g}(x)) < 1$ if and only if the coloring given by x has the same colors at positions $n, n+g$ and $n+2g$

and more generally...

$d(T^n(x), T^{n+jg}(x)) < 1$ for all $j \in \{0, 1, ..., d-1\}$ if and only if the coloring given by x assigns the same colors to the numbers $n, n+g, ..., n+(d-1)g$.

Connecting van der Waerden's theorem with dynamics

Earlier I mentioned this theorem:

Theorem

Let X be a compact metric space and T a continuous map from X to itself. Then there is a point $x \in X$ which is recurrent, i.e. for this x, for every $\epsilon > 0$, there is a natural number g such that

$$d(x, T^g(x)) < \epsilon.$$

Furstenberg, using the previous theorem as the base case, gave a proof by induction of the following:

Multiple Recurrence Theorem (Furstenberg, 1977)

Let X be a compact metric space and T a continuous map from X to itself. Then, for every $d \in \mathbb{N}$, there is a point $y \in X$ such that for every $\epsilon > 0$, there is a natural number g such that for all $j \in \{0, 1, 2, ..., d - 1\}$,

$$d(y, T^{jg}(y)) < \epsilon.$$

Such a point y is called **multiply recurrent**.

Connecting van der Waerden's theorem with dynamics

Using some other (relatively elementary) tools from topology, from the Multiple Recurrence Theorem one can deduce

Corollary

Let X be a compact metric space and T a continuous map from X to itself. Then, for every $d \in \mathbb{N}$, every $x \in X$ and every $\epsilon > 0$, there are natural numbers n and g such that for all $j \in \{0, 1, 2, ..., d-1\}$,

$$d(T^n(x), T^{n+jg}(x)) < \epsilon.$$

Sketch of proof: Restrict T to $Y = \overline{\bigcup_{j=0}^{\infty} T^j(x)}$; apply the MRT to the t.d.s. (Y, T) to find a multiply recurrent y which must be arbitrarily close to some $T^n(x)$; then use continuity of T.

Let's put all this together:

Start with a coloring of $\mathbb{N}$.

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \cdots$

Our goal is to show that there is a monochromatic AP of length d.

Start with a coloring of $\mathbb{N}$.

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 $\cdots$

First, think of this coloring as a point $x \in X$:

$$x = R, G, B, R, B, B, G, O, B, O, B, R, G, B, B, R, R, ...$$

A proof of van der Waerden's theorem

Start with a coloring of $\mathbb{N}$.

$x = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \cdots$

Second, apply the corollary of the MRT to this x, using the shift map T and $\epsilon = 1$. This gives an n and a g such that

$$d(T^n(x), T^{n+jg}(x)) < 1 \quad \text{for } j \in \{0, 1, ..., d-1\}.$$

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$$d(T^n(x), T^{n+jg}(x)) < 1 \quad \text{for } j \in \{0, 1, ..., d-1\}.$$

This means that for this n and this g, $T^n(x)$ and $T^{n+jg}(x)$ have the same color at position 0, for all $j \in \{0, 1, ..., d-1\}$.

A proof of van der Waerden's theorem

Start with a coloring of $\mathbb{N}$.

$x = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \cdots$

Having obtained n and g such that $T^n(x)$ and $T^{n+jg}(x)$ have the same color at position 0, for all $j \in \{0, 1, ..., d-1\}$,

we see that x must have the same color at positions $n, n+g, n+2g, ..., n+(d-1)g$, i.e. that

$$\{n, n+g, n+2g, ..., n+(d-1)g\}$$

forms a monochromatic AP of length d.
This proves van der Waerden's theorem!

The end

van der Waerden's theorem is not the only thing that, while seeming to have nothing to do with dynamical systems, is explained (best explained?) by rephrasing the problem in the context of dynamics.

Other stuff that (surprisingly) has to do with dynamics:

- The existence of absolutely normal numbers
- Perelman's proof of the Poincaré conjecture
- the Ising model of ferromagnetism

So you should learn dynamics!